

Consumption out of Risky Assets

Merton (1969) and Samuelson (1969) solve the optimization problem for a consumer who receives no labor income¹ and whose only available financial asset has a risky return factor \mathcal{R} which is lognormally distributed, $\log \mathcal{R}_{t+1} \sim \mathcal{N}(r - \sigma_r^2/2, \sigma_r^2)$.

With market assets m , the dynamic budget constraint is:

$$m_{t+1} = (m_t - c_t)\mathcal{R}_{t+1}. \quad (1)$$

Start with the standard Euler equation for consumption under CRRA utility:

$$1 = \beta \mathbb{E}_t \left[\mathcal{R}_{t+1} \left(\frac{c_{t+1}}{c_t} \right)^{-\rho} \right] \quad (2)$$

and postulate a solution of the form $c_t = \kappa m_t$:

$$\begin{aligned} 1 &= \beta \mathbb{E}_t \left[\mathcal{R}_{t+1} \left(\frac{\kappa m_{t+1}}{\kappa m_t} \right)^{-\rho} \right] \\ &= \beta \mathbb{E}_t \left[\mathcal{R}_{t+1} \left(\frac{(m_t - c_t)\mathcal{R}_{t+1}}{m_t} \right)^{-\rho} \right] \\ &= \beta \mathbb{E}_t \left[\mathcal{R}_{t+1} \left(\frac{(1 - \kappa)m_t \mathcal{R}_{t+1}}{m_t} \right)^{-\rho} \right] \\ &= \beta \mathbb{E}_t \left[\mathcal{R}_{t+1} ((1 - \kappa)\mathcal{R}_{t+1})^{-\rho} \right] \\ &= \beta (1 - \kappa)^{-\rho} \mathbb{E}_t \left[\mathcal{R}_{t+1}^{1-\rho} \right] \\ (1 - \kappa)^\rho &= \beta \mathbb{E}_t \left[\mathcal{R}_{t+1}^{1-\rho} \right] \\ (1 - \kappa) &= \left(\beta \mathbb{E}_t \left[\mathcal{R}_{t+1}^{1-\rho} \right] \right)^{1/\rho} \\ \kappa &= 1 - \left(\beta \mathbb{E}_t \left[\mathcal{R}_{t+1}^{1-\rho} \right] \right)^{1/\rho}. \end{aligned} \quad (3)$$

Since $\log \mathcal{R}_{t+1}^{1-\rho} = (1 - \rho) \log \mathcal{R}_{t+1}$, fact `ELogNormTimes` implies that (using the definition $\exp(\bullet) \equiv e^\bullet$),

$$\mathbb{E}_t \left[\mathcal{R}_{t+1}^{1-\rho} \right] = \exp[(1 - \rho)(r - \sigma_r^2/2) + (1 - \rho)^2 \sigma_r^2/2] \quad (4)$$

$$\begin{aligned} &= \exp[(1 - \rho)r - (1 - \rho)(\sigma_r^2/2) + (1 - \rho)(\sigma_r^2/2) - \rho(1 - \rho)\sigma_r^2/2] \\ &= \exp[(1 - \rho)r - \rho(1 - \rho)\sigma_r^2/2]. \end{aligned} \quad (5)$$

Substituting in (3):

$$\kappa = 1 - \beta^{1/\rho} \exp \left[\rho \left((1/\rho - 1)r - (1 - \rho)\sigma_r^2/2 \right) \right]^{1/\rho}$$

¹Carroll and Kimball (1996) show that the Merton-Samuelson case (CRRA utility and no labor income risk) is one of only two configurations of risk and utility (in the HARA class) for which the consumption rule is linear; the other configuration is Constant Absolute Risk Aversion with only labor income risk and no rate-of-return risk. If both rate-of-return and non-rate-of-return risk are present, the consumption function is strictly concave for any utility function in the HARA class (which encompasses all commonly used utility functions, including CARA, CRRA, and Stone-Geary; even quadratic utility is a limiting case).

$$= 1 - \beta^{1/\rho} \exp \left[(1/\rho - 1)r - (1 - \rho)\sigma_r^2/2 \right]. \quad (6)$$

Now use `OverPlus` and `TaylorOne`,

$$\begin{aligned} \beta^{1/\rho} &= \left(\frac{1}{1 + \vartheta} \right)^{1/\rho} \\ &\approx 1 - \rho^{-1}\vartheta \\ &\approx \exp(-\rho^{-1}\vartheta) \end{aligned}$$

which hold if $\rho^{-1}\vartheta$ is close to zero. Substituting into (6) and using `ExpPlus` and `LogEps` gives

$$\begin{aligned} \kappa &\approx 1 - (1 + \rho^{-1}(r - \vartheta) - r + (\rho - 1)\sigma_r^2/2) \\ &= r - \rho^{-1}(r - \vartheta) - (\rho - 1)(\sigma_r^2/2) \end{aligned} \quad (7)$$

which, when $\sigma_r^2 = 0$, reduces to the usual perfect foresight formula $\kappa = r - \rho^{-1}(r - \vartheta)$.

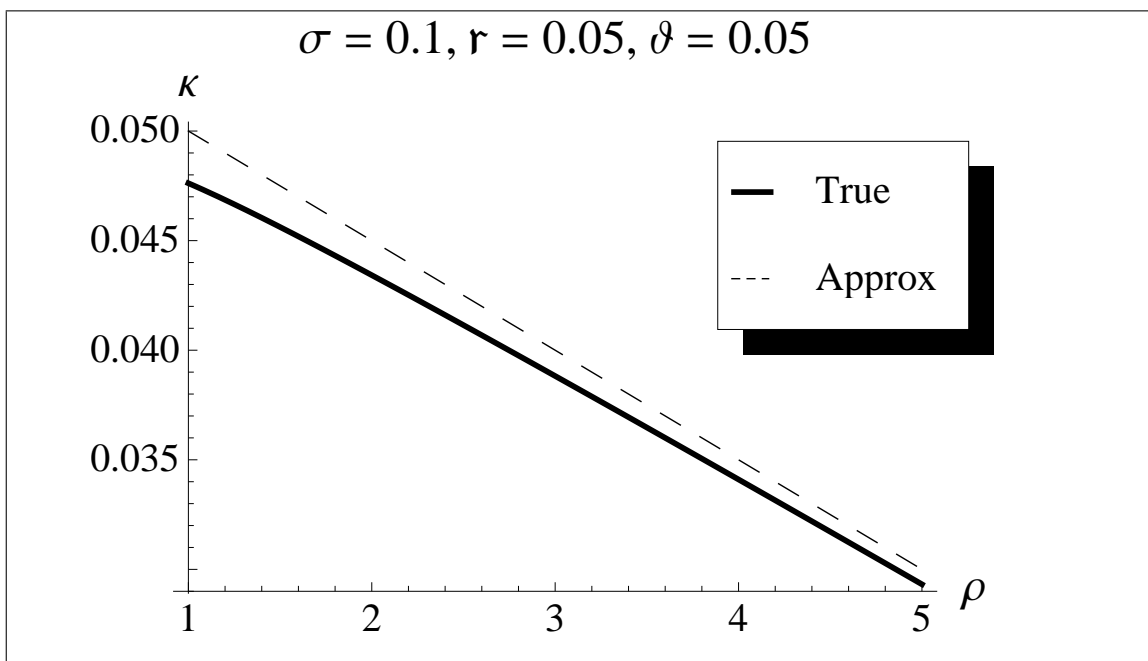
This equation implies the plausible result that as unavoidable uncertainty in the financial return goes up (σ_r^2 rises) the level of consumption falls (because $\rho > 1$, so $-(\rho - 1)$ which multiplies σ_r^2 is negative), reflecting the precautionary saving motive.²

The top figure plots the marginal propensity to consume as a function of the coefficient of relative risk aversion (for both the true MPC and the approximation derived above), under parameter values such that $\vartheta - r \approx 0$ so that a change in ρ does not affect the MPC through the intertemporal elasticity of substitution channel. As intuition would suggest, as consumers become more risk averse, they save more (the MPC is lower; that is, the plotted loci are downward-sloping).

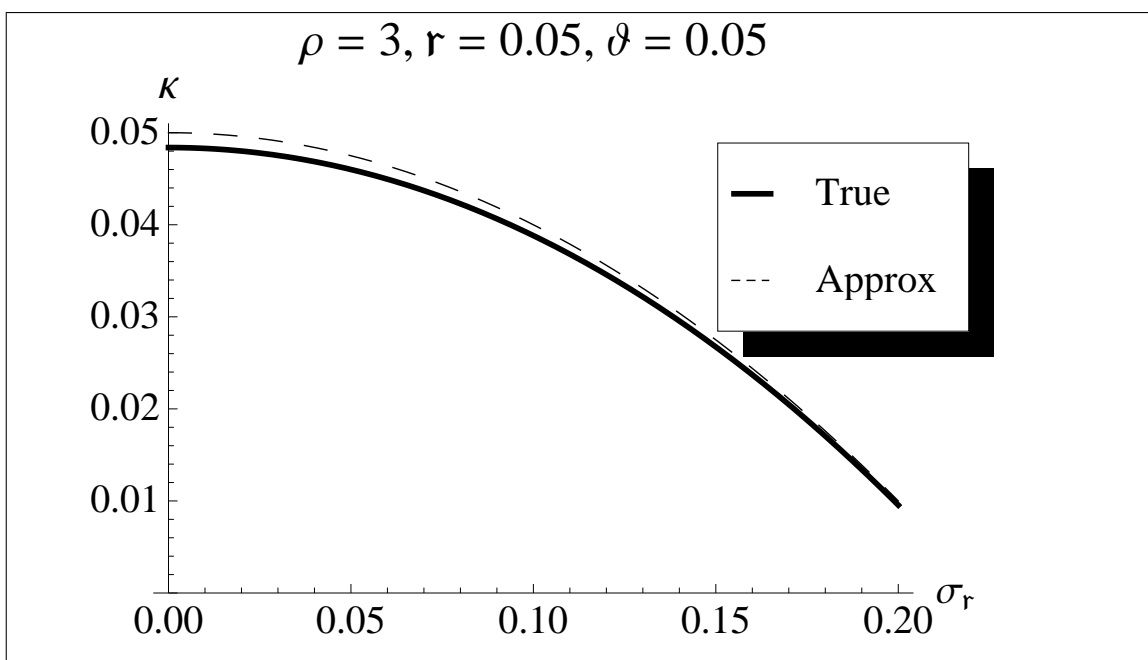
The other way to see the precautionary effect is to examine the effect on the MPC of a change in risk. For a consumer with relative risk aversion of 3, the bottom figure shows that as the size of the risk increases, the MPC κ falls.

²It is surprising to note that for a consumer with logarithmic utility, a mean-preserving spread in risk has no effect on the level of consumption (this can be seen by substituting $\rho = 1$ into (7), which causes the term involving risk σ_r^2 to disappear from the equation). The reason this is surprising is that intuition suggests that if the consumer's consumption (and therefore current saving) are unchanged, the increase in uncertainty must constitute a mean-preserving spread in future consumption, which by Jensen's inequality should yield higher expected marginal utility. The place where this argument goes wrong is that it forgets that the expectation in the Euler equation $u'(c_t) = \beta \mathbb{E}_t[\mathbf{R}_{t+1}u'(c_{t+1})]$ is also affected by a covariance between \mathbf{R}_{t+1} and $u'(c_{t+1})$; the case of log utility is the special case where this boils down to a constant times $\mathbb{E}_t[\mathbf{R}_{t+1}/\mathbf{R}_{t+1}] = 1$, which is why the expected marginal utility is unaffected by the unavoidable increase in risk. This is yet another reason (if any more were needed) to conclude that logarithmic utility does not exhibit sufficient curvature to plausibly represent attitudes toward risk. ($\rho \geq 2$ seems a plausible lower bound).

Figure 1 Relation Between MPC κ and Parameters



(a) Marginal Propensity to Consume Falls as Relative Risk Aversion ρ Rises



(b) Marginal Propensity to Consume Falls as Risk σ Rises

References

- CARROLL, CHRISTOPHER D., AND MILES S. KIMBALL (1996): “On the Concavity of the Consumption Function,” *Econometrica*, 64(4), 981–992, <http://econ.jhu.edu/people/ccarroll/concavity.pdf>.
- MERTON, ROBERT C. (1969): “Lifetime Portfolio Selection under Uncertainty: The Continuous Time Case,” *Review of Economics and Statistics*, 50, 247–257.
- SAMUELSON, PAUL A. (1969): “Lifetime Portfolio Selection By Dynamic Stochastic Programming,” *Review of Economics and Statistics*, 51, 239–46.