An Equiprobable Approximation to the Bivariate Lognormal

Economic agents face risks of many kinds, which may mutually covary. A stock broker, for example, is likely to earn a salary bonus that is positively related to the performance of the stock market; if that broker also has personal stock investments, his financial wealth and labor income will be positively correlated.

The first part of this handout presents a convenient (and empirically realistic) formulation in which a consumer faces two shocks (which can be interpreted as a shock to noncapital income and a shock to the rate of return) that are distributed according to a multivariate lognormal that allows for correlation between them. The second part describes a computationally simple and convenient method for approximating that joint distribution.

1 Theory

Consider a consumer who faces both a risk to transitory noncapital income[1]

$$\theta_{1,t+1} \equiv \log \Theta_{1,t+1} \sim \mathcal{N}(-0.5\sigma_1^2, \sigma_1^2) \tag{1}$$

and a risky log rate-of-return that is affected by following factors: the riskless rate r; a risk premium φ; an additional constant ζ (whose purpose will become clear below); a component that is linearly related to $\theta_{1,t+1}$; and an independent shock $\theta_2 \sim \mathcal{N}(-0.5\sigma_2^2, \sigma_2^2)$:

$$\mathbf{r}_{t+1} \equiv \log \mathbf{R}_{t+1} = \mathsf{r} + \varphi + \zeta + \omega\theta_{1,t+1}(\sigma_2/\sigma_1) + \theta_{2,t+1} \tag{2}$$

for some constant ω. Since $(\sigma_2/\sigma_1)\omega\theta_{1,t+1}$ is the only component of $\mathbf{r}_{t+1}$ that covaries with $\theta_{1,t+1}$,

$$\begin{aligned}
\text{cov}(\theta_{1,t+1}, \mathbf{r}_{t+1}) &= \text{cov}(\theta_{1,t+1}, (\sigma_2/\sigma_1)\omega\theta_{1,t+1}) \\
&= \omega(\sigma_2/\sigma_1)\underbrace{\text{cov}(\theta_{1,t+1}, \theta_{1,t+1})}_{=\sigma_1^2} \\
&= \omega\sigma_2\sigma_1.
\end{aligned}$$

Equation (2) yields a description of the return process in which the parameter ω controls the correlation between the risky log return shock and the risky log labor income shock. If $\omega = 0$ the processes are independent.

Now we want to find the value of ζ such that the mean risky return is unaffected by σ_1^2 (so that we will be able to understand clearly the distinct effects of labor income risk, the independent component of rate-of-return risk σ_2^2, and the correlation between labor income risk and rate-of-return risk, ω). Thus, we want to find the ζ such that

$$\mathbb{E}_t[\mathbf{R}_{t+1}] = e^{\mathsf{r}+\varphi} \tag{3}$$

[1] The assumed distribution has the property $\mathbb{E}[\Theta_{1,t+1}] = 1$, cf. MathFacts.

regardless of the values of σ_1^2 and σ_2^2. We therefore need:

$$\begin{aligned}\mathbb{E}[e^{\zeta+(\sigma_2/\sigma_1)\omega\theta_{1,t+1}+\theta_{2,t+1}}] &= 1. \\ \log\mathbb{E}[e^{\zeta+(\sigma_2/\sigma_1)\omega\theta_{1,t+1}+\theta_{2,t+1}}] &= 0.\end{aligned} \tag{4}$$

Using standard facts about lognormals (cf. `MathFacts`), and for convenience defining $\hat{\omega} = (\sigma_2/\sigma_1)\omega$, we have

$$\begin{aligned}0. &= \zeta - 0.5\hat{\omega}\sigma_1^2 - 0.5\sigma_2^2 + 0.5\hat{\omega}^2\sigma_1^2 + 0.5\sigma_2^2 \\ &= \zeta - 0.5\sigma_1^2\hat{\omega}(1-\hat{\omega}) \\ \zeta &= 0.5(\hat{\omega}-\hat{\omega}^2)\sigma_1^2 = 0.5(\omega\sigma_2\sigma_1 - \omega^2\sigma_2^2).\end{aligned} \tag{5}$$

2 Computation

A key step in the computational solution of any model with uncertainty is the calculation of expectations. Writing $\tilde{\Theta}_1 \equiv \tilde{\Theta}_{1,t+1}$ and $\tilde{\mathbf{R}} \equiv \mathbf{R}_{t+1}$ and $\mathbb{E}[\bullet] = \mathbb{E}_t[\bullet_{t+1}]$, the expectation of some function h that depends on the realization of the risky return $\tilde{\mathbf{R}}$ and the labor income shock is:

$$\mathbb{E}[\mathrm{h}(\tilde{\Theta}_1, \tilde{\mathbf{R}})] = \int_{\underline{\Theta}_1}^{\bar{\Theta}_1}\int_{\underline{\mathbf{R}}}^{\bar{\mathbf{R}}} \mathrm{h}(\tilde{\Theta}_1, \tilde{\mathbf{R}})d\mathrm{F}(\tilde{\Theta}_1, \tilde{\mathbf{R}}) \tag{6}$$

where $\mathrm{F}(\tilde{\Theta}_1, \tilde{\mathbf{R}})$ is the joint cumulative distribution function. Standard numerical computation software can compute this double integral, but at such a slow speed as to be almost unusable. Computation of the expectation can be massively speeded up by advance construction of a numerical approximation to $\mathrm{F}(\tilde{\Theta}_1, \tilde{\mathbf{R}})$.

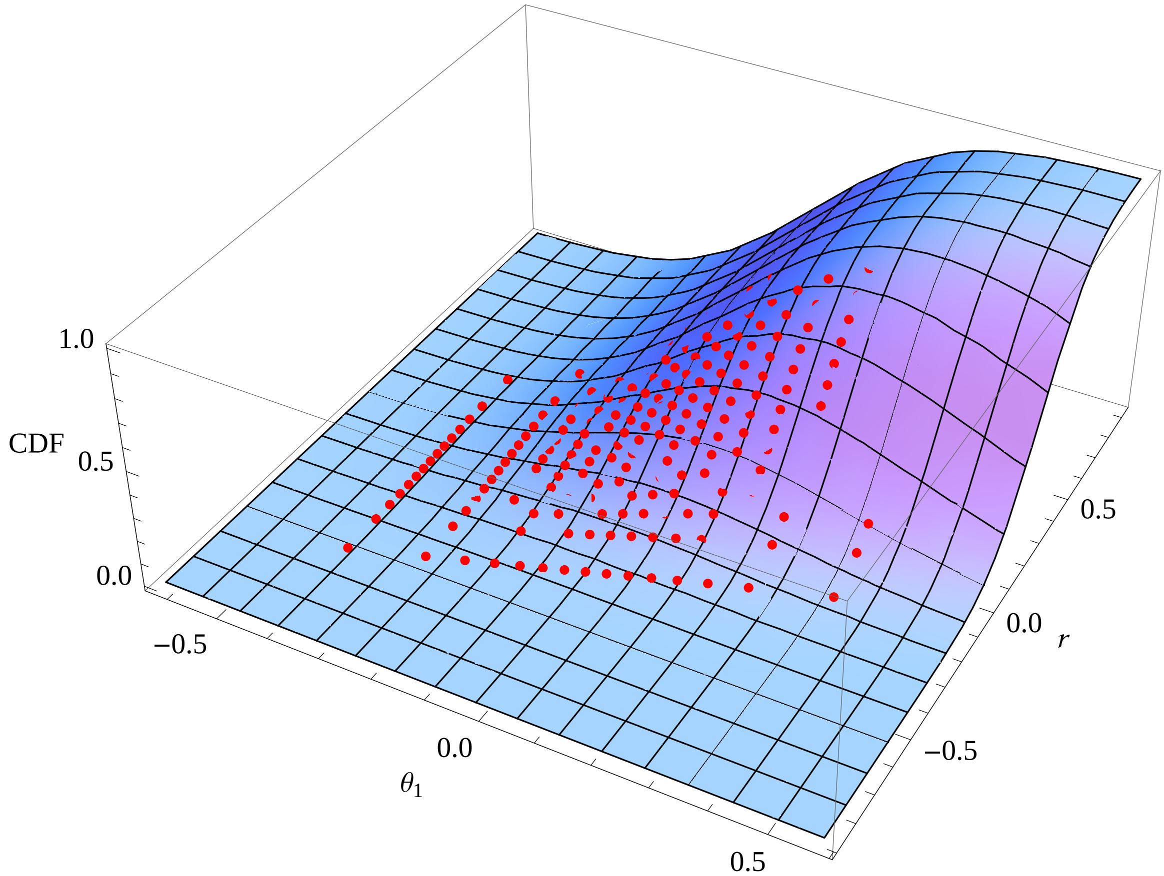
Such approximations generally take the approach of replacing the distribution function with a discretized approximation to it; appropriate weights $w_{i,j}$ are attached to each of a finite set of points indexed by i and j, and the approximation to the integral is given by:

$$\mathbb{E}[\mathrm{h}(\tilde{\Theta}_1, \tilde{\mathbf{R}})] \approx \sum_{i=1}^{n}\sum_{j=1}^{m} \mathrm{h}(\hat{\Theta}_1[i,j], \hat{\mathbf{R}}[i,j])w[i,j] \tag{7}$$

where the $\hat{\Theta}_1$ and $\hat{\mathbf{R}}$ matrices contain the conditional means of the two variables in each of the $\{i,j\}$ regions. Various methods are used for constructing the weights $w[i,j]$ and the nodes (the i and j points for Θ_1 and $\mathbf{R}$).

Perhaps the most popular such method is Gauss-Hermite interpolation (see Judd (1998) for an exposition, or Kopecky and Suen (2010) for some alternatives). Here, we will pursue a particularly intuitive alternative: Equiprobable discretization. In this method, $m = n$ and boundaries on the joint CDF are determined in such a way as to divide up the total probability mass into submasses of equal size (each of which therefore has a mass of n^{-2}). This is conceptually easier if we represent the underlying shocks as statistically independent, as with $\theta_{1,t+1}$ and $\theta_{2,t+1}$ above; in that case, each submass is a square region in the Θ_1 and Θ_2 grid. We then compute the average value of Θ_1 and $\mathbf{R}$ *conditional* on their being located in each of the subdivisions of the range of the CDF.

Figure 1 'True' CDF With Approximation Points in Red for $\omega = 0.5$



Since, in this specification, $\mathbf{R}$ is a function of Θ_1, the $\mathbf{R}$ values are indexed by both i and j, but since we have written Θ_1 as IID, the representation of the approximating summation is even simpler than (7):

$$\mathbb{E}[\mathrm{h}(\tilde{\Theta}_1, \tilde{\mathbf{R}})] \approx n^{-2} \sum_{i=1}^{n} \sum_{j=1}^{n} \mathrm{h}(\hat{\Theta}_1[i], \mathbf{R}(\hat{\Theta}_1[i], \hat{\Theta}_2[j])) \tag{8}$$

where the function $\mathbf{R}(\Theta_1, \Theta_2)$ is implicitly defined by (2).

Details can be found in the *Mathematica* notebook associated with this handout. A particular example, in which $\sigma_2^2 = \sigma_1^2$ and $\omega = 0.5$, is illustrated in figure 1; the red dots reflect the height of the approximation to the CDF above the conditional mean values for Θ_1 and $\mathbf{R}$ within each of the equiprobable regions.

References

Judd, Kenneth L. (1998): *Numerical Methods in Economics*. The MIT Press, Cambridge, Massachusetts.

Kopecky, Karen A., and Richard M.H. Suen (2010): “Finite State Markov-Chain Approximations To Highly Persistent Processes,” *Review of Economic Dynamics*, 13(3), 701–714, http://www.karenkopecky.net/RouwenhorstPaper.pdf.