# Liquidity Constraints and Precautionary Saving

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Abstract

Economists working with numerical solutions to the optimal consumption/saving problem under uncertainty have long known that there are quantitatively important interactions between liquidity constraints and precautionary saving behavior. This paper provides the analytical basis for those interactions. First, we explain why the introduction of a liquidity constraint increases the precautionary saving motive around levels of wealth where the constraint becomes binding. Second, we provide a rigorous basis for the oft-noted similarity between the effects of introducing uncertainty and introducing constraints, by showing that in both cases the effects spring from the concavity in the consumption function which either uncertainty or constraints can induce. We further show that consumption function concavity, once created, propagates back to consumption functions in prior periods. Finally, our most surprising result is that the introduction of additional constraints beyond the first one, or the introduction of additional risks beyond a first risk, can actually reduce the precautionary saving motive, because the new constraint or risk can 'hide' the effects of the preexisting constraints or risks.

**Keywords**: liquidity constraints, consumption function, uncertainty, stochastic income, precautionary saving

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# Contents

# 1 Introduction

In the past decade, numerical solutions to the optimal consumption/saving problem have become the standard theoretical tool for modelling consumption behavior. Numerical solutions have become popular because analytical solutions are not available for realistic descriptions of utility and uncertainty, nor for the plausible case where consumers face both liquidity constraints and uncertainty.

A drawback to numerical solutions is that it is often difficult to determine why results come out the way they do. A leading example of this problem comes in the relationship between precautionary saving behavior and liquidity constraints. At least since Zeldes 1984, economists working with numerical solutions have known that liquidity constraints can strictly increase precautionary saving under very general circumstances - even for consumers with quadratic utility functions that provide no inherent precautionary saving motive.[1] On the other hand, simulation results have sometimes seemed to suggest that liquidity constraints and precautionary saving are substitutes rather than complements. For example, Samwick 1995 has shown that unconstrained consumers with a precautionary saving motive in a retirement saving model behave in ways qualitatively and quantitatively similar to the behavior of liquidity constrained consumers facing no uncertainty.

This paper provides the theoretical tools needed to make sense of the interactions between liquidity constraints and precautionary saving. These tools provide a rigorous theoretical foundation that can be used to clarify the reasons for the numerical literature's apparently contrasting findings.

For example, one of the paper's simpler points is a proof that when a liquidity constraint is added to the standard consumption problem, the resulting value function exhibits increased prudence around the level of wealth where the constraint becomes binding. (Kimball 1990 defines prudence of the value function and shows that it is the key theoretical requirement to produce precautionary saving.) Constraints induce precaution basically because constrained agents have less flexibility in responding to shocks because the effects of the shocks cannot be spread out over time; thus risk has a bigger negative effect on expected utility (or value) for constrained agents than for unconstrained agents. The precautionary saving motive is heightened by the desire (in the face of risk) to make such constraints less likely to bind.

At a deeper level, we show that the effect of a constraint on prudence is an example of a more general theoretical result: Prudence is induced by concavity of the consumption function. Since a constraint causes consumption concavity around the point where the constraint binds, adding a constraint necessarily boosts prudence around that point. We show that this concavity-boosts-prudence result holds not just for quadratic utility functions but for any utility function in the Hyperbolic Absolute Risk Aversion (HARA) class (which includes Constant Relative Risk

[1] For a detailed but nontechnical discussion of simulation results on the relation between liquidity constraints and precautionary saving, see Carroll 2001. For a prominent numerical examination of some of the interactions between precautionary saving and liquidity constraints, see Deaton 1991, who also provides conditions under which the problem defines a contraction mapping.

Aversion, Constant Absolute Risk Aversion, and most other commonly used forms).

These results tie in closely with findings in our previous paper, Carroll and Kimball 1996, which shows that within the HARA class, the introduction of uncertainty causes the consumption function to become strictly concave (in the absence of constraints) for all but a few carefully chosen combinations of utility function and uncertainty. Indeed, taken together, the results of the two papers can be seen as establishing rigorously the sense in which precautionary saving and liquidity constraints are very close substitutes.[2] In this paper, in fact, we provide an example of a specific kind of uncertainty that (under CRRA utility, in the limit) induces a consumption function that is pointwise identical to the consumption function that would be induced by the addition of a liquidity constraint.

We further show that, once consumption concavity is induced (either by a constraint or by uncertainty), it propagates back to periods before the period in which the concavity is first created.[3] But in the quadratic utility case the propagation is rather subtle: the prior-period consumption rules are concave (and prudence is higher) at any level of wealth from which it is possible that the constraint will bind, but also possible that it may not bind. Precautionary saving takes place in such circumstances because a bit more saving can reduce the probability that the constraint will bind.

The fact that precautionary saving arises from the *possibility* that constraints might bind may help to explain why such a high percentage of households cite precautionary motives as the most important reason for saving (Kennickell and Lusardi 1999) even though the fraction of households who report actually having been constrained in the past is relatively low (Jappelli 1990).

Our final theoretical contribution is to show that the introduction of further liquidity constraints beyond the first one may actually *reduce* precautionary saving by 'hiding' the effects of the preexisting constraint(s); identical logic implies that uncertainty can hide the effects of a constraint, because the consumer may need to save so much for precautionary reasons that the constraint becomes irrelevant. For example, a typical perfect foresight model of retirement consumption for a consumer with Social Security income implies that the legal constraint on borrowing against Social Security benefits will cause the consumer to run assets down to zero, then set consumption equal to income for the remainder of life. Now consider adding the possibility of large medical expenses near the end of life (e.g. nursing home fees). Under reasonable assumptions the consumer may save enough against this risk to render the constraint irrelevant.

The rest of the paper is structured as follows. To fix notation and ideas, the next section presents a very brief review of the logic of precautionary saving in the standard case (without liquidity constraints). The third section sets out our general theoretical framework. The fourth section shows that concavity of the consump-

[2]See Fernandez-Corugedo 2000 for a related demonstration that 'soft' liquidity constraints bear an even closer resemblance to precautionary behavior. Mendelson and Amihud **?** provide an impressive treatment of a similar problem.

[3]Our previous paper showed that the concavity induced by uncertainty propagated backwards, but the proofs in that paper cannot be applied to concavity created by a liquidity constraint.

tion function heightens prudence. The fifth section shows how concavity, whether induced by constraints or uncertainty, propagates to previous periods. Section 6 shows how the introduction of a constraint creates a precautionary saving motive for consumers with quadratic utility, and how that precautionary motive propagates backwards; it also shows that the introduction of additional liquidity constraints beyond the first constraint does *not* necessarily further increase (and can even reduce) the precautionary motive at any given level of wealth. The next section examines the effects of introducing a constraint when utility is of the CRRA form, and contains our example in which a constraint and uncertainty have identical effects on the consumption function. It uses this example to make the point that introduction of uncertainty can hide the effects of constraints or preexisting uncertainty. The final section concludes.

# 2 A Brief Review

We begin with a very brief review of the logic of precautionary saving in the two-period case; with minor modifications this two-period model is directly applicable to the multiperiod case when the second period utility function is interpreted as the value function arising from optimal behavior from time $t+1$ on.

Consider a consumer with initial wealth $w_t$ who anticipates uncertain future income $y_{t+1} = \bar{y} + \zeta_{t+1}$ where $\zeta_{t+1}$ is stochastic. This consumer solves the unconstrained optimization problem

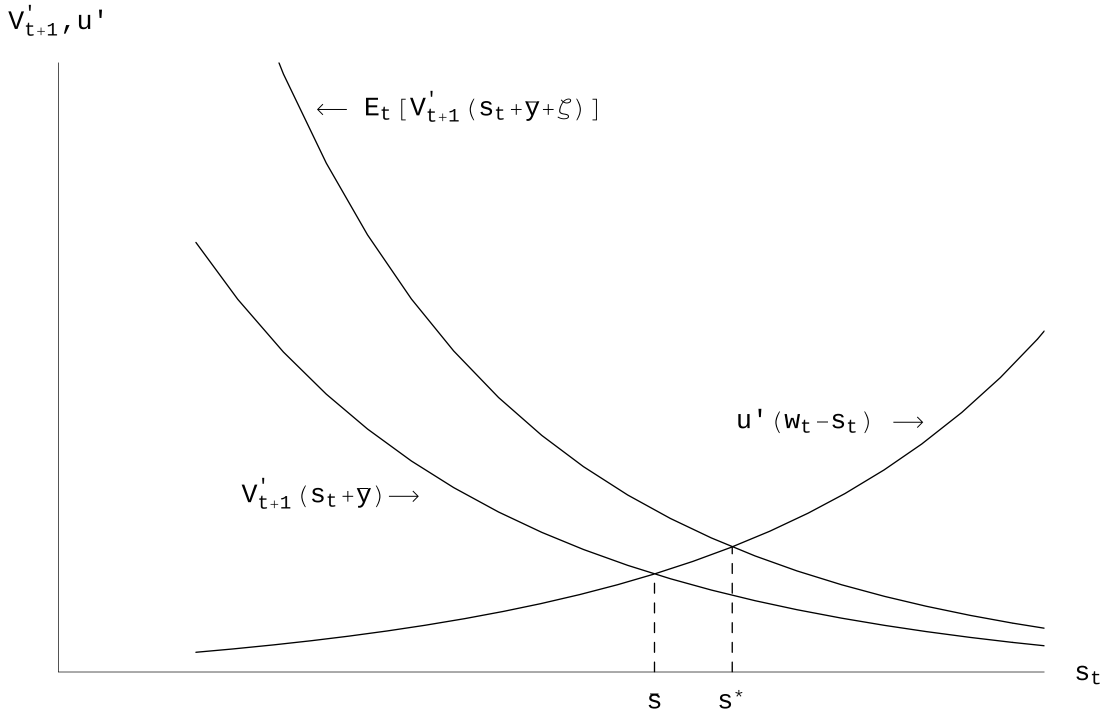
$$\max_{\{c_t\}} \; u(c_t) + E_t \left[ V_{t+1}(w_t - c_t + \bar{y} + \zeta_{t+1}) \right], \tag{1}$$

or, equivalently,

$$\max_{\{s_t\}} \; u(w_t - s_t) + E_t \left[ V_{t+1}(s_t + \bar{y} + \zeta_{t+1}) \right]. \tag{2}$$

The familiar first-order condition for this problem is to set $u'(c_t) = E_t[V'_{t+1}(w_t - c_t + \bar{y} + \zeta_{t+1})]$ or, equivalently, $u'(w_t - s_t) = E_t[V'_{t+1}(s_t + \bar{y} + \zeta_{t+1})]$.

Figure 1 shows a standard example of this problem in which both $u$ and $V_{t+1}$ are Constant Relative Risk Aversion (CRRA) utility functions. The consumer is assumed to start period $t$ with amount of wealth $w_t$. The horizontal axis represents the choice of how much the consumer saves in period $t$, and the upward-sloping curve labelled $u'(w_t - s_t)$ reflects the period-$t$ marginal utility of the consumption $(w_t - s_t)$ associated with that choice of saving. The downward-sloping curve labelled $V'_{t+1}(s_t + \overline{y})$ reflects the marginal value the consumer would experience in period $t+1$ as a function of saving $s_t$ in the previous period if she were perfectly certain to receive income $\overline{y}$ in period $t+1$. This curve is downward-sloping as a function of $s_t$ because the more the consumer saves in period $t$, the more is available for consumption in period $t+1$ and thus the lower is the marginal utility of spending in $t+1$. In this perfect-certainty case, the utility-maximizing level of consumption is found at the point of intersection between the $u'(w_t - s_t)$ and the $V'_{t+1}(s_t + \overline{y})$ curves, i.e. the level of saving that equalizes the current and future marginal utility




EquateMargUtils.eps

Figure 1: Determining Consumption in the Two Period Case Given Initial Wealth $w_t$

of consumption. In the CRRA case where the period-utility functions $u(c)$ and $V_{t+1}(w_{t+1})$ are identical, the optimal solution is to consume exactly half of total lifetime resources in the first period; the point labelled $\bar{s}$ reflects this level of saving.

In the case where period $t+1$ income is uncertain, first-period marginal utility must be equated to the expectation of the second-period marginal value function. That expectation will be a convex combination of the marginal values associated with each possible outcome, where the weights on each outcome are given by the probability of that outcome. For illustration, suppose there is a 0.5 probability that the consumer will receive income $\overline{y}+\eta$ and a 0.5 probability that she will receive income $\overline{y}-\eta$. Since the probability of each outcome is 1/2, the consumer's expected marginal value function for each $s_t$ will be traced out by the midpoint of the line segment connecting $V'_{t+1}(s_t+\bar{y}+\eta)$ and $V'_{t+1}(s_t+\bar{y}-\eta)$. Figure 2 illustrates the construction of the $E_t[V'_{t+1}(s_t+\overline{y}+\zeta_{t+1})]$ curve; for example, if the consumer chooses to save $s_t=\overline{s}$, then her expected marginal value in the second period is given by $.5V'_{t+1}(\bar{s}+\overline{y}+\eta)+.5V'_{t+1}(\bar{s}+\overline{y}-\eta)$, as shown in the figure.

The expected marginal value function traced out by this convex combination of the good and bad outcomes is reproduced and labelled $E_t[V'_{t+1}(s_t+\bar{y}+\zeta_{t+1})]$ in figure 1. The optimal level of saving $s^*$ under uncertainty is simply the level of $s_t$ at the intersection of $u'(w_t-s_t)$ and $E_t[V'_{t+1}(s_t+\bar{y}+\zeta_{t+1})]$, where the first order condition is satisfied. The magnitude of precautionary saving is the amount by which saving rises from the riskless case ($\bar{s}$) to the risky case ($s^*$).

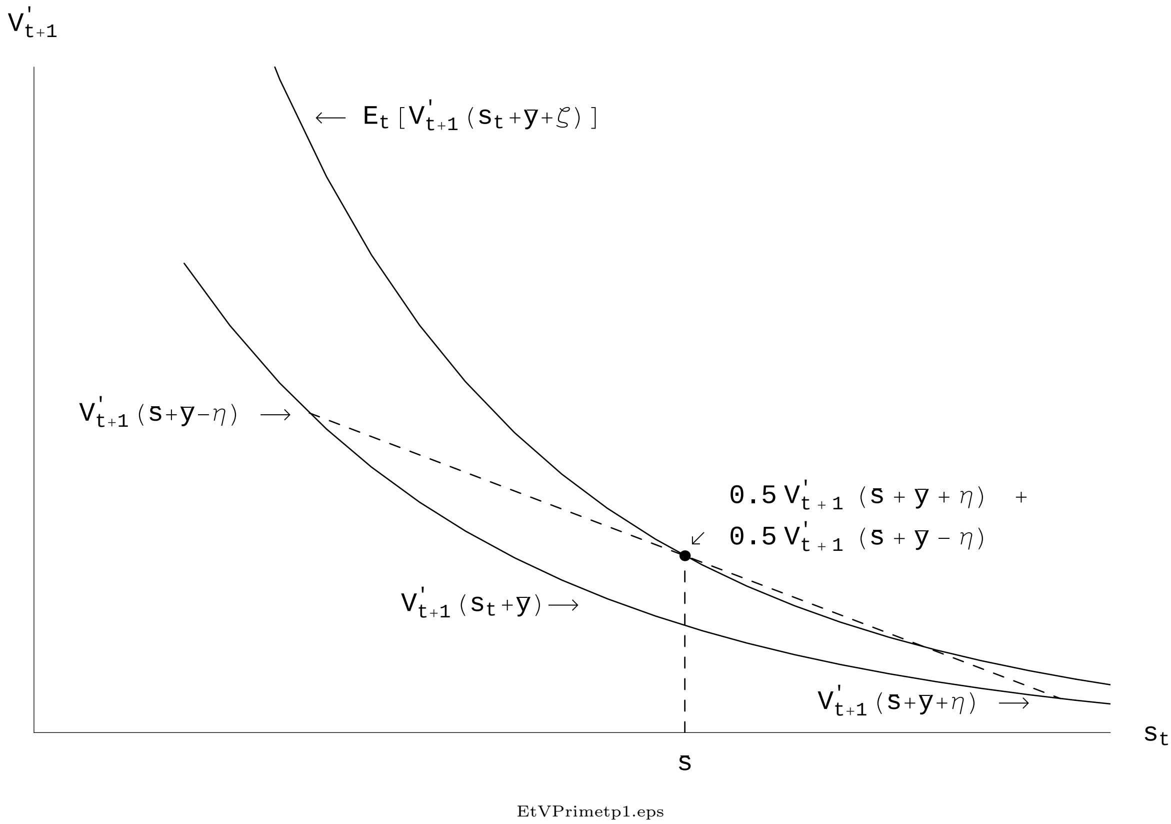



Figure 2: Construction of $E_t[V'_{t+1}]$

Figure 2 illustrates the simple point that the magnitude of precautionary saving is related to the degree of convexity of the marginal value function. Jensen's inequality guarantees that if $V'_{t+1}$ is strictly convex, then $E_t[V'_{t+1}(s_t + \bar{y} + \zeta_{t+1})] > V'_{t+1}(s_t + E_t[\bar{y} + \zeta_{t+1}])$ and consequently the intersection with $u'(w_t - s_t)$ will occur at a higher value of first-period saving. Clearly, if $V'_{t+1}$ were linear (as is true in the case of quadratic utility in the absence of liquidity constraints), mean-zero risks in period $t+1$ would not affect the expectation of the marginal value function, because the curve generated by the 'convex combination' would lie atop the original marginal value function. Thus, the convexity in the marginal value function creates a precautionary saving motive.

Formally, Kimball 1990 shows that the prudence of the value function (defined as $-V'''(w)/V''(w)$) measures the convexity of the marginal value function at $w$ and therefore the intensity of the precautionary saving motive at that point. To be precise, given two different value functions $V(w)$ and $\hat{V}(w)$, if the absolute prudence of $\hat{V}(w)$ is greater than for $V(w)$ (that is, if $-\hat{V}'''(w)/\hat{V}''(w) > -V'''(w)/V''(w)$) then the addition of a risk causes a greater rightward shift of expected $\hat{V}'(w)$ than of expected $V'(w)$. As figure 2 suggests, a greater rightward shift tends to produce a greater increase in precautionary saving.

Thus, to analyze the multiperiod case, we need to be able to characterize the degree of convexity of the marginal value function or the prudence of the value function.[4]

# 3 The Setup

Before stating and proving our main theorems, we need to lay out the basic setup of the consumption/saving problem with many periods. Consider a consumer who faces some future risks but is not subject to any current or future liquidity constraints. Assume that the consumer is maximizing the time-additive present discounted value of utility from consumption $u(c)$. Denoting the (possibly stochastic) gross interest rate and time preference factors as $R_t \in (0, \infty)$ and $\beta_t \in (0, \infty)$, respectively, and labelling consumption $c_t$, stochastic labor income $y_t$, and gross wealth (inclusive of period-t labor income) $w_t$, the consumer's problem can be writ-

[4]In order to use the prudence of the value function to gauge the effect of a risk in labor income at time $t+1$, we implicitly assume that this risk is independent of all the other risks realized in periods beyond $t+1$ that are already built into the shape of $V_{t+1}$. In other words, the effect of labor income on the value function must work entirely through its effect on wealth at time $t+1$. There are two possible approaches when the realization of $y_{t+1}$ is correlated with future risks, incomes, or rates of return. First, each period could be decomposed into two transitions, one where the information is revealed about the distribution of future incomes, rates of return, etc. and a second where the labor income at time $t+1$ is revealed. The other approach, which, when possible, is more powerful, is to capitalize all the future effects of a shock into wealth at time $t+1$. This approach is possible when the news revealed is mathematically equivalent to a particular effect on the quantity of an asset in the model.

ten as:[5]

$$\begin{aligned} V_t(w_t) &= \max_{\{c_t\}} u(c_t) + E_t\left[\sum_{s=t+1}^{T}\left(\prod_{j=t+1}^{s}\tilde{\beta}_j\right)u(\tilde{c}_s)\right] \\ s.t. & \qquad w_{t+1} = R_{t+1}(w_t - c_t) + y_{t+1}. \end{aligned} \tag{3}$$

As usual, the recursive nature of the problem makes this equivalent to the Bellman equation:

$$V_t(w_t) = \max_{\{c_t\}} u(c_t) + E_t[\tilde{\beta}_{t+1}V_{t+1}(\tilde{R}_{t+1}(w_t - c_t) + \tilde{y}_{t+1})]. \tag{4}$$

Defining

$$\Omega_t(s_t) = E_t[\tilde{\beta}_{t+1}V_{t+1}(\tilde{R}_{t+1}s_t + \tilde{y}_{t+1})] \tag{5}$$

where $s_t = w_t - c_t$ is the portion of period t resources saved, this becomes[6]

$$V_t(w_t) = \max_{\{c_t\}} u(c_t) + \Omega_t(w_t - c_t). \tag{6}$$

It is also useful to define $\breve{c}_t(\mu_t)$, $\breve{s}_t(\mu_t)$, and $\breve{w}_t(\mu_t)$ as:

$$\breve{c}_t(\mu_t) = u'^{-1}(\mu_t), \tag{7}$$

$$\breve{s}_t(\mu_t) = \Omega_t'^{-1}(\mu_t), \tag{8}$$

$$\breve{w}_t(\mu_t) = V_t'^{-1}(\mu_t). \tag{9}$$

In words, $\breve{c}_t(\mu_t)$ ('c-breve') indicates the level of consumption that yields marginal utility $\mu_t$ (note the mnemonic convenience of indicating marginal utility by the Greek letter spelled mu), $\breve{s}_t(\mu_t)$ indicates the level of end-of-period savings[7] in period $t$ that yields a discounted expected marginal value of $\mu_t$, and $\breve{w}_t(\mu_t)$ indicates the level of beginning-of-period wealth that would yield marginal value of $\mu_t$ assuming optimal (though potentially constrained) disposition of that wealth between

[5] We allow for a stochastic discount factor because some problems which contain a stochastic scaling variable (such as permanent income) can be analyzed more easily by dividing the problem through by the scale variable; this division induces a term that effectively plays the role of a stochastic discount factor.

[6] For notational simplicity we express the value function $V_t(w_t)$ and the expected discounted value function $\Omega_t(s_t)$ as functions simply of wealth and savings, but implicitly these functions reflect the entire information set as of time t; if, for example, the income process is not i.i.d., then information on lagged income or income shocks could be important in determining current optimal consumption. In the remainder of the paper the dependence of functions on the entire information set as of time $t$ will be unobtrusively indicated, as here, by the presence of the $t$ subscript. For example, we will call the policy rule in period $t$ which indicates the optimal value of consumption $c_t(w_t)$. In contrast, because we assume that the utility function is the same from period to period, the utility function has no $t$ subscript.

[7] We use the word 'savings' to indicate the level of wealth remaining in a period after that period's consumption has occurred; 'savings' is therefore a stock variable, and is distinct from 'saving' which is the difference between income and consumption.

consumption and saving.[8] In the absence of a liquidity constraint in period $t$, these definitions imply that for an optimizing consumer whose optimal choice of consumption in period $t$ yields marginal utility $\mu_t$,

$$c_t = \breve{c}_t(\mu_t), \tag{10}$$
$$s_t = \breve{s}_t(\mu_t), \tag{11}$$
$$w_t = \breve{w}_t(\mu_t). \tag{12}$$

In the presence of a liquidity constraint that requires $s_t \geq 0$, equation (11) becomes:

$$s_t = \max[0, \breve{s}_t(\mu_t)]. \tag{13}$$

Note that the budget constraint $w_t = c_t + s_t$ allows us to write:

$$\breve{w}_t(\mu_t) = \breve{c}_t(\mu_t) + \max[0, \breve{s}_t(\mu_t)]. \tag{14}$$

# 4 Prudence and Consumption Concavity

Our ultimate goal is to understand the relationship between liquidity constraints and precautionary saving. But the magnitude of precautionary saving depends on the absolute prudence of the value function. The purpose of this section is therefore to lay out the relationship between consumption concavity and prudence. Our analysis of consumption concavity is couched in general terms, and therefore applies whether the source of concavity is liquidity constraints or something else. This generality is useful, because there is a good candidate for the 'something else': uncertainty. Our treatment here will therefore alternate between discussion of the effects of imposing liquidity constraints and the effects of introducing uncertainty.

## 4.1 Defining Utility, Concavity, and Prudence

### 4.1.1 Definition of HARA Utility

Carroll and Kimball (1996) show that the introduction of uncertainty into a standard unconstrained optimal consumption problem causes the consumption policy function to become concave for consumers with utility in the Hyperbolic Absolute Risk Aversion class, defined as utility functions that satisfy

$$u'''(c)u'(c)/[(u''(c))^2] = k. \tag{15}$$

The HARA utility functions with positive, nonincreasing absolute prudence satisfy this equation with $k \geq 1$, quadratic utility satisfies it with $k = 0$, while the imprudent HARA utility functions satisfy it with $k < 0$.

[8] We chose the slightly unusual breve accent (˘) because of its rough resemblance to the shape of marginal utility $\mu$, which is the argument for the breve-accented functions.

The crucial element in the proof is to show that the value function satisfies the differential inequality

$$V'''(w)V'(w)/[(V''(w))^2] \geq k. \tag{16}$$

Since (as we show below) constraints can cause $V''$ to be discontinuous and $V'''$ to fail to exist entirely, the proof strategy of Carroll and Kimball (1996) involving condition (16) will not work when constraints exist. As a consequence, it will be more convenient to work with an alternative to (15) as our definition of the HARA class: Here we view the HARA class as those utility functions with nonnegative, nonincreasing absolute prudence that (after normalization) satisfy, for some constant $k$, either (1) $u'(c) = k - c$, with the domain of $c$ limited to $c < k$ (the quadratic case); (2) $u'(c) = (c - k)^{-\gamma}$ with $\gamma \geq 0$ and the domain of $c$ limited to $c > k$ (the main case); or (3) $u'(c) = e^{-ac}$ with $a > 0$ (the exponential case).

#### 4.1.2 Definition of Consumption Concavity

The central issue in our new approach will involve whether the value function exhibits what we will call "property CC". (The mnemonic is that "CC" stands for "concave consumption.") We will first consider property CC in a global sense, and then turn to definition of the property on a pointwise basis.

**Definition 1** *A function $F(x)$ has property CC in relation to a utility function $u(c)$ with $u' > 0$, $u'' < 0$ iff $F'(x) = u'(\phi(x))$ for some monotonically increasing concave function $\phi$.*

Thus, to say that property CC holds for a value function $V_t(w_t)$ is to say that there exists a concave $\phi(w_t)$ such that

$$V_t'(w_t) \quad = \quad u'(\phi(w_t)).$$

But the envelope theorem tells us that

$$V_t'(w_t) = u'(c_t(w_t)), \tag{17}$$

so property CC holding for $V_t(w_t)$ is equivalent to having a concave consumption function $\phi(w_t) = c_t(w_t)$.[9] We will need to use property CC with respect both to beginning-of-period value functions $V_t(w_t)$ and end-of-period value functions $\Omega_t(s_t)$; to avoid confusion we will designate the concave function associated with $\Omega_t(s_t)$ (if $\Omega_t(s_t)$ has property CC) as $\chi_t(s_t)$ and will reserve $c_t(w_t)$ for the beginning-of-period value functions.

It is easy to show by taking derivatives that if $V(w)$ satisfies property CC, then when $V'''(w)$ exists this condition reduces to the differential inequality (16), with $k = 0$ in the quadratic case, $k = 1 + (1/\gamma)$ in the main case and $k = 1$ in the exponential case.

Definition 1 did not distinguish between the case where $\phi$ was strictly concave and where it is linear (weakly concave), nor did it define the interval over which concavity was measured. For our proofs, we will need more precise definitions.

[9]Remember that the envelope theorem depends only on being able to spend *current* wealth on *current* consumption, so it holds whether or not there is a liquidity constraint.

**Definition 2** *A function $F(x)$ has property strict CC over the interval between $x_1$ and $x_2 > x_1$ in relation to a HARA utility function $u(c)$ with nonnegative, nonincreasing prudence iff*

$$F'(x) = u'(\phi(x))$$

*for some increasing function $\phi(x)$ that satisfies strict concavity over the interval from $x_1$ to $x_2$, defined by*

$$\phi(x) > \frac{x_2 - x}{x_2 - x_1}\phi(x_1) + \frac{x - x_1}{x_2 - x_1}\phi(x_2) \tag{18}$$

*for all $x \in (x_1, x_2)$.*

**Definition 3** *A function $F(x)$ has property borderline CC over the interval from $x_1$ to $x_2$ if equation* (18) *holds with equality.*

**Definition 4** *A function $F(x)$ has property CC (strict or borderline, respectively) at a point $x$ if there exists a $\delta > 0$ such that for all $x_1, x_2$ such that $x_1 < x < x_2$ and $|x_2 - x_1| < \delta$, the function exhibits property CC (strict or borderline, respectively) over the interval from $x_1$ to $x_2$.*

Note that if a function has property CC globally, then it will have either strict or borderline CC at every point.

Finally, we need to define when one function exhibits greater concavity than another.

**Definition 5** *Consider two functions $F(x)$ and $\hat{F}(x)$ that both exhibit property CC with respect to the same $u'$ at a point $x$ for some interval $(x_1, x_2)$ such that $x_1 < x < x_2$. Then $\hat{F}(x)$ exhibits property greater CC than $F(x)$ if*

$$\hat{\phi}(x) - \left(\frac{x_2 - x}{x_2 - x_1}\hat{\phi}(x_1) + \frac{x - x_1}{x_2 - x_1}\hat{\phi}(x_2)\right) \geq \phi(x) - \left(\frac{x_2 - x}{x_2 - x_1}\phi(x_1) + \frac{x - x_1}{x_2 - x_1}\phi(x_2)\right) \tag{19}$$

*for all $x \in (x_1, x_2)$, and property strictly greater CC if* (20) *holds as a strict inequality.*

The importance of strictly greater CC is its relationship to prudence.

**Lemma 1** *If $\hat{V}_t$ exhibits strictly greater CC than $V_t$ at point $w_t$, then absolute prudence of $\hat{V}_t(w_t)$ is greater than absolute prudence of $V_t(w_t)$.*

*Proof.* Kimball 1990 following Pratt 1964 shows that greater prudence can be defined as $\hat{V}_t'(w_t)$ being a convex function of $V_t'(w_t)$. But since $V_t'(w_t) = u'(c_t(w_t))$ and $\hat{V}_t'(w_t) = u'(\hat{c}_t(w_t))$ for the same monotonically downward sloping $u'$, greater CC of $\hat{V}_t$ than $V_t$ at $w_t$ implies $\hat{V}_t'(w_t)$ is a convex function of $V_t(w_t)$.

## 4.2 How Does Consumption Concavity Heighten Prudence?

Our method in this section will be to compare prudence in a *baseline* case where the consumption function $c_t(w_t)$ is linear to prudence in a *modified* situation in which the consumption function $\hat{c}_t(w_t)$ is a concavification of the baseline consumption function.

### 4.2.1 The CRRA Case

Our first baseline $c_t(w_t)$ will be the linear consumption function that arises under CRRA utility in the absence of labor income risk or constraints.[10] Below we show that imposing a constraint concavifies the consumption function. Similarly, Carroll and Kimball 1996 show that the addition of labor income risk renders the risk-modified consumption rule concave. In either case it is possible to show that as wealth approaches infinity the consumption rule in the modified situation $\hat{c}_t(w_t)$ approaches the consumption rule in the baseline situation. When the experiment is the imposition of a liquidity constraint, $\hat{c}_t(w_t)$ approaches $c_t(w_t)$ because as wealth approaches infinity the constraint becomes irrelevant because the probability that it will ever bind becomes zero. When the treatment is the addition of labor income risk, $\hat{c}_t(w_t)$ approaches $c_t(w_t)$ because as wealth approaches infinity the portion of future consumption that the consumer plans on financing out of the uncertain labor income stream becomes vanishingly small.[11] Formally, we can capture both the liquidity constraint and the precautionary saving cases with the assertion that

$$\lim_{w_t \to \infty} \hat{c}(w_t) - c(w_t) = 0.$$

**Theorem 1** *Consider an agent who has a utility function with $u'(c) > 0$, $u''(c) < 0$, $u'''(c) > 0$ and nonincreasing absolute prudence $-u'''(c)/u''(c)$ in two different situations. If optimal consumption in the baseline situation is described by a neoclassical consumption function $c_t(w_t)$ that is linear, while optimal behavior in the modified situation (indicated by a hat) is described by a concave neoclassical consumption function $\hat{c}_t(w_t)$ and if $\lim_{w_t \to +\infty} \hat{c}_t(w_t) - c_t(w_t) = 0$, then at any given level of wealth $w_t$ the value function in the modified situation exhibits greater absolute prudence than in the baseline situation. Prudence at $w_t$ in the modified situation is strictly greater if and only if the modified consumption function is strictly concave at some wealth level at or above $w_t$.*

*Proof.* By the envelope theorem, the marginal value of wealth is always equal to the marginal utility of consumption as long as it is possible to spend *current* wealth

[10]The analysis below goes through even if there is rate-of-return risk in the problem, so long as the rate-of-return risk is not modified when the labor income risk is added.

[11]Since in the CRRA case the proportionate effect of risk on consumption depends on the *square* of the standard deviation of the risk relative to wealth, as this ratio gets small as wealth approaches infinity, the absolute size of the effect of the risk in reducing consumption approaches zero.

for *current* consumption. That is,

$$\begin{aligned} V_t'(w_t) &= u'(c_t(w_t)) && (20) \\ \hat{V}_t'(w_t) &= u'(\hat{c}_t(w_t)). && (21) \end{aligned}$$

Differentiating each of these equations with respect to $w_t$,[12]

$$\begin{aligned} V_t''(w_t) &= u''(c_t(w_t))c_t'(w_t) && (22) \\ \hat{V}_t''(w_t) &= u''(\hat{c}_t(w_t))\hat{c}_t'(w_t). && (23) \end{aligned}$$

Taking another derivative can run afoul of the possible discontinuity in $\hat{c}_t'(w_t)$ that we will show below can arise from liquidity constraints, but to establish intuition it is useful to consider first the case where $\hat{c}_t''(w_t)$ exists; we will then adapt the proof for the case where $\hat{c}_t''(w_t)$ does not exist. For the baseline linear consumption function,

$$\begin{aligned} V_t'''(w_t) &= u'''(c_t(w_t))[c_t'(w_t)]^2 + u''(c_t(w_t))[c_t''(w_t)] && (24) \\ &= u'''(c_t(w_t))[c_t'(w_t)]^2, && (25) \end{aligned}$$

where the second line follows because with a linear consumption function $c_t''(w_t) = 0$. Thus,

$$\text{Absolute Prudence} = \frac{-V_t'''(w_t)}{V_t''(w_t)} = \left(\frac{-u'''(c_t(w_t))}{u''(c_t(w_t))}\right) c_t'(w_t).$$

In the modified situation with a concave consumption function, where $\hat{c}_t''(w_t)$ exists,

$$\begin{aligned} \hat{V}_t'''(w_t) &= u'''(\hat{c}_t(w_t))[\hat{c}_t'(w_t)]^2 + u''(\hat{c}_t(w_t))[\hat{c}_t''(w_t)] && (26) \\ -\frac{\hat{V}_t'''(w_t)}{\hat{V}_t''(w_t)} &= -\left(\frac{u'''(\hat{c}_t(w_t))[\hat{c}_t'(w_t)]^2 + u''(\hat{c}_t(w_t))[\hat{c}_t''(w_t)]}{u''(\hat{c}_t(w_t))\hat{c}_t'(w_t)}\right) && (27) \\ -\frac{\hat{V}_t'''(w_t)}{\hat{V}_t''(w_t)} &= \left(\frac{-u'''(\hat{c}_t(w_t))}{u''(\hat{c}_t(w_t))}\right)\hat{c}_t'(w_t) - \frac{\hat{c}_t''(w_t)}{\hat{c}_t'(w_t)}. && (28) \end{aligned}$$

As can be seen from Figure 3,[13] the assumption that the two consumption functions converge asymptotically, $\lim_{w_t \to +\infty} \hat{c}_t(w_t) - c_t(w_t) = 0$, together with the linearity of $c_t(w_t)$ and concavity of $\hat{c}_t(w_t)$, guarantees that the marginal propensity to

[12] Since $\hat{c}(w_t)$ is concave, it has left-hand and right-hand derivatives at every point, though the left-hand and right-hand derivatives may not be equal. Equation (23) should be interpreted accordingly as applying to left-hand and right-hand derivatives separately. (Reading (23) in this way implies that $\hat{c}_t'(w_t^-) \geq \hat{c}_t'(w_t^+)$; therefore $\hat{V}''(w_t^-) \leq \hat{V}''(w_t^+)$).

[13] This figure was generated using simulation programs written for Carroll 2001; these programs are available on Carroll's web page. The parameterization is as follows. The coefficient of relative risk aversion is $\rho = 2$, the time preference factor is $\beta = 0.95$, the gross interest factor is $R = 1.04$, the growth factor for permanent income is $G = 1.01$. The stochastic process for transitory income for $\hat{c}(w)$ involves a small probabilitly (0.005) that income will be zero; if it is not zero, then the transitory shock is lognormally distributed with standard deviation of 0.2. Both rules reflect the limit as the number of remaining periods of life approaches infinity.

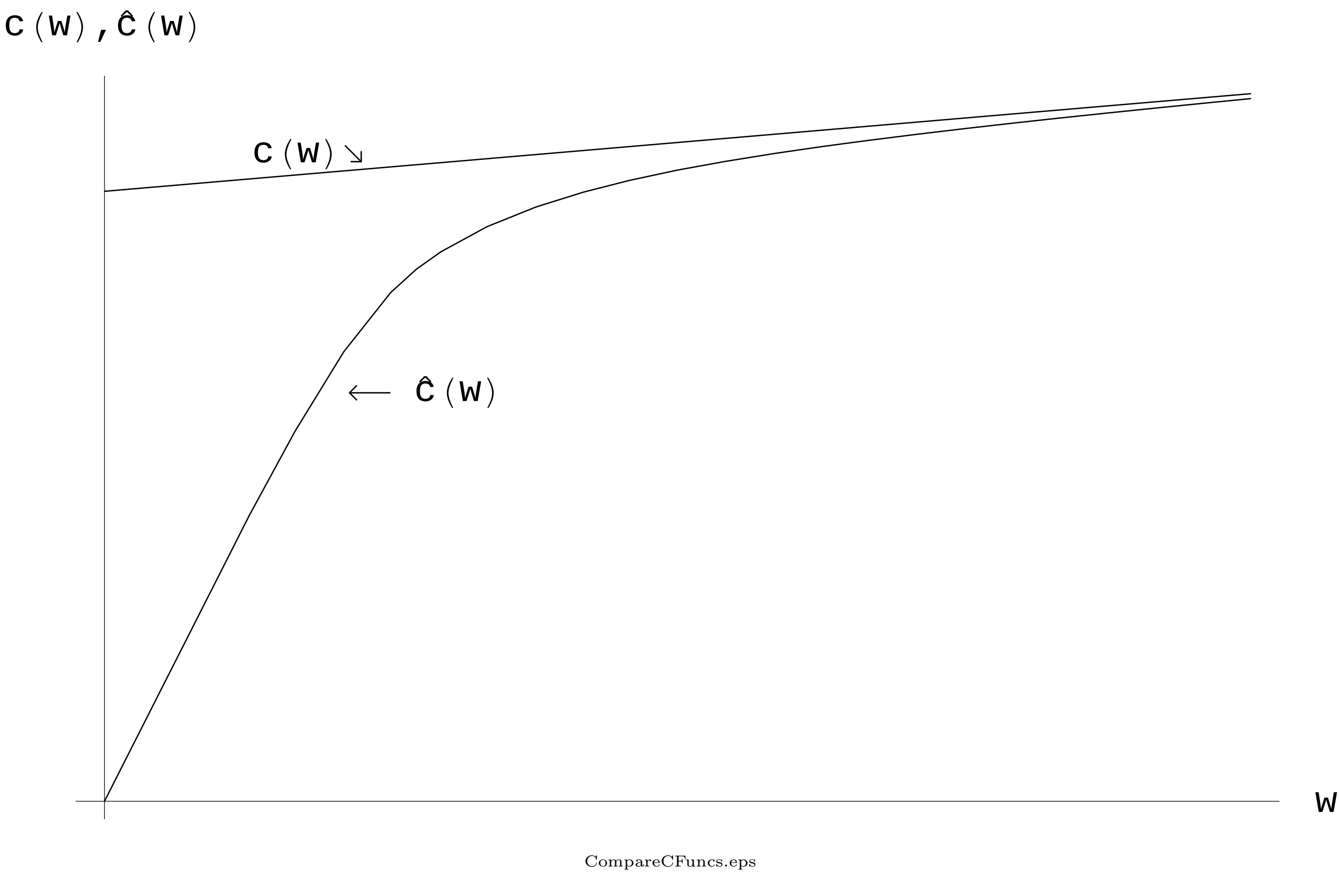



Figure 3: Consumption Functions in the Baseline and Modified Cases

consume is higher and the level of consumption lower in the modified situation, $\hat{c}_t'(w_t) \geq c_t'(w_t)$ and $\hat{c}_t(w_t) \leq c_t(w_t)$. The inequalities are strict if there is any strictness to the concavity of $\hat{c}_t(\cdot)$ at any level of wealth above $w_t$.

In conjunction with the assumption of nonincreasing absolute prudence of the utility function, $\hat{c}_t(w_t) \leq c_t(w_t)$ implies that

$$\frac{-u'''(\hat{c}_t(w_t))}{u''(\hat{c}_t(w_t))} \geq \frac{-u'''(c_t(w_t))}{u''(c_t(w_t))}. \tag{29}$$

Therefore, where $\hat{c}_t''(w_t)$ exists,

$$\begin{aligned}
-\frac{\hat{V}_t'''(w_t)}{\hat{V}_t''(w_t)} &= \left(\frac{-u'''(\hat{c}_t(w_t))}{u''(\hat{c}_t(w_t))}\right)\hat{c}_t'(w_t) - \underbrace{\overbrace{\hat{c}_t''(w_t)}^{\leq 0} / \overbrace{\hat{c}_t'(w_t)}^{>0}}_{\leq 0} &(30)\\
&\geq \left(\frac{-u'''(c_t(w_t))}{u''(c_t(w_t))}\right)c_t'(w_t) &(31)\\
&= -\frac{V_t'''(w_t)}{V_t''(w_t)}. &(32)
\end{aligned}$$

That is, concavity of $\hat{c}_t(w_t)$ along with $\lim_{w_t\to\infty} c_t(w_t) - \hat{c}_t(w_t) = 0$ implies that the absolute prudence of $\hat{V}_t(w_t)$ is greater than the absolute prudence of $V_t(w_t)$.

Even when the absolute prudence of the utility function is constant, (31) is strict whenever either (1) $\hat{c}_t(\cdot)$ is strictly concave at some level of wealth above $w_t$ (because, with weak concavity everywhere, strict concavity anywhere above $w_t$ implies that $\hat{c}_t'(w_t) > c_t'(w_t)$); or (2) $\hat{c}_t(\cdot)$ is strictly concave exactly *at* $w_t$ (because strict concavity at $w_t$ implies that $-\frac{\hat{c}_t''(w_t)}{\hat{c}_t'(w_t)} > 0$). Conversely, if $\hat{c}_t(\cdot)$ is linear at $w_t$ and all higher levels of wealth, (31) clearly holds with equality. We can summarize by saying that the inequality (31) which expresses the result of the theorem is *strict* if and only if $\hat{c}_t(\cdot)$ is strictly concave at or above $w_t$.

What if $\hat{c}_t''(w_t)$ and $\hat{V}_t'''(w_t)$ do not exist? Informally, if nonexistence is caused by a constraint binding at $w_t$, the effect will be a discrete decline in the marginal propensity to consume at $w_t$, which can be thought of as $\hat{c}_t''(w_t) = -\infty$, implying positive infinite prudence at that point (see (30)). Formally, if $\hat{c}_t''(w_t)$ does not exist greater prudence of $\hat{V}_t$ than $V_t$ is defined as $\frac{\hat{V}_t''(w_t)}{V_t''(w_t)}$ being a decreasing function of $w_t$. By (22) and (23),

$$\frac{\hat{V}_t''(w_t)}{V_t''(w_t)} \equiv \left(\frac{u''(\hat{c}_t(w_t))}{u''(c_t(w_t))}\right)\left(\frac{\hat{c}_t'(w_t)}{c_t'(w_t)}\right). \tag{33}$$

The second factor, $\frac{\hat{c}_t'(w_t)}{c_t'(w_t)}$, is globally decreasing (see Figure 3; it declines monotonically toward 1). At any specific value of $w_t$ where $\hat{c}_t''(w_t)$ does not exist because the left and right hand values of $\hat{c}_t'$ are different, we say that $\hat{c}_t'$ is decreasing if

$$\lim_{w^-\to w} \hat{c}_t'(w_t) \quad > \quad \lim_{w^+\to w} \hat{c}_t'(w_t). \tag{34}$$

As for the first factor, note that nonexistence of $\hat{V}_t^{'''}(w_t)$ and/or $\hat{c}_t^{''}(w_t)$ do not spring from nonexistence of either $u^{'''}(c)$ or $\lim_{w \uparrow w_t} \hat{c}_t^{'}(w)$ (for our purposes, when the left and right derivatives of $\hat{c}_t(w_t)$ differ at a point, the relevant derivative is the one coming from the left; rather than carry around the cumbersome limit notation, read the following derivation as applying to the left derivative). To discover whether $\frac{\hat{V}_t''(w_t)}{V_t''(w_t)}$ is decreasing we can simply differentiate:

$$\frac{d}{dw_t}\left(\frac{u''(\hat{c}_t(w_t))}{u''(c_t(w_t))}\right) = \frac{u^{'''}(\hat{c}_t(w_t))\hat{c}_t^{'}(w_t)u^{''}(c_t(w_t)) - u^{''}(\hat{c}_t(w_t))u^{'''}(c_t(w_t))c_t^{'}(w_t)}{[u''(c_t(w_t))]^2}. \tag{35}$$

Since the denominator is always positive, this will be negative if the numerator is negative, i.e. if

$$u^{'''}(\hat{c}_t(w_t))u^{''}(c_t(w_t))\hat{c}_t^{'}(w_t) \leq u^{''}(\hat{c}_t(w_t))u^{'''}(c_t(w_t))c_t^{'}(w_t) \tag{36}$$

$$\left(\frac{u^{'''}(\hat{c}_t(w_t))}{u^{''}(\hat{c}_t(w_t))}\right)\hat{c}_t^{'}(w_t) \leq \left(\frac{u^{'''}(c_t(w_t))}{u^{''}(c_t(w_t))}\right)c_t^{'}(w_t) \tag{37}$$

$$\underbrace{\left(\frac{-u^{'''}(\hat{c}_t(w_t))}{u^{''}(\hat{c}_t(w_t))}\right)}_{\text{Absolute prudence at } \hat{c}_t(w_t)}\hat{c}_t^{'}(w_t) \geq \underbrace{\left(\frac{-u^{'''}(c_t(w_t))}{u^{''}(c_t(w_t))}\right)}_{\text{Absolute prudence at } c_t(w_t)}c_t^{'}(w_t). \tag{38}$$

Recall that $\hat{c}_t(w_t) \leq c_t(w_t)$ (see figure 3), so the assumption of nonincreasing absolute prudence tells us that the absolute prudence term on the LHS of (38) is greater than that on the RHS. But by the assumption of concavity of $\hat{c}_t(w_t)$ we also know that $\hat{c}_t^{'}(w_t) \geq c_t^{'}(w_t)$. Hence both terms on the LHS are greater than or equal to the corresponding terms on the RHS. The inequality is strict at any point for which $\hat{c}_t^{'}(w_t) > c_t^{'}(w_t)$.

Note finally that condition (38) is equivalent to our definition of property greater CC for consumption functions for which $c^{'}(w_t)$ and $\hat{c}^{'}(w_t)$ exist in the sense of left and right derivatives.

Thus, combining all of the factors involved in comparing the prudence of $\hat{V}_t(w_t)$ to the prudence of $V_t(w_t)$, we have shown that the value function in the modified situation will exhibit strictly greater prudence at any given $w_t$ than the value function in the baseline situation if and only if $\hat{c}_t(w_t)$ is strictly concave at $w_t$ or at some level of wealth above $w_t$.

### 4.2.2 Counterclockwise Concavification Causes a Strict Increase in Prudence

We assumed above that the baseline consumption function was linear. It will be useful for later purposes to have a slightly more general analysis. The idea is to think of the consumption function in the modified situation as being a twisted version of the consumption function in the baseline situation, where the kind of twisting allowed is a progressively larger increase in the MPC as the level of wealth gets lower. We call this a 'counterclockwise' concavification, to capture the sense that at any specific level of wealth, we can think of the increase in the MPC at lower levels of wealth as being a counterclockwise rotation of the lower portion of the consumption function around that level of wealth.

**Definition 6** *Function $\hat{c}_t(w_t)$ is a counterclockwise concavification of $c_t(w_t)$ around $\omega^{\#}$ if the following conditions hold:*

1. $\hat{c}_t(\omega) = c_t(\omega)$ *for* $\omega \geq \omega^{\#}$
2. $\lim_{\omega \uparrow w_t} \left( \frac{\hat{c}_t'(\omega)}{c_t'(\omega)} \right)$ *is weakly decreasing in* $w_t$ *everywhere below* $\omega^{\#}$
3. $\lim_{\omega \uparrow \omega^{\#}} \left( \frac{\hat{c}_t'(\omega)}{c_t'(\omega)} \right) \geq 1$
4. *If* $\lim_{\omega \uparrow \omega^{\#}} \left( \frac{\hat{c}_t'(\omega)}{c_t'(\omega)} \right) = 1$*, then* $\lim_{\omega \uparrow \omega^{\#}} \left( \frac{\hat{c}_t''(\omega)}{c_t''(\omega)} \right) > 1$

where the limits using $\omega$ are necessary to allow for the possibility of discrete drops in the MPC at potential 'kink points' in the two consumption functions. (This is a generalization of the original situation considered in theorem 1 in the sense that the original proof can be thought of as a specialization of this setup in the case where $\omega^{\#}$ approaches infinity and where the initial consumption function is restricted to linearity).

Given this definition, we have

**Theorem 2** *Consider an agent who satisfies the conditions of theorem 1 except that, rather than being linear, the optimal neoclassical consumption function in the baseline situation $c_t(w_t)$ is concave. If $\hat{c}_t(w_t)$ is a counterclockwise concavification of $c_t(w_t)$ around $\omega^{\#}$ then the value function associated with $\hat{c}_t(w_t)$ exhibits greater prudence than the value function associated with $c_t(w_t)$. Prudence at $w_t$ is strictly greater in the modified situation than in the baseline situation all levels of wealth $w_t$ below $\omega^{\#}$.*

*Proof.* The proof is identical to the proof of theorem 1, except where that proof demonstrates that $\left( \frac{\hat{c}_t'(w_t)}{c_t'(w_t)} \right)$ is weakly decreasing for the setup described in the theorem; that requirement is now assumed directly.

We will also need to define a sense in which $\hat{c}_t(w_t)$ is a global counterclockwise concavification of $c_t(w_t)$:

**Definition 7** *Function $\hat{c}_t(w_t)$ is a global counterclockwise concavification of $c_t(w_t)$ if $\hat{c}_t(w_t)$ can be constructed from $c_t(w_t)$ by sequence counterclockwise concavifications around a set of points $\vec{\omega}$.*

#### 4.2.3 The Exponential Case

The assumption $\lim_{w_t \to \infty} \hat{c}_t(w_t) - c_t(w_t) = 0$ will be true if consumers have CRRA utility and if the difference between the baseline and the modified situations is the addition of either labor income risk or a liquidity constraint. However, if the consumer's utility function is of the CARA form, a labor income risk simply shifts the entire consumption function down by an equal amount at all levels of $w_t$, and so the level of consumption in the modified case does not approach the level in the baseline case as wealth approaches infinity. We therefore need a modified version of the theorem to apply in this case.

**Corollary 1** *Consider an agent who has a utility function with $u'(c) > 0$, $u''(c) < 0$, $u'''(c) > 0$ and nonincreasing absolute prudence $-u'''(c)/u''(c)$ in two different situations. If the consumption function in the modified situation $\hat{c}_t(w_t)$ is a counter-clockwise concavification of the consumption function in the baseline situation and $\lim_{w_t \to +\infty} \hat{c}_t(w_t) - c_t(w_t) \leq 0$, then the value function in the modified situation has greater absolute prudence at $w_t$ than does the value function for baseline situation. The inequality of prudence is strict if the modified consumption function is strictly concave at or above $w_t$.*

The proof of the corollary follows the proof of the main theorem, except where $\lim_{w_t \to +\infty} \hat{c}_t(w_t) - c_t(w_t) = 0$ and concavity of $\hat{c}_t(w_t)$ were used to demonstrate that $\hat{c}_t'(w_t) \geq c_t'(w_t)$ and that $\hat{c}_t(w_t) \leq c_t(w_t)$; here we assume both propositions.

#### 4.2.4 The Quadratic Case

The quadratic case requires a somewhat different approach. First, the limit $w_t \to \infty$ is not as meaningful, since it goes beyond the bliss point. Second, since $u'''(\cdot) = 0$, strict inequality between the prudence of $\hat{V}$ and the prudence of $V$ will hold only *at* those points where $\hat{c}_t(\cdot)$ is strictly concave.

To gain intuition for the quadratic problem, consider the Euler equation in the second-to-last period of a lifetime that ends at $T$, under the assumption that there is no chance that wealth in period $T$ will be greater than the bliss-point level of consumption:[14]

$$u'(c_{T-1}) = E_{T-1}\left[\tilde{\beta}_T \tilde{R}_T u'(\tilde{R}_T(w_{T-1} - c_{T-1}) + \tilde{y}_T)\right] \tag{39}$$

$$\alpha(\kappa - c_{T-1}) = E_{T-1}\left\{\tilde{\beta}_T \tilde{R}_T \alpha\left(\kappa - \left[\tilde{R}_T(w_{T-1} - c_{T-1}) + \tilde{y}_T\right]\right)\right\} \tag{40}$$

$$c_{T-1} = \frac{E_{T-1}[\tilde{\beta}_T \tilde{R}_T^2 w_{T-1}] + E_{T-1}[\tilde{\beta}_T \tilde{R}_T \tilde{y}_T] + \kappa(1 - E_{T-1}[\tilde{\beta}_T \tilde{R}_T])}{1 + E_{T-1}[\tilde{\beta}_T \tilde{R}_T^2]}. \tag{41}$$

This equation illustrates the well-known fact that in the quadratic case in the absence of liquidity constraints and rate-of-return risk, the solution exhibits certainty equivalence with respect to risks to labor income $y_T$.[15]

[14] If there is a chance that $w_T$ could exceed the bliss point, then the kink point in the period-$T$ consumption rule can impart concavity to the period-$T-1$ consumption rule.

[15] An interesting subtlety is that even though the solution is linear in wealth, it does *not* exhibit certainty equivalence with respect to rate-of-return risk, since the level of consumption is related to the expectation of the *square* of the gross return, in a way that implies that an increase in rate-of-return risk increases the marginal propensity to consume. Note also that interactions between rate-of-return risk and income risk can cause the consumption function to shift up or down by a potentially large amount.

Recall now from equation (33) that greater prudence of $\hat{V}_t(w_t)$ than $V_t(w_t)$ occurs if

$$\frac{\hat{V}_t''(w_t)}{V_t''(w_t)} \equiv \frac{u''(\hat{c}_t(w_t))}{u''(c_t(w_t))}\frac{\hat{c}_t'(w_t)}{c_t'(w_t)} \tag{42}$$

$$= \frac{\hat{c}_t'(w_t)}{c_t'(w_t)} \tag{43}$$

is a decreasing function of $w_t$ (the second line follows because for quadratic utility $u''(c)$ is a constant).

Thus, prudence of the value function can be increased in the quadratic case only by something that causes the MPC to decrease as wealth rises. We will show below that in the quadratic case $\hat{c}_t'(w_t)$ experiences a discrete decline at values of $w_t$ where a future liquidity constraint potentially begins to impinge on current consumption.

**Corollary 2** *Consider an agent who has a quadratic utility function in two different situations. If the baseline situation has a consumption function that is concave over some range $w_t < \omega$ and the consumption function in the modified situation is a counterclockwise concavification of $c_t(w_t)$, prudence of $\hat{V}_t(w_t)$ will be strictly greater than prudence of $V_t(w_t)$ at points where $\hat{c}_t'(w_t)/c_t'(w_t)$ strictly declines.*

The proof is simply to note that equation (43) holds only at points where $\hat{c}_t'(w_t)/c_t'(w_t)$ declines with $w_t$.

# 5 The Recursive Propagation of Consumption Concavity

In this section, we provide conditions guaranteeing that if the consumption function is concave in period $t+1$, it will be concave in period $t$ and earlier, whatever the source of that concavity may be.

## 5.1 Horizontal Aggregation of Pointwise Strict and Borderline CC

First we establish that property CC of the value function is preserved through the process we call "horizontal aggregation," in which the utility from optimal current consumption and the expected utility from optimal saving are aggregated to yield the value function for current wealth.[16] Rather than stating results separately for strict and borderline CC, we state the results once under the convention that if words or expressions in brackets are ignored the result stated applies for strict CC, while if the expressions in brackets are retained but the immediately preceding text is ignored, the result applies for borderline CC.

[16]We call the intertemporal summing of utility 'horizontal aggregation' because it is easy to visualize as the sum of a series of (expected) marginal values laid out horizontally through time. See Carroll and Kimball 1996 for a more detailed justification of this terminology.

**Lemma 2** *If $\Omega_t(s_t)$ exhibits property strict [borderline] CC at level of saving $s_t$ and no liquidity constraint applies at the end of period $t$, then $V_t(w_t)$ exhibits property strict [borderline] CC at the (unique) level of wealth $w_t$ such that optimal consumption at that level of wealth yields $s_t = w_t - c_t(w_t)$.*

*Proof.* If $\Omega_t(s_t)$ exhibits strict [borderline] CC at a specific point $s_t$, then for any $s_1 < s_t < s_2$ which are close enough to $s_t$ (e.g. satisfying $|s_2 - s_1| < \delta$ as per definition 4) we can write

$$\Omega_t'(s_t) = u'(\chi(s_t)) \tag{44}$$

for some monotonically strictly increasing function $\chi(s_t)$ for which

$$\chi(ps_1 + (1-p)s_2) \quad > [=] \quad p\chi(s_1) + (1-p)\chi(s_2) \tag{45}$$

holds for $0 < p < 1$. Now take $\chi^{-1}$ of both sides, yielding

$$ps_1 + (1-p)s_2 \quad > [=] \quad \chi^{-1}(p\chi(s_1) + (1-p)\chi(s_2)). \tag{46}$$

Now note that the first order condition implies generically that

$$\begin{aligned} u'(c) &= \Omega_t'(s) && (47)\\ &= u'(\chi(s)) && (48)\\ c &= \chi(s) && (49)\\ \chi^{-1}(c) &= s. && (50) \end{aligned}$$

This can be used to find the levels of beginning-of-period consumption corresponding to $s_1$ and $s_2$.[17] Substituting (49) and (50) into (46) yields

$$p\overbrace{\chi^{-1}(c_1)}^{s_1} + (1-p)\overbrace{\chi^{-1}(c_2)}^{s_2} \quad > [=] \quad \chi^{-1}(pc_1 + (1-p)c_2) \tag{51}$$

which means that $\chi^{-1}$ satisfies the definition of a strictly [weakly] convex increasing function in a neighborhood from $c_1$ to $c_2$ around $c_t$.

But wealth is divided between savings and consumption,

$$\begin{aligned} w_t &= \chi^{-1}(c_t) + c_t && (52)\\ \omega_t(c_t) &\equiv \chi^{-1}(c_t) + c_t, && (53) \end{aligned}$$

[17]This first order condition holds with equality if there are no constraints that apply in the current period. It does *not* hold with equality at every point if there is a constraint in force at the end of the current period, because in that case there will be a level of wealth $\omega^{\#}$ at which the constraint becomes binding and below which all levels of wealth lead to zero savings; hence when there is a constraint at the end of period-$t$ there is not a one-to-one mapping from $s_t$ to a unique corresponding $c_t$ and $w_t$. As noted above, we defer to later sections discussion of what happens when a such an additional constraint is imposed.

and since $\omega_t(c_t)$ is the sum of the increasing convex [linear] function and an increasing linear function, it is itself an increasing convex [linear] function, so by the definition of an increasing convex [linear] function we have

$$
\begin{aligned}
p\omega_t(c_1) + (1-p)\omega_t(c_2) \quad &>[=] \quad \omega_t(pc_1 + (1-p)c_2) &(54)\\
\omega_t^{-1}(pw_1 + (1-p)w_2) \quad &>[=] \quad pc_1 + (1-p)c_2 &(55)\\
c_t(pw_1 + (1-p)w_2) \quad &>[=] \quad pc_t(w_1) + (1-p)c_t(w_2) &(56)
\end{aligned}
$$

where (55) follows from (54) because the inverse of an increasing convex [linear] function is an increasing concave [linear] function and (56) follows because the definition of $\omega_t^{-1}$ implies that it yields the level of consumption that satisfies the first order condition of the maximization problem for the given level of wealth. Thus, $c_t(w_t)$ satisfies the definition of a strictly [borderline] concave function at wealth level $w = \breve{w}_t(\Omega_t'(s_t))$.

This can be stated more formally by defining a set $\mathcal{S}_t$ which contains the points $s_t$ at which $\Omega_t$ exhibits property strict CC. Since we are assuming that $\Omega_t$ satisfies property global CC, knowing the set $\mathcal{S}_t$ tell us the concavity status of all feasible values of $s_t$, since global CC means that either strict CC or borderline CC must hold at each point.

Since the consumption function in the absence of a liquidity constraint is a one-to-one mapping, we can now easily construct a set $\mathcal{W}_t$ which contains the values of beginning-of-period wealth $w_t$ at which $c_t$ exhibits property strict CC, from

$$
\mathcal{W}_t \quad = \quad \breve{w}_t(\Omega_t'(\mathcal{S}_t)), \tag{57}
$$

while at the feasible values of wealth not in $\mathcal{W}_t$ the value function must exhibit property weak CC.

## 5.2 Vertical Aggregation

Our next result specifies when and how property CC of $V_{t+1}$ translates into property CC for $\Omega_t$. When a risk could intervene between the end of $t$ and the beginning of $t+1$, this involves taking expectations; we refer to the taking of expectations as 'vertical aggregation' because it is easy to visualize as the vertical stacking and summation of all possible outcomes at a point in time, weighted by their probabilities.[18] We assume here that $V_{t+1}$ exhibits property global CC, with the points at which it exhibits property strict CC contained in the set $\mathcal{W}_{t+1}$.

In the case without risk, if next period's nonstochastic income is $y_{t+1} = \bar{y}$ and the nonstochastic interest factor is $R_{t+1}$, vertical aggregation is simple:

**Lemma 3** *In the absence of any risk that could intervene between periods $t$ and $t+1$, $\mathcal{S}_t$ consists of the set of points $s_t$ such that $R_{t+1}s_t + \bar{y} \in \mathcal{W}_{t+1}$. That is, $\Omega_t$ exhibits property strict [borderline] CC at the points $s_t$ such that $V_{t+1}$ exhibits property strict [borderline] CC at $w_{t+1} = R_{t+1}s_t + \bar{y}$.*

[18] Again, see Carroll and Kimball 1996 for a more detailed justification of this terminology.

The proof is identical to the proof for horizontal aggregation presented earlier, since here there is a one-to-one mapping between values of $w_{t+1}$ and $s_t$ just as in the horizontal aggregation case there was a one-to-one mapping between $s_t$ and $w_t$.

Now assume that the interest factor is nonstochastic and equal to 1, (equivalent results go through for $R_{t+1} \neq 1$, but exposition is messier). Next assume that next period's income process is

$$y_{t+1} = \bar{y} + \zeta \tag{58}$$

for $\zeta$ which has a maximum realiztion with positive probability of $\bar{\zeta}$ and a minimum realization with positive probability of $\underline{\zeta} < \bar{\zeta}$ and $E_t[\zeta] = 0$.

### 5.2.1 The Quadratic Case

In the quadratic case, linearity of marginal utility implies that

$$\begin{aligned} u'(\chi_t(s_t)) &= E_t[u'(c_{t+1}(s_t + \tilde{y}))] && (59) \\ \chi_t(s_t) &= E_t[c_{t+1}(s_t + \tilde{y})] && (60) \end{aligned}$$

where $\tilde{y}$ represents the various possible realizations of $y_{t+1}$. So $\chi_t$ is simply the weighted sum of a set of concave functions where the weights correspond to the probabilities of the various possible outcomes for $y$. The sum of concave functions is itself strictly concave at any point at which any of the functions being summed is strictly concave, and weakly concave elsewhere. If we denote as $\zeta^+$ all values of $\zeta$ for which there a positive probability mass (in the case of a discrete distribution) or positive probability density (in the case of a continuous distribution),

$$\mathcal{S}_t = \{s | s + \bar{y} + \zeta^+ \in \mathcal{W}_{t+1} \text{ for some } \zeta^+\}. \tag{61}$$

That is, $\mathcal{S}_t$ is the set of values of $s_t$ from which there is a positive probability of arriving next period at a value of $w_{t+1} \in \mathcal{W}_{t+1}$.

### 5.2.2 The CRRA Case

In the CRRA case,

$$\begin{aligned} \Omega_t'(s_t) &= E_t\left[V_{t+1}'(s_t + \tilde{y})\right] && (62) \\ &= E_t\left[c_{t+1}(s_t + \tilde{y})^{-\gamma}\right]. && (63) \end{aligned}$$

Concavity of $c_{t+1}(w_{t+1})$ implies that

$$c_{t+1}(s_t + \tilde{y}) \geq p c_{t+1}(s_1 + \tilde{y}) + (1 - p) c_{t+1}(s_2 + \tilde{y}) \tag{64}$$

for all $\tilde{y}$ if $s_t = p s_1 + (1 - p) s_2$ with $p \in [0, 1]$. Since this holds for all $\tilde{y}$, we know that

$$\left\{E_t\left[c_{t+1}(s_t + \tilde{y})^{-\gamma}\right]\right\}^{-1/\gamma} \geq \left\{E_t\left[\{p c_{t+1}(s_1 + \tilde{y}) + (1 - p) c_{t+1}(s_2 + \tilde{y})\}^{-\gamma}\right]\right\}^{-1/\gamma}, \tag{65}$$

and the inequality is strict if (64) is strict for any possible realization of $\tilde{y}$.

Now we need to use Minkowski's inequality, which says that

$$\left\{E_t\left[(\tilde{a}_{t+1}+\tilde{b}_{t+1})^{-\gamma}\right]\right\}^{-1/\gamma} \quad \geq \quad \left\{E_t[\tilde{a}_{t+1}^{-\gamma}]\right\}^{-1/\gamma}+\{E_t[\tilde{b}_{t+1}^{-\gamma}]\}^{-1/\gamma} \tag{66}$$

for $\gamma > 1$ if $a_{t+1}$ and $b_{t+1}$ are positive, and the expression holds with equality iff $\tilde{a}_{t+1}$ and $\tilde{b}_{t+1}$ are proportional.[19]

Minkowski's inequality implies that

$$\begin{aligned}
&\left\{E_t\left[\{\overbrace{pc_{t+1}(s_1+\tilde{y})}^{=\tilde{a}_{t+1}}+\overbrace{(1-p)c_{t+1}(s_2+\tilde{y})}^{=\tilde{b}_{t+1}}\}^{-\gamma}\right]\right\}^{-1/\gamma} \\
&\quad \geq \left\{E_t[\{pc_{t+1}(s_1+\tilde{y})\}^{-\gamma}]\right\}^{-1/\gamma}+\left\{E_t[\{(1-p)c_{t+1}(s_2+\tilde{y})\}^{-\gamma}]\right\}^{-1/\gamma} \\
&\quad = p\left\{E_t[c_{t+1}(s_1+\tilde{y})^{-\gamma}]\right\}^{-1/\gamma}+(1-p)\left\{E_t[c_{t+1}(s_2+\tilde{y})^{-\gamma}]\right\}^{-1/\gamma} \\
&\quad = p\{\Omega_t^{'}(s_1)\}^{-1/\gamma}+(1-p)\{\Omega_t^{'}(s_2)\}^{-1/\gamma}.
\end{aligned} \tag{67}$$

Combining (63) with (65) and (67),

$$\{\Omega_t^{'}(s_t)\}^{-1/\gamma} \quad \geq \quad p\{\Omega_t^{'}(s_1)\}^{-1/\gamma}+(1-p)\{\Omega_t^{'}(s_2)\}^{-1/\gamma}. \tag{68}$$

where the inequality is strict unless $c_{t+1}(s_2+\tilde{y})/c_{t+1}(s_1+\tilde{y})$ is a constant for all realizations of $\tilde{y}$. But for this to be true for any $s_1$ and $s_2$ it must be the case that

$$\left(\frac{d}{ds_2}\right)\left(\frac{c_{t+1}(s_2+\tilde{y})}{c_{t+1}(s_1+\tilde{y})}\right)\Bigg|_{s_2=s_1} \quad = \quad \left(\frac{c_{t+1}^{'}(s_1+\tilde{y})}{c_{t+1}(s_1+\tilde{y})}\right)=\nu \tag{69}$$

for some constant $\nu$, which is true only if the consumption function is $c_{t+1}(w) = \exp(\nu w)$, which is not true. Thus, defining $\chi_t(s_t) = \{\Omega_t^{'}(s_t)\}^{-1/\gamma}$, (68) becomes

$$\chi_t(s_t) > p\chi_t(s_1)+(1-p)\chi_t(s_2) \tag{70}$$

for all $s_t$, so in the CRRA case $\Omega_t(s_t)$ if the risk is nondegenerate then $\chi_t(s)$ exhibits property strict CC for all feasible values of $s$.

### 5.2.3 The Exponential Case

For the exponential case, property CC holds at $s_t$ if

$$\exp(-\gamma\chi_t(s_t)) \quad = \quad E_t[\exp(-\gamma c_{t+1}(s_t+\tilde{y}))] \tag{71}$$

for some $\chi_t(s_t)$ which is strictly concave at $s_t$.

[19] For a proof, see Hardy, Littlewood, and Polya 1967, page 146, Theorem 198, equation (6.13.2).

Consider first a case where $c_{t+1}$ is linear over the range of possible values of $w_{t+1} = s_t + \bar{y} + \zeta$; then (setting absolute risk aversion $\gamma = 1$ to reduce clutter; results hold for $\gamma \neq 1$),

$$\begin{aligned}
\chi_t(s_t) &= -\log E_t[e^{-c_{t+1}(s_t+\tilde{y})}] && (72)\\
&= -\log E_t[e^{-(c_{t+1}(s_t+\bar{y})+(\tilde{y}-\bar{y})c'_{t+1})}] && (73)\\
&= c_{t+1}(s_t+\bar{y}) - \log E_t[e^{-\zeta c'_{t+1}}] && (74)
\end{aligned}$$

which is linear in $s_t$ since the second term is a constant.

Now consider a value of $s_t$ for which $s_t + \bar{y} + z \in \mathcal{W}_{t+1}$ for some $z \in (\underline{\zeta}, \bar{\zeta})$; that is, $c_{t+1}$ is strictly concave for some $z$ between the minimum and maximum possible draws of $\zeta$. (Note that if $\zeta$ has a discrete distribution, $z$ need not correspond to one of the possible draws).

Global weak concavity of $c_{t+1}$ tells us that for every $\zeta$

$$\begin{aligned}
-c_{t+1}(s_t+\tilde{y}) &\leq -((1-p)c_{t+1}(s_1+\tilde{y}) + pc_{t+1}(s_2+\tilde{y}))\\
E_t[e^{-c_{t+1}(s_t+\tilde{y})}] &\leq E_t[e^{-((1-p)c_{t+1}(s_1+\tilde{y})+pc_{t+1}(s_2+\tilde{y}))}]. && (75)
\end{aligned}$$

Meanwhile, the arithmetic-geometric mean inequality states that for positive $a$ and $b$, if $\bar{a} = E_t[\tilde{a}]$ and $\bar{b} = E_t[\tilde{b}]$, then

$$E_t\left[(\tilde{a}/\bar{a})^p(\tilde{b}/\bar{b})^{1-p}\right] \leq E_t\left[p(\tilde{a}/\bar{a}) + (1-p)(\tilde{b}/\bar{b})\right] = 1, \quad (76)$$

implying that

$$E_t[\tilde{a}^p\tilde{b}^{1-p}] \leq \bar{a}^p\bar{b}^{1-p}, \quad (77)$$

where the expression holds with equality only if $b$ is proportional to $a$. Substituting in $a = e^{-c_{t+1}(s_1+\tilde{y})}$ and $b = e^{-c_{t+1}(s_2+\tilde{y})}$, this means that

$$E_t[e^{-pc_{t+1}(s_1+\tilde{y})-(1-p)c_{t+1}(s_2+\tilde{y})}] \leq \left\{E_t[e^{-c_{t+1}(s_1+\tilde{y})}]\right\}^p \left\{E_t[e^{-c_{t+1}(s_2+\tilde{y})}]\right\}^{1-p} \quad (78)$$

and we can substitute for the LHS from (75), obtaining

$$\begin{aligned}
E_t[e^{-c_{t+1}(s_t+\tilde{y})}] &\leq \left\{E_t[e^{-c_{t+1}(s_1+\tilde{y})}]\right\}^p \left\{E_t[e^{-c_{t+1}(s_2+\tilde{y})}]\right\}^{1-p} && (79)\\
\log E_t[e^{-c_{t+1}(s_t+\tilde{y})}] &\leq p\log E_t[e^{-c_{t+1}(s_1+\tilde{y})}] + (1-p)\log E_t[e^{-c_{t+1}(s_2+\tilde{y})}] && (80)
\end{aligned}$$

which holds with equality only when $e^{-c_{t+1}(s_1+y_{t+1})}/e^{-c_{t+1}(s_2+y_{t+1})}$ is a constant, which will happen only if $c_{t+1}(s_1+y_{t+1}) - c_{t+1}(s_2+y_{t+1})$ is constant, which (given that the MPC is strictly positive everywhere) requires $c_{t+1}(s_t+\bar{y}+z)$ to be linear for $z \in (\underline{\zeta}, \bar{\zeta})$. For an $s_t$ from which $c_{t+1}$ is strictly concave for some $z$, (80) becomes

$$\chi_t(s_t) > p\chi_t(s_1) + (1-p)\chi_t(s_2). \quad (81)$$

Thus, in the exponential case, $\mathcal{S}_t$ includes any value of $s_t$ from which, for some point $w \in \mathcal{W}_{t+1}$, there is positive probability of arriving at a $w_{t+1} > w$ and a positive probability of arriving at a $w_{t+1} < w$. Formally,

$$\mathcal{S}_t = \{s | s + \bar{y} + \zeta \in \mathcal{W}_{t+1}\} \quad (82)$$

for some $\zeta \in [\underline{\zeta}, \bar{\zeta}]$. In words, $\mathcal{S}_t$ is the set of values of $s$ from which the outcome of the risk affects which side of $w$ the consumer ends up on for some $w \in \mathcal{W}_{t+1}$, or from which there is a positive probability of ending up exactly at some $w \in \mathcal{W}_{t+1}$.

# 6 Liquidity Constraints, Consumption Concavity, and Precautionary Saving

We next show how liquidity constraints create strict convexity of the marginal value function.

## 6.1 Piecewise Linearity of the Perfect Foresight Consumption Function Under Constraints

Consider a consumer in an initial situation in which he is solving a perfect foresight optimization problem with a finite horizon that begins in period $t$ and ends in period $T$. The consumer begins with wealth $w_t$ and earns a nonstochastic but potentially time-varying income $\bar{y}_{\tau+1}$ in each period; wealth accumulates according to $w_{\tau+1} = R_{\tau+1}s_\tau + \bar{y}_{\tau+1}$ where $R_{\tau+1}$ is nonstochastic. We are interested in how this consumer's behavior in period $t$ changes from this unconstrained initial situation to a final situation in which a given set of liquidity constraints, $\mathcal{T}_t$, has been imposed.

### 6.1.1 The Simplest Case

We begin by considering a simpler context, in which the consumer's income is not only nonstochastic but is not time-varying, $\bar{y}_{t+1} = \bar{y} \ \forall \ t$; the consumer is assumed to be impatient in the sense of having a time preference rate greater than the interest rate in every period, $R_\tau \beta_\tau < 1 \ \forall \ \tau$; and only the simplest kind of constraint exists: If there is a constraint at date $t$, the constraint requires $c_t \leq w_t$.

Define $\mathcal{T}_t$ as an ordered set of dates $\geq t$ at which, in the final situation, a relevant constraint exists (that is, a constraint that will bind for a consumer who starts from some feasible value of initial wealth $w_t$). We define $\mathcal{T}_t[1] = T$ because we think of the intertemporal budget constraint as being the first 'constraint' that must be satisfied. $\mathcal{T}_t[2]$ is the date of the last period before $T$ in which a constraint exists, $\mathcal{T}_t[3]$ is the second-to-last period before $T$ in which a constraint exists, and so on. For any $\tau$ such that $t \leq \tau < T$, define $c_{\tau,n}$ as the optimal consumption function in period $\tau$ assuming that the first $n$ constraints in $\mathcal{T}_t$ have been imposed; thus, $c_{\tau,1}(w)$ is the consumption function in period $\tau$ when no constraints (aside from the IBC) have been imposed, $c_{\tau,2}(w)$ is the consumption function after the chronologically last constraint has been imposed, and so on through $c_{t,k_t}$ where $k_t$ is the number of dates in $\mathcal{T}_t$. Define $\breve{w}_{t,n}, \Omega_{t,n}, V_{t,n}$ and other functions correspondingly.

Now consider imposing the constraints sequentially, as follows. Start with the unconstrained case, for which the consumption rule in the last period is $c_{T,1} = w_T$. For a perfect foresight unconstrained problem, the marginal propensity to consume in period $t$ can be obtained from the MPC in period $t+1$ from the Euler equation,

$$u'(c_{t,n}(w_t)) \quad = \quad R_{t+1}\beta_{t+1}u'(c_{t+1,n}(R_{t+1}(w_t - c_{t,n}(w_t)) + \bar{y}_{t+1})). \tag{83}$$

Differentiating both sides with respect to $w_t$, dropping subscripts on $R$ and $\beta$

and omitting arguments to reduce clutter we obtain

$$\begin{aligned} u''(c_{t,n})c'_{t,n} &= R\beta u''(c_{t+1,n})c'_{t+1,n}R(1-c'_{t,n}) && (84)\\ c'_{t,n} &= \left(\frac{R\beta u''(c_{t+1,n})Rc'_{t+1,n}}{u''(u'^{-1}(R\beta u'(c_{t+1,n}))) + R\beta u''(c_{t+1,n})Rc'_{t+1,n}}\right) && (85)\\ &= \left(\frac{c'_{t+1,n}}{c'_{t+1,n} + u''(u'^{-1}(R\beta u'(c_{t+1,n}))/(R\beta u''(c_{t+1,n})R))}\right) && (86) \end{aligned}$$

and it is straightforward to verify that for all three of our utility options this is a number less than $c'_{t+1,n}$.

Now setting $\tau = \mathcal{T}_t[2]$, note first that if $\tau < T-1$ then behavior in periods $\tau+1$ through $T-1$ is unaffected by this constraint, so we have $c_{\tau+1,2} = c_{\tau+1,1}$ and so on through $c_{\tau+1,k_t} = c_{\tau+1,1}$. For period $\tau$ we can calculate the level of consumption at which the constraint binds by realizing that a consumer for whom the constraint binds will save nothing, and that the maximum amount of consumption at which the constraint binds will satisfy the Euler equation (only points where the constraint is strictly binding violate the Euler equation; the point on the cusp does not). Thus, if we define $c^{\#}_{\tau,n}$ as the level of consumption in period $\tau$ at which the $n$'th constraint stops binding (with $c^{\#}_{\tau,1} = \infty$ because the IBC always binds), we have

$$\begin{aligned} u'(c^{\#}_{\tau,2}) &= R_{\tau+1}\beta_{\tau+1}u'(c_{\tau+1,1}(\bar{y})) && (87)\\ c^{\#}_{\tau,2} &= u'^{-1}\left(R_{\tau+1}\beta_{\tau+1}u'(c_{\tau+1,1}(\bar{y}))\right), && (88) \end{aligned}$$

and the level of wealth at which the constraint stops binding can be obtained from

$$\omega_{\tau,2} = \breve{w}_{\tau,1}(u'(c^{\#}_{\tau,2})). \qquad (89)$$

Below this level of wealth, we have $c_{\tau,2}(w) = w$ so the MPC is one, while above it we have $c_{\tau,2}(w) = c_{\tau,1}(w)$ where the MPC $c'_{\tau,1}$ equals the constant MPC that obtains for an unconstrained perfect foresight optimization problem with a horizon of $T-\tau$. Thus, $c_{\tau,2}$ satisfies our definition of a counterclockwise concavification of $c_{\tau,1}$ around $\omega_{\tau,2}$.

We can obtain the value of period $\tau-1$ consumption at which the period $\tau$ constraint stops affecting period $\tau-1$ behavior from

$$u'(c^{\#}_{\tau-1,2}) = R_\tau\beta_\tau u'(c^{\#}_{\tau,2}) \qquad (90)$$

and we can obtain $\omega_{\tau-1,2}$ via the analogue to (89). Iteration generates the remaining $c^{\#}_{\cdot,2}$ and $\omega_{\cdot,2}$ values back to period $t$.

We have thus established an apparatus that can determine how a constraint in any future period affects consumption in a current period. In order to have a distinct terminology for the effects of current-period and future-period constraints, we will restrict the use of the term 'binds' to the effects of a constraint in the period in which it applies, and will use the term 'impinges' to describe the effect of a future constraint on current consumption.

We now define a 'kink point' in a consumption function as a point like $\omega_{\tau,2}$ at which there is a discrete decline in the marginal propensity to consume that corresponds to a transition from a level of wealth where a current constraint binds, or a future constraint impinges, to a level of wealth where that constraint no longer binds or impinges.

Now consider the behavior of a consumer in period $\tau - 1$ with a level of wealth $w < \omega_{\tau-1,2}$. This consumer knows he will be constrained and will spend all of his resources next period, so at $w$ his behavior will be identical to the behavior of a consumer whose entire horizon ends at time $\tau$. For all three of our utility classes, the consumption function is linear for perfect foresight consumers with finite horizons, and the MPC declines with the horizon. The MPC for this consumer is therefore strictly less than the MPC of the unconstrained consumer whose horizon ends at $T > \tau$. Thus, in each period $\underline{\tau}$ before $\tau + 1$, the consumption function $c_{\underline{\tau},2}$ generated by imposition of the constraint constitutes a counterclockwise concavification of the unconstrained consumption function around the kink point $\omega_{\underline{\tau},2}$.

Now consider imposing the second constraint (that is, the constraint that applies in the second-to-last period in which there is a constraint), and suppose for concreteness that it applies at the end of period $\tau - 1$. It will stop binding at a level of consumption defined by

$$\begin{aligned} u'(c^{\#}_{\tau-1,3}) &= R_\tau \beta_\tau u'(c_{\tau,2}(\bar{y})) \qquad (91)\\ &= R_\tau \beta_\tau u'(\bar{y}) \qquad (92)\end{aligned}$$

where the second line follows because it is easy to show in a context like this that an impatient consumer with total resources $\bar{y}$ will be constrained. But note that we can combine (90) and (87), along with our assumption that $R_{\tau+1}\beta_{\tau+1} < 1$ to obtain

$$\begin{aligned} u'(c^{\#}_{\tau-1,2}) &= R_\tau \beta_\tau R_{\tau+1}\beta_{\tau+1} u'(\bar{y}) \qquad (93)\\ &< R_\tau \beta_\tau u'(\bar{y}) \qquad (94)\\ &= u'(c^{\#}_{\tau-1,3}) \qquad (95)\end{aligned}$$

which, from the assumption of declining marginal utility, tells us

$$c^{\#}_{\tau-1,2} > c^{\#}_{\tau-1,3}. \qquad (96)$$

This means that the constraint is relevant: The preexisting constraint in period $\tau$ does not force the consumer to do so much saving in period $\tau - 1$ that the $\tau - 1$ constraint could fail to bind. (Below we consider circumstances in which a later constraint *can* render an earlier constraint irrelevant.)

The prior-period levels of consumption and wealth at which constraint number 2 stops impinging on consumption can again be calculated recursively from

$$\begin{aligned} u'(c^{\#}_{s,3}) &= R_{s+1}\beta_{s+1} u'(c^{\#}_{s+1,3}) \qquad (97)\\ \omega_{s,3} &= \breve{w}_{s,2}(u'(c^{\#}_{s,3})). \qquad (98)\end{aligned}$$

Furthermore, once again we can think of the constraint as terminating the horizon of a finite-horizon consumer in an earlier period than it is terminated for the less-constrained consumer, with the implication that the MPC below $\omega_{s,3}$ is strictly greater than the MPC above $\omega_{s,3}$. Thus, the consumption function $c_{s,3}$ constitutes a counterclockwise concavification of the consumption function $c_{s,2}$ around the kink point $\omega_{s,3}$.

The same logic applies in the case of each of the remaining constraints in $\mathcal{T}_t$. Defining

$$c'_{\tau,n} \quad = \quad \lim_{w \downarrow \omega_{\tau,n}} c'_{\tau,n}(w), \tag{99}$$

we can summarize all of this by

**Proposition 1** *For all three of our utility classes, define $c_{\tau,1}$ as the optimal neoclassical consumption function that solves an unconstrained perfect foresight finite horizon problem ending in period $T$, assuming that $R_\tau \beta_\tau < 1$ and the remainder of the problem is as specified above. Consider an ordered set of constraints $\mathcal{T}_t$ that may be imposed between periods $t \leq \tau$ and $T$. If the first $n$ constraints have been imposed, then imposition of constraint $n+1$ has the following effects:*

- *For each period $\tau$ weakly prior to $\mathcal{T}_t[n+1]$, there will be a level of wealth $\omega_{\tau,n+1}$ above which imposition of the constraint has no effect on period-$\tau$ consumption but below which imposition of the constraint causes consumption to fall;*
- $c'_{\tau,n+1} > c'_{\tau,n}$*;*
- *The consumption function $c_{\tau,n+1}$ is a counterclockwise concavification of $c_{\tau,n}$ around the kink point $\omega_{\tau,n+1}$.*

leading to the conclusion that

**Proposition 2** *Under the circumstances described in proposition 1, once all $k_t$ constraints have been imposed, the consumption function in period $t$ is a piecewise linear increasing concave function with kink points at the successively larger values of wealth $\vec{\omega}_{t,k_t} = \{\omega_{t,k_t}, \omega_{t,k_t-1}, ..., \omega_{t,2}\}$ at which future constraints successively stop impinging on current consumption.*

### 6.1.2 Increasing the Number of Constraints

The previous section analyzed a case where there was a preordained set of constraints $\mathcal{T}_t$ under consideration, which were applied sequentially. We now examine how behavior will be modified if we add a new date to the set of dates at which the consumer is constrained.

Call the new set of dates $\grave{\mathcal{T}}_t$, and call the consumption rules corresponding to the new set of dates $\grave{c}_{t,1}$ through $\grave{c}_{t,k_t+1} = \grave{c}_{t,\grave{k}_t}$. If the date of the new constraint is $\grave{\tau}$, then behavior after period $\grave{\tau}$ is not affected by imposition of the new constraint. Now call $m$ the number of constraints in $\mathcal{T}_t$ at dates strictly greater than $\tau'$. Then note that that $\grave{c}_{\tau',m} = c_{\tau',m}$, because until the new constraint (number $m+1$) is imposed consumption is the same as in the absence of the constraint. Now recall that, as discussed above, imposition of the constraint at $\tau'$ causes a counterclockwise

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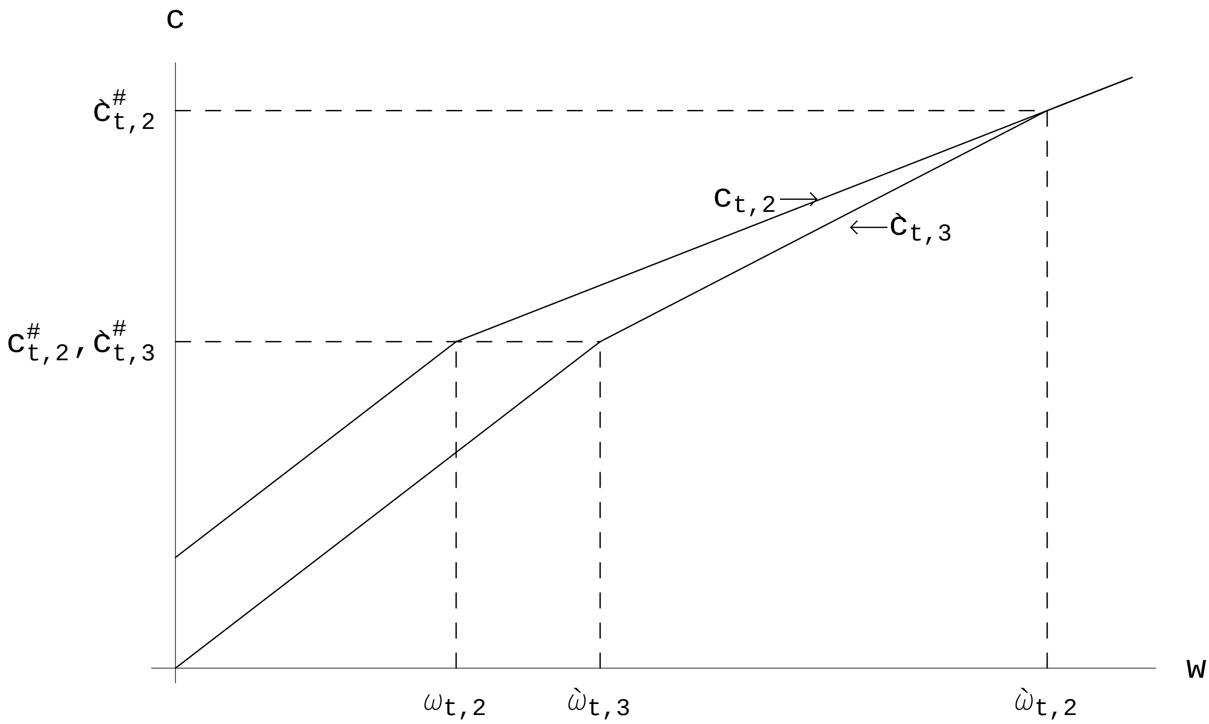



Figure 4: How a Current Constraint Can Hide a Future Kink

concavification of the consumption function around a new kink point, $\omega_{\tau',m+1}$; that is, $\grave{c}_{\tau',m+1}$ is a counterclockwise concavification of $\grave{c}_{\tau,m}$. But since $\grave{c}_{\tau',m} = c_{\tau',m}$, $\grave{c}_{\tau',m+1}$ is also a counterclockwise concavification of $c_{\tau',m}$.

The most interesting observation, however, is that behavior under constraints $\grave{\mathcal{T}}_t$ in periods strictly before $\grave{\tau}$ *cannot* be described as a counterclockwise concavification of behavior under $\mathcal{T}_t$. The reason is that the values of wealth at which the earlier constraints caused kink points in the consumption functions before period $\grave{\tau}$ probably will not correspond to kink points once the extra later constraint has been added.

An example is presented in figure 4. The original $\mathcal{T}_t$ contains only a single constraint, at the end of period $t+1$, inducing a kink point at $\omega_{t,2}$ in the consumption rule $c_{t,2}$. The expanded set of constraints, $\grave{\mathcal{T}}_t$, adds one constraint, at period $t+2$. $\grave{\mathcal{T}}_t$ induces two kink points in the newly optimal consumption rule $\grave{c}_{t,3}$, at $\grave{\omega}_{t,2}$ and $\grave{\omega}_{t,3}$. It is true that imposition of the new constraint causes consumption to be lower than before at every level of wealth below $\grave{\omega}_{t,2}$. However, this does not imply higher prudence of the value function at every $w < \grave{\omega}_{t,2}$. In particular, note that the original consumption function $c_{t,2}$ exhibited property strict CC at $w = \omega_{t,2}$, or effectively infinite prudence at this point, while the new consumption function only exhibits property weak CC at $\grave{c}_{t,3}(\omega_{t,2})$, so for this particular level of wealth prudence was greater before than after imposition of the new constraint.

The intuition is simple: At levels of initial wealth below $\grave{\omega}_{t,2}$, the consumer had been planning to end period $\grave{\tau}$ with negative wealth. After the new constraint has been imposed, the old plan of ending up with negative wealth at the end of $\tau'$ is no longer feasible - the consumer will save more for any given level of current wealth below $\grave{\omega}_{t,2}$, including $\omega_{t,2}$. But the reason $\omega_{t,2}$ was a kink point in the initial situation was that it was the level of wealth where consumption would have been equal to wealth in period $t+1$. Now, because of the extra savings induced by the constraint in $\tau'$, the larger savings induced by wealth $\omega_{t,2}$ implies that the period $t+1$ constraint will no longer bind for a consumer who begins period $t$ with wealth $\omega_{t,2}$. In other words, at wealth $\omega_{t,2}$ the extra savings induced by the new constraint 'hides' the original constraint and prevents it from being relevant any more at $\omega_{t,2}$.

Notice, however, that all constraints that existed in $\mathcal{T}_t$ will remain relevant at *some* level of wealth under $\mathcal{T}_t$ even after the new constraint is imposed - they just induce kink points at different levels of wealth than before, e.g. the first constraint causes a kink at $\grave{\omega}_{t,2}$ rather than at $\omega_{t,2}$.

### 6.1.3 A More General Analysis

We now want to allow time variation in the level of income, $\bar{y}_\tau$, and in the location of the liquidity constraint (e.g. a constraint in period $\tau$ might require the consumer to end period $\tau$ with savings $s_\tau$ greater than $\sigma$ where $\sigma$ is a negative number). We also drop the restriction that $\beta_\tau R_\tau < 1$, thus allowing the consumer to want to have consumption growth over time.

Under these more general circumstances, a constraint imposed in a given period can render constraints in either earlier or later periods irrelevant. For example, consider a CRRA utility consumer with $R_\tau \beta_\tau = 1 \ \forall \ \tau$ who earns income of 1 in each period, but who is required to arrive at the end of period $T-2$ with savings of 5. Then a constraint that requires savings to be greater than zero at the end of period $T-3$ will have no effect, because with an income of only 1 and a CRRA utility function that requires positive consumption, the consumer is required by the constraint in period $T-2$ to end period $T-3$ with savings greater than 4. Also, a constraint that requires the consumer to end period $T-1$ with positive net worth will also not have any significance for behavior in periods prior to $T-2$, because any consumer who satisfies the $T-2$ constraint will optimally choose to have positive savings in period $T-1$ anyway.

Formally, consider now imposing the first constraint, which applies in period $\tau < T$. The simplest case, analyzed before, was a constraint that requires the minimum level of end-of-period wealth to be $s_\tau \geq 0$. Here we generalize this to $s_\tau \geq \sigma_{\tau,2}$ where in principle we can allow borrowing by choosing $\sigma$ to be a negative number. Now for constraint $i = 2$ calculate the kink points for prior periods from

$$u'(c^{\#}_{\tau,i}) = R_{\tau+1}\beta_{\tau+1}u'(c_{\tau+1,i-1}(R_{\tau+1}\sigma_{\tau,i} + \bar{y}_{t+1})) \tag{100}$$

$$\omega_{\tau,i} = \breve{w}_{\tau,i-1}(u'(c^{\#}_{\tau,i})). \tag{101}$$

In addition, for constraint $i = 2$ recursively calculate

$$\underline{\sigma}_{\tau,i} = (\sigma_{\tau+1,i} - \bar{y}_{\tau+1,i} + \underline{c})/R_{\tau+1} \tag{102}$$

where $\underline{c}$ is the lowest value of consumption permitted by the model (independent of constraints). For example, CRRA utility is well defined only on the positive real numbers, so for a CRRA utility consumer $\underline{c} = 0$. In the exponential and quadratic cases, there is nothing to prevent consumption of $-\infty$, so for those models $\underline{c} = -\infty$, unless there is a desire to restrict the model to positive values of consumption, in which case the $c \geq 0$ constraint will be implemented through the use of (102).

Now assume that the first $n$ constraints in $\mathcal{T}_t$ have been imposed, and consider imposing constraint number $n+1$, which we assume applies in period $\tau$. The first thing to check is whether constraint number $n+1$ is relevant given the already-imposed set of constraints. This is simple: A constraint that requires $s_\tau \geq \sigma_{\tau,n+1}$ will be irrelevant if $\min_i[\underline{\sigma}_{\tau,i}] \leq \sigma_{\tau,n+1}$. If the constraint is irrelevant then the analysis proceeds simply by dropping this constraint and renumbering the constraints in $\mathcal{T}_t$ so that the former constraint $n+2$ becomes constraint $n+1$, $n+3$ becomes $n+2$, and so on.

Now consider the other possible problem: That constraint number $n+1$ imposed in period $\tau$ will render irrelevant some of the constraints that have already been imposed. This too is simple to check: It will be true if the proposed $\sigma_{\tau,n+1} \geq \sigma_{\tau,i}$ for any $i \leq n$. The fix is again simple: Counting down from $i = n$, find the smallest value of $i$ for which $\sigma_{\tau,n+1} \geq \sigma_{\tau,i}$. Then we know that constraint $n+1$ has rendered constraints $i$ through $n$ irrelevant. The solution is to drop these constraints from $\mathcal{T}_t$ and start the analysis over again with the modified $\mathcal{T}_t$.

If this set of procedures is followed until the chronologically earliest relevant constraint has been imposed, the result will be a $\mathcal{T}_t$ that contains a set of constraints that can be analyzed as in the simpler case. In particular, proceeding from the final $\mathcal{T}_t[2]$ through $\mathcal{T}_t[k_t]$, the imposition of each successive constraint in $\mathcal{T}_t$ now causes a counterclockwise concavification of the consumption function around successively lower values of wealth as progressively earlier constraints are applied, and the result is again a piecewise linear and strictly concave consumption function, with the number of kink points equal to the number of constraints that are relevant at any feasible level of wealth in period $t$.

Finally, consider adding a new constraint to the problem, and call the new set of constraints $\grave{\mathcal{T}}_t$. Suppose the new constraint applies in period $\grave{\tau}$. Then analysis of the new situation will be like analysis of an added constraint in the simpler case examined above if the new constraint is relevant given the constraints that apply after period $\grave{\tau}$, and if the new constraint does not render any of those later constraints irrelevant. If the new constraint fails either of these tests, the analysis of $\grave{\mathcal{T}}_t$ can proceed from the ground up as described above.

## 6.2 Liquidity Constraints, Prudence, and Precautionary Premia

### 6.2.1 When and Where Do Liquidity Constraints Increase Prudence?

Having determined the effects of constraints on the shape of the perfect foresight consumption function, we are now in position to discuss the effect of constraints on prudence at a given level of wealth.

**Definition 8** *Consider a consumer subject to a given set of constraints $\mathcal{T}_t$. In period $\tau \leq T$, for a consumer with wealth $w$ define the number of 'relevant' constraints $\mathcal{R}_\tau(w)$ as $i-1$ where $i$ is the index on the smallest value of $\omega_{\tau,i}$ strictly greater than $w$. That is, if $\omega_{\tau,i} > w > \omega_{\tau,i+1}$ then the consumer with wealth $w$ in period $\tau$ faces $i-1$ relevant constraints (we subtract 1 because the IBC is the first 'constraint'). If $w > \omega_{\tau,2}$ then we say that a consumer with wealth $w$ in period $\tau$ faces no relevant constraints.*

**Lemma 4** *Consider a consumer with wealth $w$ with power utility or exponential utility in period $\tau$ where $t \leq \tau < T$. Then the successive imposition of each of the first $\mathcal{R}_\tau(w)$ constraints in $\mathcal{T}_t$ strictly increases the absolute prudence of the consumer's value function in period $\tau$ at wealth $w$. Imposition of any remaining constraints beyond constraint number $\mathcal{R}_\tau(w)$ has no effect on the absolute prudence of the value function at $w$ in period $\tau$.*

*Proof.* The proof is simply that, if constraints 1 through $n$ have been imposed, then imposition of constraint $n+1$ in $\mathcal{R}_\tau(w)$ constitutes a counterclockwise concavification of the consumption function around a level of wealth greater than $w$, and theorem 2 tells us for consumers with either power or exponential utility that such a concavification increases the absolute prudence of the value function at any level of wealth below the point around which the concavification is performed.

In contrast, for consumers with quadratic utility we have

**Lemma 5** *Consider a consumer with quadratic utility in period $\tau \leq T$ with wealth $w$. Then the imposition of each of the first $\mathcal{R}_\tau(w)$ constraints strictly increases the absolute prudence of the consumer's value function at each of the associated kink points but not elsewhere; that is, absolute prudence is positive at points $\omega_{\tau,2}$ through $\omega_{\tau,\mathcal{R}_\tau(w)}$, but zero elsewhere.*

which holds by application of theorem 1.

#### 6.2.2 Resemblance Between Precautionary Saving and a Liquidity Constraint

Before proceeding we need to analyze a particular sense in which the introduction of a risk resembles the introduction of a constraint.

An example will make the point. Consider the second-to-last period of life for two CRRA utility consumers, and assume for simplicity that $R_T = \beta_T = 1$.

The first consumer is subject to a liquidity constraint $c_{T-1} \geq w_{T-1}$, and earns nonstochastic income of $\bar{y} = 1$ in period $T$. This consumer's saving rule will be

$$s_{T-1,2}(w_{T-1}) = \begin{cases} 0 & \text{if } w_{T-1} \leq 1 \\ (w_{T-1} - 1)/2 & \text{if } w_{T-1} > 1. \end{cases} \tag{103}$$

The second consumer is not subject to a liquidity constraint, but faces a stochastic income process,

$$y_T = \begin{cases} 0 & \text{with probability } p \\ \frac{1}{1-p} & \text{with probability } (1-p). \end{cases} \tag{104}$$

If we write the consumption rule for the unconstrained consumer facing the risk as $\tilde{s}_{T-1,1}$, the key result is that in the limit as $p \downarrow 0$, behavior of the two consumers becomes the same. That is, defining $\tilde{s}_{T-1,1}(w)$ as the optimal saving rule for the consumer facing the risk,

$$\lim_{p \downarrow 0} \tilde{s}_{T-1,1}(w_{T-1}) = s_{T-1,2}(w_{T-1}) \tag{105}$$

for every $w_{T-1}$.

To see this, start with the Euler equations for the two consumers given wealth $w$,

$$\begin{aligned} u'(w - s_{T-1,2}(w)) &= u'(s_{T-1,2}(w) + 1) && (106) \\ u'(w - \tilde{s}_{T-1,1}(w)) &= pu'(\tilde{s}_{T-1,1}(w)) + (1-p)u'(\tilde{s}_{T-1,1}(w) + 1). && (107) \end{aligned}$$

Consider first the case where $w$ is large enough that the constraint does not bind for the constrained consumer, $w > 1$. In this case the limit of the Euler equation for the second consumer is identical to the Euler equation for the first consumer (because for $w > 1$ savings are positive for the consumer facing the risk, implying that the limit of the first $u'$ term on the RHS of (107) is finite), implying that the limit of (107) is (106) for $w > 1$.

Now consider the case where $w < 1$ so that the first consumer would be constrained. This consumer spends her entire resources $w$, and by the definition of constrained we know that

$$u'(w) > u'(1). \tag{108}$$

Now consider the consumer facing the risk. If this consumer were to save exactly zero and then experienced the bad shock in period $T$, she would experience negative infinite utility (the Inada condition). Therefore we know that for any fixed $p$ and any $w > 0$ the consumer will save some positive amount. For a fixed $w$, hypothesize that there is some positive amount $\delta$ such that no matter how small $p$ became the consumer would always choose to save at least $\delta$. But for any positive $\delta$, the limit of the RHS of (107) is $u'(\delta + 1)$. But we know from concavity of the utility function that $u'(1+\delta) < u'(1)$ and we know from (108) that $u'(w) > u'(1) > u'(1+\delta)$, so as $p \downarrow 0$ there must always come a point at which the consumer can improve her total utility by shifting some resources from the future to the present, i.e. by saving less. Since this argument holds for any $\delta > 0$ it demonstrates that as $p$ goes to zero there is no positive level of saving that would make the consumer better off. But saving of zero or a negative amount is ruled out by the Inada condition at $u'(0)$. Hence saving must approach, but never equal, zero as $p \downarrow 0$.

Thus, we have shown that for $w \leq 1$ and for $w > 1$ in the limit as $p \downarrow 0$ the consumer facing the risk but no constraint behaves identically to the consumer facing the constraint but no risk. This argument can be generalized to show that for the CRRA utility consumer, spending must always be strictly less than the sum of current wealth and the minimum possible value of human wealth. Thus, the addition of a risk to the problem can rule out certain levels of wealth as feasible, and can also render either future or past constraints irrelevant, just as the imposition of a new constraint can. We take this point into account when necessary below.

#### 6.2.3 Prudence and Compensating Precautionary Premia

It is now time to be explicit about the relationship between prudence and precautionary saving. Begin by defining two marginal value functions $\mu(z)$ and $\grave{\mu}(z)$ which are globally (weakly) convex, downward sloping, and continuous in some variable $z$.

Consider a risk $\xi$ whose minimum realization with positive probability (or positive probability density; henceforth, we will say 'with positive probability' when we mean 'with positive probability or positive probability density') is $\underline{\zeta}$ and whose maximum realization with positive probability is $\bar{\zeta}$, and follow Kimball 1990 by defining the Compensating Precautionary Premia as the values $\kappa$ and $\grave{\kappa}$ such that

$$\mu(0) = E[\mu(\kappa + \tilde{\zeta})] \tag{109}$$

$$\grave{\mu}(0) = E[\grave{\mu}(\grave{\kappa} + \tilde{\zeta})]. \tag{110}$$

The relevant part of Pratt's 1964 theorem 1 as reinterpreted using Kimball's 1990 lemma (p. 57) can be restated as

**Lemma 6** *Let $A(z)$ and $\grave{A}(z)$ be absolute prudence of the marginal utility functions $\mu$ and $\grave{\mu}$ respectively at $z$,*[20] *and let $\kappa$ and $\grave{\kappa}$ be the respective compensating precautionary premia associated with imposition of a given risk $\xi$ as per* (109) *and* (110)*. Then the following conditions are equivalent, either with the bracketed material omitted, or with the bracketed material replacing the material immediately prior to the bracket:*

1. *$\grave{A}(x + \kappa) \geq A(x + \kappa)$ for all points $x$ in $\xi$, and $\grave{A}(x + \kappa) > A(x + \kappa)$ for at least one [for no] point $x \in (\underline{\zeta}, \bar{\zeta})$.*

2. *The compensating precautionary premium for marginal value function $\grave{\mu}$ with respect to risk $\xi$ is strictly greater than [exactly equal to] the CPP for marginal value function $\mu$ with respect to the risk $\xi$; that is, $\grave{\kappa} > [=]\kappa$.*

Note finally that precautionary premia are not equivalent to precautionary saving effects, because (as Kimball 1990 points out), precautionary premia apply at a given level of consumption, while precautionary saving applies at a given level of wealth.

[20] A small technicality: Absolute prudence of value functions is infinite at kink points in the consumption function, so if both $c(z)$ and $\grave{c}(z)$ had a kink point at exactly the same $z$ but the amount by which the slope declined were different, the comparison of prudence would not yield a well-defined answer. Under these circumstances we will say that $\grave{A}(z) > A(z)$ if the decline in the MPC is greater for $\grave{c}$ at $z$ than for $c$.

### 6.3 Constraints, Risks, Precautionary Premia, and Precautionary Saving

We now have all the pieces necessary to tease out the relationship between constraints, risks, and precautionary saving.

#### 6.3.1 A First Liquidity Constraint and Precautionary Saving

We now take up the question of how the introduction of a risk $\xi_{t+1}$ that will be realized between period $t$ and $t+1$ affects consumption in period $t$ in the presence and in the absence of a subsequent constraint. To simplify the discussion, consider a consumer for whom $\beta_{t+1} = R_{t+1} = 1$, with mean income $\bar{y}$ in $t+1$.

Assume that the realization of the risk $\xi_{t+1}$ will be some value $\zeta$ for which the minimum draw with positive probability is $\underline{\zeta}$ and the maximum draw with positive probability is $\bar{\zeta}$, and signify a decision rule that takes account of the presence of the immediate risk by a $\sim$. Thus, the perfect foresight unconstrained consumption function is $c_{t,1}(w)$, the perfect foresight consumption function in the presence of the future constraint is $c_{t,2}(w)$, the consumption function with no constraints but with the risk is $\tilde{c}_{t,1}(w)$ and the consumption function with both risk and constraint is $\tilde{c}_{t,2}(w)$, and similarly for the other functions (e.g. $\tilde{\Omega}'_{t,2}(s)$ is the end-of-period-$t$ marginal value of a consumer facing the constraint and the risk). Now define

$$\tilde{\bar{\omega}}_{t,2} = \breve{\tilde{w}}_{t,2}(\tilde{\Omega}'_{t,1}(\omega_{t+1,2} - (\bar{y} + \underline{\zeta}))) \quad (111)$$

$$\underline{\tilde{\omega}}_{t,2} = \breve{\tilde{w}}_{t,2}(\tilde{\Omega}'_{t,2}(\omega_{t+1,2} - (\bar{y} + \bar{\zeta}))). \quad (112)$$

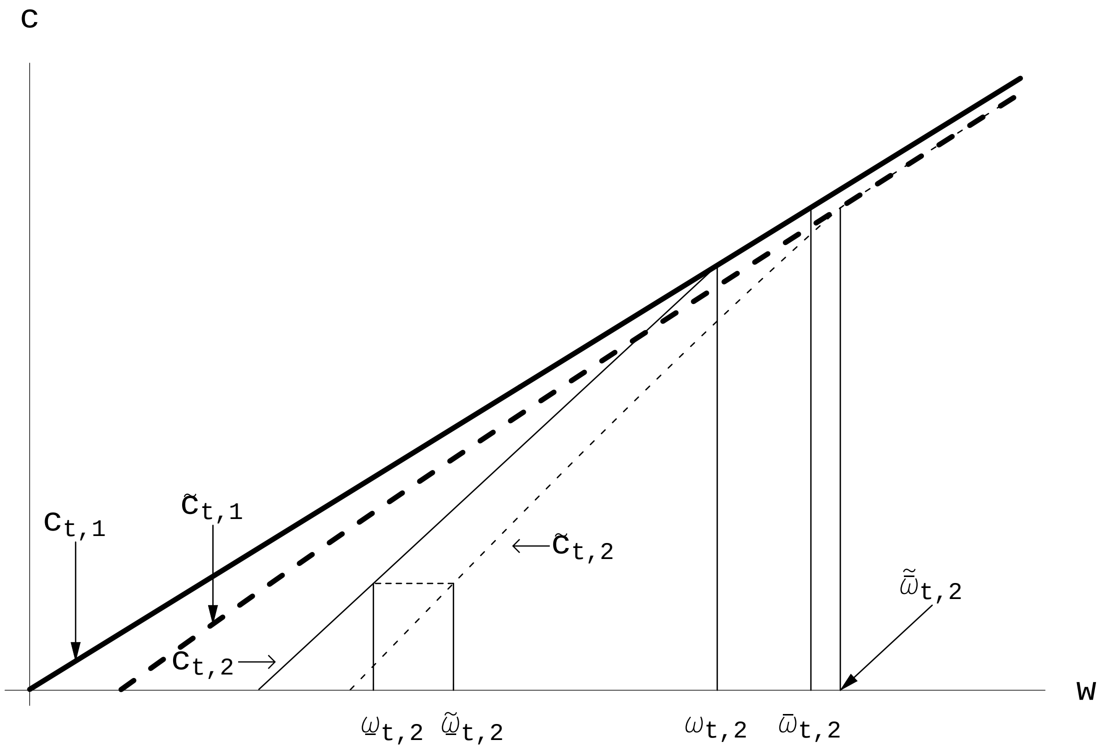
In words, $\tilde{\bar{\omega}}_{t,2}$ is the level of wealth such that a constrained optimizing consumer with this amount of wealth facing shock $\xi_{t+1}$ will save enough to guarantee that even under the worst possible realization of the shock, wealth next period will be greater than the level $\omega_{t+1,2}$ at which the constraint would bind, while $\underline{\tilde{\omega}}_{t,2}$ is the level of wealth such that the constrained consumer facing the risk would save so little that he will be constrained next period even under the best possible draw of $\zeta$. Similarly, define

$$\bar{\omega}_{t,2} = \breve{w}_{t,2}(\tilde{\Omega}'_{t,1}(\omega_{t+1,2} - (\bar{y} + \underline{\zeta}))) \quad (113)$$

$$\underline{\omega}_{t,2} = \breve{w}_{t,2}(\tilde{\Omega}'_{t,2}(\omega_{t+1,2} - (\bar{y} + \bar{\zeta}))), \quad (114)$$

which are the levels of wealth for the perfect foresight consumers that correspond to the same levels of consumption as are generated by $\tilde{\bar{\omega}}_{t,2}$ and $\underline{\tilde{\omega}}_{t,2}$ for the risk-bearing consumers.

Note that we must be careful to check that $\omega_{t+1,2} - \bar{y} - \underline{\zeta}$ is inside the realm of feasible values of $s_t$, in the sense of values that permit the consumer to guarantee that future levels of consumption will be within the permissible range (e.g. positive for consumers with CRRA utility). If this is not true for some level of wealth, then any constraint that binds at or below that level of wealth is irrelevant, because the restriction on wealth imposed by the risk is more stringent than the restriction imposed by the constraint.




SolveConstrCompare4Cases.eps

Figure 5: Consumption Functions With and Without a Constraint and a Risk

Supposing $\mathcal{T}_t[2]$ is not irrelevant in the presence of the risk, we are now in position to state

**Lemma 7** *For all three utility classes, the introduction of the risk $\xi_{t+1}$ has a larger precautionary effect on the level of consumption (induces more precautionary saving) for a consumer for whom there is a positive probability that the constraint $\mathcal{T}_t[2]$ will bind than for a consumer who will never be constrained before the end of his horizon. That is,*

$$c_{t,2}(w) - \tilde{c}_{t,2}(w) \geq c_{t,1}(w) - \tilde{c}_{t,1}(w), \tag{115}$$

*and the inequality is strict at levels of wealth $\tilde{\underline{\omega}}_{t,2} < w < \tilde{\bar{\omega}}_{t,2}$. For the exponential and power utility cases, the inequality is also strict for $w_t \leq \tilde{\underline{\omega}}_{t,2}$, while in the quadratic utility case* (115) *holds with equality for $w_t \leq \tilde{\underline{\omega}}_{t,2}$.*

Figure 5 shows an example of optimal consumption rules in period $t$ under different combinations of an immediate risk (realized between $t$ and $t+1$) and a future constraint (applying between periods $t+1$ and $t+2$).[21] The darker loci reflect behavior of consumers who do not face the future constraint, and the dashed loci

[21]Specifically, this depicts optimal behavior in period $T-2$ of a model in which there is a symmetric two-point mean zero shock that may apply between periods $T-2$ and $T-1$. If the shock were distributed continuously, the change in slope for the constrained consumer facing the risk would be continuous. The *Mathematica* notebook that generates the figure is available on Carroll's home page.

reflect behavior of consumers who *do* face the immediate risk. As expected, for levels of wealth above $\omega_{t,2}$ where the future constraint stops impinging on current behavior for perfect foresight consumers, behavior of the constrained and unconstrained perfect foresight consumers is the same; similarly, $\tilde{c}_{t,2}(w_t) = \tilde{c}_{t,1}(w_t)$ for levels of wealth above $\tilde{\bar{\omega}}_{t,2}$ beyond which the probability of the future constraint binding is zero. And for both constrained and unconstrained consumers, the introduction of the risk reduces the level of consumption (the dashed loci are below their solid counterparts).

The import of lemma 7 in this context is that for levels of wealth below $\tilde{\bar{\omega}}_{t,2}$ the vertical distance between the solid and the dashed loci is greater for the constrained (thin line) than for the unconstrained (thick line) consumers, because of the interaction between the liquidity constraint and the precautionary motive.

Our proof proceeds by constructing behavior of consumers facing the risk from the behavior of the perfect foresight consumers. We will be considering matters from the perspective of some level of wealth $\mathbf{w}$ for the perfect foresight consumers. Because the same marginal utility function $u'$ applies to all four consumption rules, the Compensating Precautionary Premia associated with the introduction of the risk $\xi_{t+1}$, $\kappa_{t,1}$ and $\kappa_{t,2}$, must satisfy

$$\begin{aligned} c_{t,1}(\mathbf{w}) &= \tilde{c}_{t,1}(\mathbf{w}+\kappa_{t,1}) && (116)\\ c_{t,2}(\mathbf{w}) &= \tilde{c}_{t,2}(\mathbf{w}+\kappa_{t,2}). && (117)\end{aligned}$$

Define the amounts of precautionary saving induced by the risk $\xi_{t+1}$ at an arbitrary level of wealth $w$ in the two cases as

$$\begin{aligned} \psi_{t,1}(w) &= c_{t,1}(w) - \tilde{c}_{t,1}(w) && (118)\\ \psi_{t,2}(w) &= c_{t,2}(w) - \tilde{c}_{t,2}(w) && (119)\end{aligned}$$

where the mnemonic is that the first two letters of the Greek letter psi stand for **p**recautionary **s**aving.

We can rewrite (117) (resp. (116)) as

$$\begin{aligned} c_{t,2}(\mathbf{w}+\kappa_{t,2}) + \int_{\mathbf{w}+\kappa_{t,2}}^{\mathbf{w}} c'_{t,2}(v)dv &= \tilde{c}_{t,2}(\mathbf{w}+\kappa_{t,2})\\ c_{t,2}(\mathbf{w}+\kappa_{t,2}) - \tilde{c}_{t,2}(\mathbf{w}+\kappa_{t,2}) &= \int_{\mathbf{w}}^{\mathbf{w}+\kappa_{t,2}} c'_{t,2}(v)dv = \psi_{t,2}(\mathbf{w}+\kappa_{t,2})\\ c_{t,1}(\mathbf{w}+\kappa_{t,1}) - \tilde{c}_{t,1}(\mathbf{w}+\kappa_{t,1}) &= \int_{\mathbf{w}}^{\mathbf{w}+\kappa_{t,1}} c'_{t,1}(v)dv = \psi_{t,1}(\mathbf{w}+\kappa_{t,1})\end{aligned}$$

and

$$\psi_{t,1}(\mathbf{w}+\kappa_{t,2}) = \psi_{t,1}(\mathbf{w}+\kappa_{t,1}) - \int_{\mathbf{w}+\kappa_{t,1}}^{\mathbf{w}+\kappa_{t,2}} (\tilde{c}'_{t,1}(v) - c'_{t,1}(v))dv$$

so the difference between precautionary saving for the constrained and unconstrained consumers at $w = \mathbf{w} + \kappa_{t,2}$ is

$$\psi_{t,2}(\mathbf{w} + \kappa_{t,2}) - \psi_{t,1}(\mathbf{w} + \kappa_{t,2}) = \int_{\mathbf{w}}^{\mathbf{w}+\kappa_{t,1}} (c'_{t,2}(v) - c'_{t,1}(v))dv + \int_{\mathbf{w}+\kappa_{t,1}}^{\mathbf{w}+\kappa_{t,2}} \left(c'_{t,2}(v) + (\tilde{c}'_{t,1}(v) - c'_{t,1}(v))\right) dv. \quad (120)$$

If we can show that (120) is a positive number for all feasible levels of $\mathbf{w}$ satisfying $\mathbf{w} < \bar{\omega}_{t,2}$, then we will have proven the lemma.

### 6.3.2 Some Definitions

Throughout the next few sections, we will consider matters from the perspective of perfect foresight consumers with fixed wealth $\mathbf{w}$. It will be useful to have the following defined:

$$\begin{aligned} \mathbf{c}_{t,n} &= c_{t,n}(\mathbf{w}), & (121)\\ \mathbf{s}_{t,n} &= \mathbf{w} - c_{t,n}(\mathbf{w}) = s_{t,n}(\mathbf{w}) & (122)\\ c(z) &= c_{t+1,1}(\mathbf{s}_{t,1} + \bar{y} + z) & (123)\\ \grave{c}(z) &= c_{t+1,2}(\mathbf{s}_{t,2} + \bar{y} + z). & (124) \end{aligned}$$

### 6.3.3 Perfect Foresight Wealth $\mathbf{w} > \bar{\omega}_{t,2}$ or $(w > \tilde{\bar{\omega}}_{t,2})$

Recall that $\bar{\omega}_{t,2}$ was defined as the level of wealth corresponding, for a perfect foresight consumer, to the value of consumption at which, for the consumer facing both risk and constraint, the probability of the constraint binding reaches zero. Obviously, then, for $\mathbf{w} > \bar{\omega}_{t,2}$, for the consumer facing both risk and constraint, the constraint does not affect consumption because the probability that it will bind is zero. We present the proof of this proposition because it previews techiques that will be less transparent in the more complicated cases.

Note first that for such a $\mathbf{w}$, $c'_{t,1}(v) = c'_{t,2}(v)$ for all $v > \mathbf{w}$, so the first integral in (120) is zero (since $\kappa_{t,1} \geq 0$).

The second integral will be zero if $\kappa_{t,1} = \kappa_{t,2}$. The compensating precautionary premia are defined from

$$\begin{aligned} u'(\mathbf{c}_{t,1}) &= E_t[u'(c(\kappa_{t,1} + \zeta))] & (125)\\ u'(\mathbf{c}_{t,2}) &= E_t[u'(\grave{c}(\kappa_{t,2} + \zeta))], & (126) \end{aligned}$$

but for $\mathbf{w} \geq \bar{\omega}_{t,2}$ we have $\mathbf{c}_{t,1} = \mathbf{c}_{t,2}$ so the LHS's of these two equations are the same. The definitions (123)-(124) translate here to

$$\begin{aligned} c(\kappa_{t,2} + \zeta) &= c_{t+1,1}(\overbrace{\mathbf{s}_{t,1}}^{=\mathbf{s}_{t,2}} + \bar{y} + \kappa_{t,2} + \zeta) & (127)\\ \grave{c}(\kappa_{t,2} + \zeta) &= c_{t+1,2}(\mathbf{s}_{t,2} + \bar{y} + \kappa_{t,2} + \zeta). & (128) \end{aligned}$$

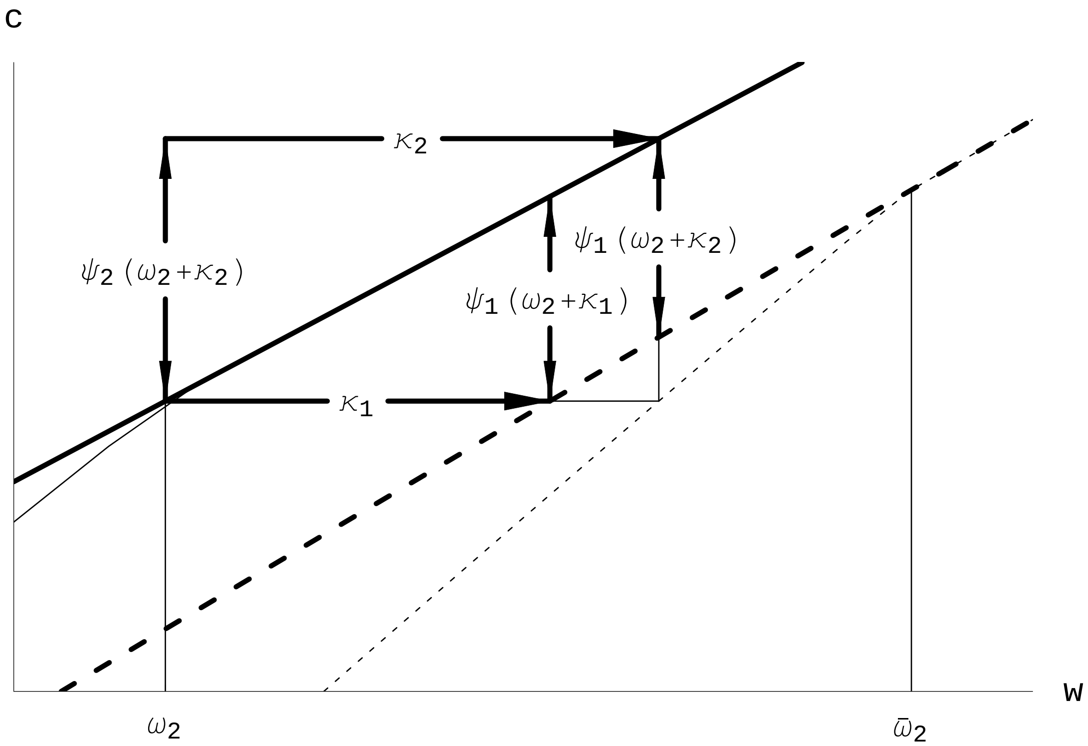



Figure 6: Interaction of a Liquidity Constraint and Precautionary Saving

Now recall that lemma 6 tells us that if absolute prudence of $u'(c(\kappa_{t,2}+\zeta))$ is identical to absolute prudence of $u'(\grave{c}(\kappa_{t,2}+\zeta))$ for every realization of $\zeta$ then $\kappa_{t,1} = \kappa_{t,2}$. But we defined $\bar{\omega}_{t,2}$ as the lowest level of wealth for which $w_{t+1} = \mathbf{s}_{t,2} + \bar{y} + \kappa_{t,2} + \zeta \geq \omega_{t+1,2}$ which means that the constraint cannot bind for any realization $\zeta \in (\underline{\zeta}, \bar{\zeta})$, which means that $c_{t+1,1}(w_{t+1}) = c_{t+1,2}(w_{t+1})$ for all $\zeta$, which means that the RHS of (127) and (128) are identical for all $\zeta$, implying identical absolute prudence of $u'(c(\kappa_{t,2}+\zeta))$ and $u'(\grave{c}(\kappa_{t,2}+\zeta))$ for all $\zeta$. Thus by lemma 6, $\kappa_{t,1} = \kappa_{t,2}$.

Therefore, as expected, for consumers who are rich enough that the constraint could not possibly bind, the presence or absence of the constraint makes no difference to the magnitude of precautionary saving.

### 6.3.4 $\omega_{t,2} \leq \mathbf{w} < \bar{\omega}_{t,2}$

Next consider $\mathbf{w}$ such that $\omega_{t,2} \leq \mathbf{w} < \bar{\omega}_{t,2}$; that is, a level of $\mathbf{w}$ corresponding to a level of consumption at which the future constraint will not bind for the perfect foresight constrained consumer, but will bind with positive probability for the constrained consumer facing the risk. (The specific case of $\mathbf{w} = \omega_{t,2}$ is depicted in figure 6, which magnifies the upper part of figure 5. In the figure, the ubiquitous $t$ subscripts have been dropped to reduce clutter). At these levels of wealth, since the constraint does not bind for the constrained perfect foresight consumer, the

MPC's for the constrained and unconstrained perfect foresight consumers are identical, so the first integral in (120) is zero. And since $c'_{t,2}(\upsilon) > 0$, the second integral will certainly be positive if $\tilde{c}'_{t,1}(\upsilon) \geq c'_{t,1}(\upsilon)$. For both quadratic and exponential unconstrained consumers, the introduction of a risk has no effect on the marginal propensity to consume (for quadratic, the risk has no effect on consumption; for exponential, the level drops but the MPC remains the same, cf. (74)), so for these consumers this term is zero. For CRRA consumers, Carroll and Kimball 1996 show that at a given $\upsilon$ the MPC is higher in the presence than in the absence of uncertainty, $\tilde{c}_{t,1}(\upsilon)' > c'_{t,1}(\upsilon)$. Thus the whole second integral will be strictly positive so long as the upper limit of integration exceeds the lower limit, which will be true so long as $\kappa_{t,2} > \kappa_{t,1}$. We will prove this by showing that assuming $\kappa_{t,2} \leq \kappa_{t,1}$ leads to the conclusion that $\kappa_{t,2} > \kappa_{t,1}$.

Now define $\cancel{\zeta}^{m}_{t+1,n}$ (read $\cancel{\zeta}$ as 'cancel $\zeta$') as the value of $\zeta$ such that

$$\mathbf{s}_{t,n} + \bar{y} + \kappa_{t,n} + \zeta \quad = \quad \omega_{t+1,m}. \tag{129}$$

In words, $\cancel{\zeta}^{m}_{t+1,n}$ is the value of $\zeta$ that 'cancels' the values of $\mathbf{s}_{t,n}$, $\bar{y}$, and $\kappa_{t,n}$ relative to the $m$'th constraint in the sense that given the values of these variables, a draw of $\zeta = \cancel{\zeta}^{m}_{t+1,n}$ restores wealth to exactly the value at which constraint $m$ stops binding.

In this case where $\mathbf{w} \geq \omega_{t,2}$, $\mathbf{s}_{t,1} = \mathbf{s}_{t,2}$. Now assume $\kappa_{t,2} = \kappa_{t,1}$. In this case $c_{t+1,2}(\mathbf{s}_{t,2} + \bar{y} + \kappa_{t,2} + \zeta)$ and $c_{t+1,1}(\mathbf{s}_{t,1} + \bar{y} + \kappa_{t,1} + \zeta)$ are identical for $\zeta > \cancel{\zeta}^{2}_{t+1,2}$ while for $\zeta \leq \cancel{\zeta}^{2}_{t+1,2}$ the MPC of $c_{t+1,2}$ exceeds that of $c_{t+1,1}$, which means by the arguments in section 4.2.2 that absolute prudence of $\grave{c}(\kappa_{t,1} + \zeta)$ exceeds absolute prudence of $c(\kappa_{t,1} + \zeta)$ for some $\zeta \in (\underline{\zeta}, \bar{\zeta})$, which by application of lemma 6 gives us the required result that $\kappa_{t,2} > \kappa_{t,1}$. This is a contradiction of our assumption that $\kappa_{t,2} = \kappa_{t,1}$; we can similarly rule out $\kappa_{t,2} < \kappa_{t,1}$, leaving $\kappa_{t,2} > \kappa_{t,1}$ as the only possibility. We will use a similar argument repeatedly in what follows; for short, we will call this the 'contradiction' argument.

#### 6.3.5 $\mathbf{w} < \omega_{t,2}$

For $\mathbf{w} < \omega_{t,2}$ the perfect foresight consumer subject to the constraint will anticipate (with certainty) being constrained in period $t+1$, and so the appropriate level of consumption will be lower than for the unconstrained consumer. However, for both constrained and unconstrained perfect foresight consumers the Euler equation will hold between periods $t$ and $t+1$,

$$\begin{aligned} u'(\mathbf{c}_{t,1}) &= u'(c_{t+1,1}(\mathbf{s}_{t,1} + \bar{y})) && (130) \\ u'(\mathbf{c}_{t,2}) &= u'(c_{t+1,2}(\mathbf{s}_{t,2} + \bar{y})), && (131) \end{aligned}$$

and strict monotonicity of $u'$ implies that[22]

$$\begin{aligned} \mathbf{c}_{t,1} &= c_{t+1,1}(\mathbf{s}_{t,1} + \bar{y}) && (132) \\ \mathbf{c}_{t,2} &= c_{t+1,2}(\mathbf{s}_{t,2} + \bar{y}). && (133) \end{aligned}$$

[22] If we were to relax the assumption $R_{t+1}\beta_{t+1} = 1$ the levels of consumption satisfying the Euler equations would differ for different utility specifications; this has no substantive implications but would considerably complicate the exposition.

Linearity of the constrained consumption rule below the point at which the constraint impinges means that

$$\mathbf{c}_{t,2} \;\; = \;\; \mathbf{c}_{t,1} - (\omega_{t,2} - \mathbf{w})(c'_{t,2} - c'_{t,1}). \tag{134}$$

The compensating precautionary premium for the unconstrained consumer is defined implicitly from

$$u'(\mathbf{c}_{t,1}) \;\; = \;\; E_t[u'(c_{t+1,1}(\mathbf{s}_{t,1} + \bar{y} + \kappa_{t,1} + \zeta))], \tag{135}$$

but the linearity of the unconstrained consumption rule in $t+1$ means that

$$\begin{aligned} c_{t+1,1}(\mathbf{s}_{t,1} + \bar{y} + \kappa_{t,1} + \zeta) \;\; &= \;\; c_{t+1,1}(\mathbf{s}_{t,1} + \bar{y}) + (\kappa_{t,1} + \zeta)c'_{t+1,1} && (136) \\ &= \;\; \mathbf{c}_{t,1} + (\kappa_{t,1} + \zeta)c'_{t+1,1}, && (137) \end{aligned}$$

so (135) can be rewritten

$$u'(\mathbf{c}_{t,1}) \;\; = \;\; E_t[u'(\mathbf{c}_{t,1} + (\kappa_{t,1} + \zeta)c'_{t+1,1})]. \tag{138}$$

For the constrained consumer we can similarly write

$$c_{t+1,2}(\mathbf{s}_{t,2} + \bar{y} + \kappa_{t,2} + \zeta) = \begin{cases} \mathbf{c}_{t,2} + (\kappa_{t,2} + \zeta)c'_{t+1,2} & \text{for } \zeta < \not\zeta^2_{t+1,2} \\ c^{\#}_{t+1,2} + (\zeta - \not\zeta^2_{t+1,2})c'_{t+1,1} & \text{for } \zeta \geq \not\zeta^2_{t+1,2} \end{cases} \tag{139}$$

so if we define $\pi$ as the probability of a draw of $\zeta < \not\zeta^2_{t+1,2}$ and $E_t[\bullet_{t+1}|\ <]$ (resp. $E_t[\bullet_{t+1}|\ \geq])$ as the expectation of $\bullet_{t+1}$ conditional on a draw of $\zeta$ below (resp. weakly above) $\not\zeta^2_{t+1,2}$, we can write the implicit equation for the CPP $\kappa_{t,2}$ as

$$\begin{aligned} u'(\mathbf{c}_{t,2}) = \pi E_t[u'(\mathbf{c}_{t,2} + (\kappa_{t,2} + \zeta)c'_{t+1,2})|\ <] \\ + (1-\pi)E_t[u'(c^{\#}_{t+1,2} + (\zeta - \not\zeta^2_{t+1,2})c'_{t+1,1})|\ \geq]. \end{aligned} \tag{140}$$

The next two sections use this formula to consider separately the cases where wealth is so low that the constraint always binds, and where the constraint might or might not bind.

### 6.3.6 Wealth Low Enough that Constraint Will Bind with Certainty

If $\mathbf{w}$ is so low that the constraint next period is certain to bind for the constrained consumer facing the risk, then $\pi = 1$ and the RHS of (139) reduces to

$$\mathbf{c}_{t,2} + (\kappa_{t,2} + \zeta)c'_{t+1,2} \;\; = \;\; \mathbf{c}_{t,1} + (\kappa_{t,2} + \zeta)c'_{t+1,2} + (\omega_{t,2} - \mathbf{w})(c'_{t,2} - c'_{t,1}). \tag{141}$$

Now we can restate our earlier result (38) for the current context: If $P(\bullet)$ is absolute prudence of the *utility* function at consumption $\bullet$, then absolute prudence of the *value function* associated with the consumption function $\grave{c}(z)$ is greater than absolute prudence of the value function associated with $c(z)$ at $z$ if

$$P(\grave{c}(z))\grave{c}'(z) \;\; > \;\; P(c(z))c'(z). \tag{142}$$

Lemma 6 then tells us that $\grave{\kappa} = \kappa_{t,2} > \kappa_{t,1} = \kappa$ if

$$P(\grave{c}(\kappa_{t,1} + \zeta))\grave{c}'(\kappa_{t,1} + \zeta) \;\; \geq \;\; P(c(\kappa_{t,1} + \zeta))c'(\kappa_{t,1} + \zeta) \tag{143}$$

for all $\zeta \in (\underline{\zeta}, \bar{\zeta})$ and (143) is a strict inequality for some $\zeta \in (\underline{\zeta}, \bar{\zeta})$.

For exponential utility, absolute prudence is constant, so (143) reduces to

$$c'_{t+1,2}(\mathbf{s}_{t,2} + \bar{y} + \kappa_{t,1} + \zeta) \;\; \geq \;\; c'_{t+1,1}(\mathbf{s}_{t,1} + \bar{y} + \kappa_{t,1} + \zeta). \tag{144}$$

Now we apply the 'contradiction' argument. Assume $\kappa_{t,2} = \kappa_{t,1}$. In this case the LHS of (144) is constant at $c'_{t+1,2}$ (by the definition of $\mathbf{w} < \underline{\omega}_{t,2}$); the RHS is constant at $c'_{t+1,1} < c'_{t+1,2}$, so the condition is satisfied everywhere, and so by lemma 6 $\kappa_{t,2} > \kappa_{t,1}$, which is the contradiction; the contradiction also holds if we assume $\kappa_{t,2} < \kappa_{t,1}$, leaving $\kappa_{t,2} > \kappa_{t,1}$ as the only possibility.

For CRRA utility, absolute prudence at $c$ is $(\rho + 1)/c$, so (143) becomes

$$\left(\frac{\grave{c}'(\kappa_{t,1} + \zeta)}{c'(\kappa_{t,1} + \zeta)}\right) \;\; \geq \;\; \left(\frac{\grave{c}(\kappa_{t,1} + \zeta)}{c(\kappa_{t,1} + \zeta)}\right). \tag{145}$$

Again assuming $\kappa_{t,2} = \kappa_{t,1}$ and recognizing that for the values of $\mathbf{w}$ under consideration the MPC's are constant, (145) translates to a requirement that

$$\left(\frac{c'_{t+1,2}}{c'_{t+1,1}}\right) \;\; \geq \;\; \left(\frac{c_{t+1,2}(\mathbf{s}_{t,2} + \bar{y} + \kappa_{t,1} + \zeta)}{c_{t+1,1}(\mathbf{s}_{t,1} + \bar{y} + \kappa_{t,1} + \zeta)}\right). \tag{146}$$

Substituting (141) and (137) for the levels of consumption on the RHS, this becomes

$$\left(\frac{c'_{t+1,2}}{c'_{t+1,1}}\right) \;\; \geq \;\; \left(\frac{\mathbf{c}_{t,1} + (\kappa_{t,1} + \zeta)c'_{t+1,2} + (\omega_{t,2} - \mathbf{w})(c'_{t,2} - c'_{t,1})}{\mathbf{c}_{t,1} + (\kappa_{t,1} + \zeta)c'_{t+1,1}}\right). \tag{147}$$

The RHS of (147) takes its maximum value at $\mathbf{w} = \omega_{t,2}$, so if the inequality is satisfied there it will be satisfied for all other values of $\mathbf{w} < \omega_{t,2}$. Thus, a sufficient condition is that

$$\left(\frac{c'_{t+1,2}}{c'_{t+1,1}}\right) \;\; \geq \;\; \left(\frac{\mathbf{c}_{t,1} + (\kappa_{t,2} + \zeta)c'_{t+1,2}}{\mathbf{c}_{t,1} + (\kappa_{t,2} + \zeta)c'_{t+1,1}}\right) \tag{148}$$

$$\left(\mathbf{c}_{t,1} + (\kappa_{t,2} + \zeta)c'_{t+1,1}\right) c'_{t+1,2} \;\; \geq \;\; \left(\mathbf{c}_{t,1} + (\kappa_{t,2} + \zeta)c'_{t+1,2}\right) c'_{t+1,1} \tag{149}$$

$$c'_{t+1,2} \;\; \geq \;\; c'_{t+1,1} \tag{150}$$

which holds as a strict inequality. Again we have shown $\kappa_{t,2} > \kappa_{t,1}$ after assuming $\kappa_{t,2} = \kappa_{t,1}$, and again the same result can be shown if we assume $\kappa_{t,2} < \kappa_{t,1}$, leaving $\kappa_{t,2} > \kappa_{t,1}$ as the only possibility.

For quadratic utility, there are no kink points in next period's consumption function for $\mathbf{w} < \underline{\omega}_{t,2}$, so absolute prudence is zero everywhere and $\kappa = \grave{\kappa} = \kappa_{t,2} = \kappa_{t,1} = 0$. Note that in the quadratic case the fact that $\kappa$ is zero implies that $\underline{\omega}_{t,2} = \underline{\tilde{\omega}}_{t,2}$, thus justifying the claim in lemma 7 that for $w < \underline{\tilde{\omega}}_{t,2}$ the size of precautionary saving in response to $\xi_{t+1}$ is the same for the constrained as for the unconstrained consumer (zero in both cases).

#### 6.3.7 Before

Now consider intermediate values of wealth for which the perfect foresight constrained consumer saves more than the perfect foresight unconstrained consumer, and for which there is a chance that the constraint will not bind for the constrained consumer facing the risk.

For exponential utility, the question boils down to whether the prudence condition (144) holds weakly everywhere and strictly somewhere, which again reduces to whether

$$c'_{t+1,2}(\mathbf{s}_{t,2} + \bar{y} + \kappa_{t,2} + \zeta) \quad > \quad c'_{t+1,1}(\mathbf{s}_{t,1} + \bar{y} + \kappa_{t,1} + \zeta) \tag{151}$$

for some $\zeta \in (\underline{\zeta}, \bar{\zeta})$. As usual assuming $\kappa_{t,2} = \kappa_{t,1}$, note that we are considering values of $\zeta$ for which the constraint strictly binds for $\underline{\zeta} < \zeta \leq \not\zeta^{2}_{t+1,2} < \bar{\zeta}$, so (151) does indeed hold strictly somewhere, and since it holds weakly for $\zeta > \not\zeta^{2}_{t+1,2}$, lemma 6 gives us $\kappa_{t,2} > \kappa_{t,1}$ by the contradiction argument.

The CRRA case is considerably more difficult. Define $c(z)$ and $\grave{c}(z)$ as in (123)-(124), and restate (135) and (140) as

$$\begin{aligned} \mathbf{c}_{t,1}^{-\rho} &= E[(c(\kappa_{t,1} + \zeta))^{-\rho}] && (152) \\ \mathbf{c}_{t,2}^{-\rho} &= E[(\grave{c}(\kappa_{t,2} + \zeta))^{-\rho}]. && (153) \end{aligned}$$

Define

$$\nu \quad = \quad \mathbf{c}_{t,2}/\mathbf{c}_{t,1} \tag{154}$$

which is strictly less than one by the assumption $\mathbf{w} < \omega_{t,2}$, and note that both sides of (152) can be multiplied by $\nu^{-\rho}$ to produce

$$\mathbf{c}_{t,2}^{-\rho} \quad = \quad E[(\nu c(\kappa_{t,1} + \zeta))^{-\rho}]. \tag{155}$$

Now define

$$\begin{aligned} \Gamma &= \left( \frac{c^{\#}_{t+1,2}}{\nu c(\omega_{t+1,2} - (\bar{y} + \mathbf{s}_{t,2}))} \right) && (156) \\ &= \left( \frac{c^{\#}_{t+1,2}}{\nu c(\omega_{t+1,2} - (\bar{y} + \mathbf{s}_{t,1} + (\mathbf{s}_{t,2} - \mathbf{s}_{t,1})))} \right) && (157) \\ &= \left( \frac{c^{\#}_{t+1,2}}{\left(c(\omega_{t+1,2} - (\bar{y} + \mathbf{s}_{t,1})) - (\mathbf{s}_{t,2} - \mathbf{s}_{t,1})c'_{t+1,1}\right)\nu} \right) && (158) \\ &= \left( \frac{c^{\#}_{t+1,2}}{\left(c^{\#}_{t+1,2} - (\mathbf{s}_{t,2} - \mathbf{s}_{t,1})c'_{t+1,1}\right)\nu} \right) && (159) \end{aligned}$$

and note that since $\mathbf{s}_{t,2} > \mathbf{s}_{t,1}$ we have $\Gamma\nu > 1$.

Finally, defining

$$\not{z}^m_{t+1,n} \quad = \quad \kappa_{t,n} + \not{\zeta}^m_{t+1,n} \tag{160}$$

construct

$$\check{c}(z) = \begin{cases} \mathbf{c}_{t,2} + z c'_{t+1,2} & \text{if } z < \not{z}_{t+1,2} \\ c^{\#}_{t+1,2} + (z - \not{z}^2_{t+1,2}) c'_{t+1,1} \Gamma \nu & \text{if } z \geq \not{z}^2_{t+1,2}, \end{cases} \tag{161}$$

The CPP associated with this consumption function in period $t+1$, $\check{\kappa}$, is defined implicitly from using $z = \check{\kappa} + \zeta$ and writing the Euler equation as

$$\begin{aligned} u'(\mathbf{c}_{t,2}) = \pi E_t[u'(\mathbf{c}_{t,2} + (\check{\kappa} + \zeta) c'_{t+1,2})| <] \\ + (1-\pi) E_t[u'(c^{\#}_{t+1,2} + (\zeta - \not{\zeta}^2_{t+1,2} + (\check{\kappa} - \kappa_{t,2})) c'_{t+1,1} \Gamma \nu)| \geq] \end{aligned} \tag{162}$$

and note that if we hypothesize that $\check{\kappa} = \kappa_{t,2}$ then the only difference between (162) and (140) is the presence of the $\Gamma\nu$ term in the second term on the RHS of (162). But since $\Gamma\nu > 1$, for $E[\bullet| \geq]$ we know that $\zeta > \not{\zeta}^2_{t+1,2}$, so if $\check{\kappa} = \kappa_{t,2}$ the RHS of (162) is strictly less than the RHS of (140). But the LHS of (140) and (162) are identical. So the assumption $\check{\kappa} = \kappa_{t,2}$ generates a contradiction. Similar logic rules out $\check{\kappa} > \kappa_{t,2}$, leaving $\check{\kappa} < \kappa_{t,2}$ as the only possibility.

We will now apply lemma 6 to show $\check{\kappa} > \kappa_{t,1}$. To do so we need the prudence condition

$$\left( \frac{\check{c}'(\kappa_{t,1} + \zeta)}{\nu c'(\kappa_{t,1} + \zeta)} \right) \quad \geq \quad \left( \frac{\check{c}(\kappa_{t,1} + \zeta)}{\nu c(\kappa_{t,1} + \zeta)} \right) \tag{163}$$

to hold weakly everywhere ($\forall \zeta \in (\underline{\zeta}, \overline{\zeta})$) and strictly somewhere. Notice first that if we define $z = \kappa_{t,1} + \zeta$ then this translates into

$$\left( \frac{\check{c}'(z)}{\nu c'(z)} \right) \quad \geq \quad \left( \frac{\check{c}(z)}{\nu c(z)} \right) \tag{164}$$

for $z \in (\underline{\zeta} + \kappa_{t,1}, \bar{\zeta} + \kappa_{t,1})$.

Our method will be to consider separately the cases where $z < \not{z}^2_{t+1,2}$, $z = \not{z}^2_{t+1,2}$, and $z > \not{z}^2_{t+1,2}$. Start with the $z > \not{z}^2_{t+1,2}$ case, noting first that

$$c(z) \quad = \quad c(\not{z}^2_{t+1,2}) + (z - \not{z}^2_{t+1,2}) c'_{t+1,1} \tag{165}$$

and similarly, for $z \geq z^2_{t+1,2}$ we have

$$\check{c}(z) \quad = \quad \check{c}(\not{z}_{t+1,2}) + (z - \not{z}_{t+1,2}) c'_{t+1,2} \Gamma \nu \tag{166}$$

so (163) becomes

$$\begin{aligned} \left( \frac{\Gamma \nu c'_{t+1,1}}{\nu c'_{t+1,1}} \right) \quad &\geq \quad \left( \frac{\check{c}(\not{z}^2_{t+1,2}) + (z - \not{z}^2_{t+1,2}) c'_{t+1,1} \Gamma \nu}{\nu c(\not{z}^2_{t+1,2}) + (z - \not{z}^2_{t+1,2}) c'_{t+1,1} \nu} \right) &(167) \\ \Gamma \quad &\geq \quad \left( \frac{\Gamma (\nu c(\not{z}^2_{t+1,2}) + (z - \not{z}^2_{t+1,2}) c'_{t+1,1} \nu)}{\nu c(\not{z}^2_{t+1,2}) + (z - \not{z}^2_{t+1,2}) c'_{t+1,1} \nu} \right) &(168) \\ &= \quad \Gamma &(169) \end{aligned}$$

so prudence of $\nu c(z)$ and $\check{c}(z)$ is identical for $z > \not{z}^2_{t+1,2}$.

Now consider points for which $z < \not{z}^2_{t+1,2}$. We know that

$$\begin{aligned} \nu c_{t+1,1}(\mathbf{s}_{t,1} + \bar{y} + z) &= (\mathbf{c}_{t,1} + z c'_{t+1,1})\nu & (170) \\ &= \mathbf{c}_{t,2} + z c'_{t+1,1}\nu & (171) \end{aligned}$$

where the only difference with the first part of (161) is the substitution of $c'_{t+1,1}\nu$ for $c'_{t+1,2}$. The prudence comparision is thus whether

$$\left(\frac{c'_{t+1,2}}{c'_{t+1,1}\nu}\right) \geq \left(\frac{\mathbf{c}_{t,2} + z c'_{t+1,2}}{\mathbf{c}_{t,2} + z c'_{t+1,1}\nu}\right) \quad (172)$$

which boils down to whether

$$c'_{t+1,2} > c'_{t+1,1}\nu \quad (173)$$

which is true for all $z$ below the kink point since $\nu < 1$ and $c'_{t+1,2} > c'_{t+1,1}$.

Finally, at the kink point $z = \not{z}^2_{t+1,2}$ the slope of $\check{c}$ undergoes a discrete decline, which means that it exhibits infinite prudence, clearly greater than the finite prudence exhibited by $c(\not{z}^2_{t+1,2})$, at that point.

Thus prudence for $\check{c}(z)$ is strictly greater than for $c(z)$ for $z$ at and below the kink point $\not{z}^2_{t+1,2}$ and prudence is equal above the kink point, so by lemma 6 we have $\check{\kappa} > \kappa_{t,1}$. Hence we now have $\kappa_{t,2} > \check{\kappa} > \kappa_{t,1}$ which was what we needed.

There is one subtle circumstance that could invalidate this argument: In order to apply lemma 6 we need

$$\not{z}^2_{t+1,2} \in (\underline{\zeta} + \kappa_{t,1}, \bar{\zeta} + \kappa_{t,1}), \quad (174)$$

which we have not demonstrated. Consider first the possibility

$$\begin{aligned} \not{z}^2_{t+1,2} &< \underline{\zeta} + \kappa_{t,1} & (175) \\ \omega_{t+1,2} - (\mathbf{s}_{t,2} + \bar{y}) &< \underline{\zeta} + \kappa_{t,1} & (176) \\ \omega_{t+1,2} &< \underline{\zeta} + \kappa_{t,1} + \mathbf{s}_{t,2} + \bar{y} & (177) \\ \omega_{t+1,2} &< \underline{\zeta} + \kappa_{t,1} + \mathbf{s}_{t,1} + \bar{y}. & (178) \end{aligned}$$

But this means that a constrained consumer saving only the amount associated with the unconstrained consumer's CPP would have zero probability of hitting the constraint, which contradicts our assumption that $\mathbf{w} < \bar{\omega}_{t,2}$. On the other hand consider

$$\begin{aligned} \not{z}^2_{t+1,2} &> \bar{\zeta} + \kappa_{t,1} & (179) \\ \omega_{t+1,2} &> \bar{\zeta} + \kappa_{t,1} + \mathbf{s}_{t,2} + \bar{y} & (180) \end{aligned}$$

and note that if $\kappa_{t,2} \leq \kappa_{t,1}$ then (180) would hold for every realization of $\zeta$ which means the constrained consumer would be constrained in every state, which contradicts our assumption that $\mathbf{w} > \underline{\omega}_{t,2}$ (as well as our earlier result that $\kappa_{t,2} > \kappa_{t,1}$

for consumers who are constrained with probability one). The remaining possibility $\kappa_{t,2} > \kappa_{t,1}$ cannot be ruled out, but if $\not{z}^2_{t+1,2}$ fails to be in $(\kappa_{t,1} + \underline{\zeta}, \kappa_{t,1} + \bar{\zeta})$ because $\kappa_{t,2} > \kappa_{t,1}$, that leaves us with the result we need anyway.

Finally, consider the quadratic case. Here the prudence condition (142) reduces to $0 = 0$ at all $\zeta$ other than $\zeta = \not{\zeta}_{t+1,2}$, the kink point, where prudence is infinite. But the alternative definition of greater absolute prudence applies: For the range of values of $\mathbf{w}$ under consideration here, $\mathbf{s}_{t,2}$ will be such that $\mathbf{s}_{t,2} + \bar{y} + \zeta$ contains a kink point at which there is a discrete decline in the MPC, so absolute prudence of the value function in the presence of the constraint is greater than in its absence at that point, leading again by lemma 6 to the conclusion that $\kappa_{t,2} > \kappa_{t,1}$.

This completes the proof of lemma 7.

### 6.3.8 Further Constraints

The foregoing analysis was specialized to the comparison of a situation with no constraints to the case of a single future constraint. Consider now the case where we have imposed $n$ constraints and are considering imposition of constraint $n + 1$ (and where constraint $n + 1$ applies at the end of some future period; below we examine the case where $\mathcal{T}_t[k_t] = t$).

Assuming constraint $n+1$ is relevant in the presence of the risk, parallel derivations to those that produced (120) imply that the key question is whether $\kappa_{t,n+1} > \kappa_{t,n}$. Analysis of the cases where $\mathbf{w} < \underline{\omega}_{t,n+1}$ and $\mathbf{w} > \omega_{t,n+1}$ is identical to the corresponding cases above, and need not be repeated. Thus, we examine only the case of values of wealth $\mathbf{w}$ such that $\underline{\omega}_{t,n+1} < \mathbf{w} < \omega_{t,n+1}$. And we examine only the CRRA utility case, as the extension to the exponential and quadratic cases follows the same patterns as in section 6.3.7.

Consider the following definitions which are parallel to those in section 6.3.7:

$$\nu_{n+1} = \mathbf{c}_{t,n+1}/\mathbf{c}_{t,n} \tag{181}$$

$$c(z) = c_{t+1,n}(\mathbf{s}_{t,n} + \bar{y} + z) \tag{182}$$

$$\grave{c}(z) = c_{t+1,n+1}(\mathbf{s}_{t,n+1} + \bar{y} + z), \tag{183}$$

Now we wish to define $\check{c}(z)$ so that $\check{c}(z) = \grave{c}(z)$ for $z < \not{z}_{t+1,n+1}$, while the absolute prudence of $\check{c}(z)$ is identical to that of $c(z)$ for $z \geq \not{z}_{t+1,n+1}$. Setting

$$\Gamma_{t+1,n+1} = \left( \frac{c^{\#}_{t+1,n+1}}{c(\not{z}_{t+1,n+1})\nu_{t+1}} \right), \tag{184}$$

this definition of $\check{c}$ requires an MPC of

$$\check{c}'(z) = \nu_{t+1}\Gamma_{t+1,n+1}c'(\not{z}^{n+1}_{t+1,n+1}) \tag{185}$$

at $z = \not{z}_{t+1,n+1}$.

Equal absolute prudence of $\check{c}(z)$ and $\nu_{t+1}c(z)$ requires

$$\left( \frac{\check{c}'(z)}{\check{c}(z)} \right) = \left( \frac{c'(z)}{c(z)} \right) \tag{186}$$

but since $c'(z) = c'_{t+1,n}$ is constant over the range $\not{z}^{n+1}_{t+1,n+1} \leq z < \not{z}^{n}_{t+1,n}$ if we choose a constant $\check{c}'(z)$ equal to (185) over this range the LHS of (186) reduces to

$$\begin{aligned}
\left(\frac{\nu_{t+1}\Gamma_{t+1,n+1}c'_{t+1,n}}{c^{\#}_{t+1,n+1} + (z - \not{z}^{n+1}_{t+1,n+1})\nu_{t+1}\Gamma_{t+1,n+1}c'_{t+1,n}}\right) &= \left(\frac{c'_{t+1,n}}{c(\not{z}^{n+1}_{t+1,n+1}) + (z - \not{z}^{n+1}_{t+1,n+1})c'_{t+1,n}}\right) \\
&= \left(\frac{c'(z)}{c(z)}\right) \qquad (187)
\end{aligned}$$

as required. Similarly, if we define

$$\Gamma^{n}_{t+1,n} = \left(\frac{\check{c}(\not{z}^{n}_{t+1,n})}{c(\not{z}^{n}_{t+1,n})\nu_{t+1}}\right), \qquad (188)$$

and set

$$\check{c}'(z) = \nu_{t+1}\Gamma^{n}_{t+1,n}c'(\not{z}^{n}_{t+1,n}) \qquad (189)$$

for $z > \not{z}^{n}_{t+1,n}$ then prudence of $\check{c}(z)$ will again equal that of $c(z)$ at $\not{z}^{n}_{t+1,n}$. Proceeding in this way to define successive $\Gamma$'s corresponding to the successive kink points in $c(z)$ defines a function $\check{c}(z)$ that has prudence identical to that of $c(z)$ for all $z \geq \not{z}^{n+1}_{t+1,n+1}$. Since prudence of $\check{c}(z)$ is greater than prudence of $c(z)$ for $z < \not{z}^{n+1}_{t+1,n+1}$ and equal above this point, by lemma 6 we can conclude that $\check{\kappa} > \kappa$ by the same arguments as before.

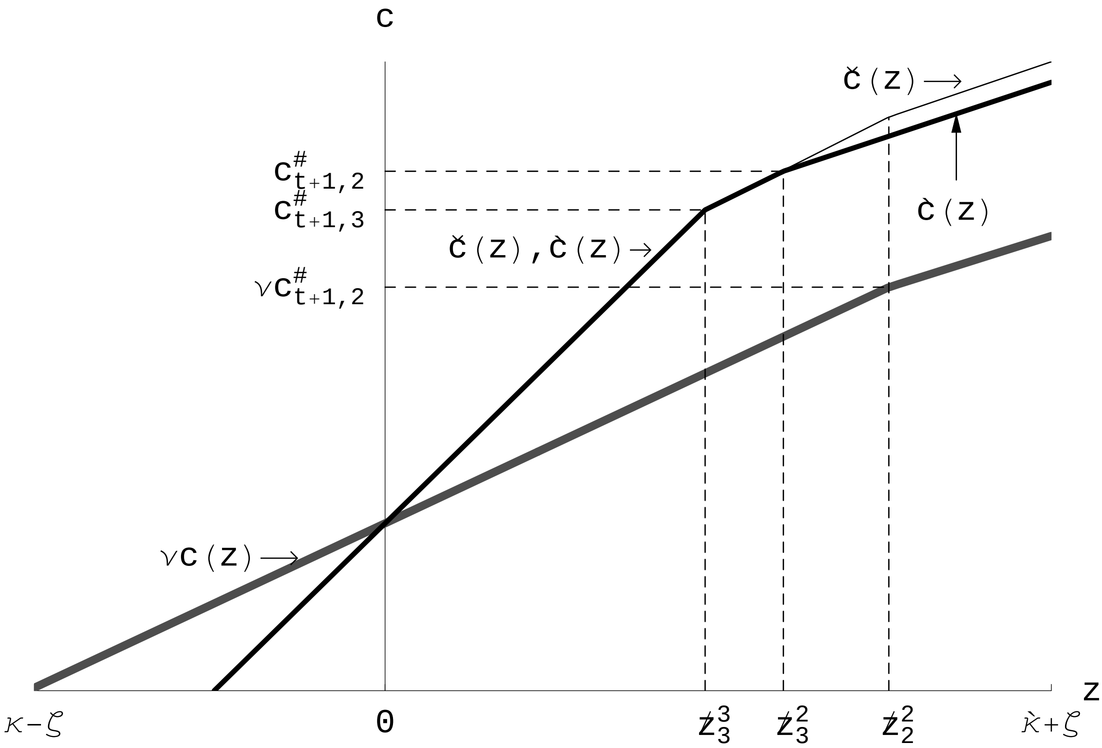
Before proceeding to show $\grave{\kappa} > \check{\kappa}$, to build intuition consider figure 7, which depicts an example of a problem in which a second constraint is being added to a problem with a single initial constraint (the ubiquitous $t+1$ subscripts have been dropped to reduce clutter). For values of $z \leq \not{z}^{3}_{t+1,3}$ (that is, the point where the final constraint stops binding for the more-constrained consumer), $\check{c}(z) = \grave{c}(z)$. For $z^{3}_{t+1,3}$ to $z^{2}_{t+1,2}$ we have $\check{c}'(z) = \Gamma_{t+1,n+1}\nu_{t+1}c'(z) > c'(z)$ but the difference does not become visible until $z$ rises above the point where the second constraint stops binding for the multiply-constrained consumer, $\not{z}^{2}_{t+1,3}$. Between $\not{z}^{2}_{t+1,3}$ and the point $\not{z}^{2}_{t+1,2}$ where the constraint stops binding for the less-constrained consumer, the slope of $\check{c}(z)$ is visibly greater than that of $\grave{c}(z)$.

The definitions of $\not{z}^{3}_{t+1,3}$ and $\not{z}^{2}_{t+1,2}$ also make sense: These are visibly the values of $z$ below which the final constraint is always binding for the two functions. It also apparent that the kink point in $\nu_{t+1}c(z)$ occurs at $\nu_{t+1}c^{\#}_{t+1,2}$ as required, and that $\nu_{t+1}c(0) = \grave{c}(0)$ (because both are equal to $\mathbf{c}_{t,2}$, corresponding to (132) and (133)).

Returning to the theory, the remaining question is whether $\grave{\kappa} > \check{\kappa}$. As before, this is proven by contradiction. Defining $\pi$ as the probability of drawing a $\zeta < \not{\zeta}^{3}_{t+1,3}$ and $E[\bullet| <]$ and $E[\bullet| \geq]$ correspondingly, the definition of the CPPs tells us

$$\begin{aligned}
u'(\mathbf{c}_{t,2}) &= \pi E[u'(\grave{c}(\grave{\kappa} + \zeta))| <] + (1-\pi)E[u'(\grave{c}(\grave{\kappa} + \zeta))| \geq] \qquad (190) \\
u'(\mathbf{c}_{t,2}) &= \pi E[u'(\check{c}(\check{\kappa} + \zeta))| <] + (1-\pi)E[u'(\check{c}(\check{\kappa} + \zeta))| \geq]. \qquad (191)
\end{aligned}$$

Under the assumption $\check{\kappa} = \grave{\kappa}$ the first terms on the RHS of these equations are identical. But for $\zeta > \not{\zeta}^{3}_{t+1,3}$ we know that if $\check{\kappa} = \grave{\kappa}$ then $\check{c}(\grave{\kappa} + \zeta) > \grave{c}(\grave{\kappa} + \zeta)$ (see




SolveConstrHardCase.eps

Figure 7: Definition of $\check{c}(z)$

the figure). Thus assuming $\breve{\kappa} = \grave{\kappa}$ the LHS and the first terms on the RHS in (190) and (191) are identical but the second term on the RHS of (190) is greater than the corresponding term in (191), generating a contradiction that can be resolved only by assuming $\breve{\kappa} < \grave{\kappa}$.

Thus, we can state a more general version of lemma 7,

**Lemma 8** *For all three utility classes, the introduction of the risk $\xi_{t+1}$ has a larger precautionary effect on the level of consumption (induces more precautionary saving) for a consumer who faces the first $n+1$ liquidity constraints in $\mathcal{T}_t$ than for a consumer who faces only the first $n$ constraints. That is,*

$$c_{t,n+1}(w) - \tilde{c}_{t,n+1}(w) \geq c_{t,n}(w) - \tilde{c}_{t,n}(w), \tag{192}$$

*and the inequality is strict at levels of wealth $\underline{\tilde{\omega}}_{t,n+1} < w < \tilde{\tilde{\omega}}_{t,n+1}$. For the exponential and power utility cases, the inequality is also strict for $w_t \leq \underline{\tilde{\omega}}_{t,n+1}$, while in the quadratic utility case* (192) *holds with equality for $w_t \leq \underline{\tilde{\omega}}_{t,n+1}$.*

Finally, if there are no constraints that could apply, or risks that could be realized, between period $s < t$ and period $t$, it is simple to extend this to period $s$; we state this final version as a theorem,

**Theorem 3** *Consider a consumer in period $s < t$ facing a set of liquidity constraints $\mathcal{T}_s = \mathcal{T}_t$, and a risk $\xi_{t+1}$ that will be realized between the end of period $t$ and the beginning of period $t+1$. The introduction of the risk $\xi_{t+1}$ has a larger precautionary effect on the level of consumption (induces more precautionary saving) in period $s$ for a consumer who faces the first $n+1$ liquidity constraints in $\mathcal{T}_s$ than for a consumer who faces only the first $n$ constraints. That is,*

$$c_{s,n+1}(w_s) - \tilde{c}_{s,n+1}(w_s) \geq c_{s,n}(w_s) - \tilde{c}_{s,n}(w_s), \tag{193}$$

*and the inequality is strict at levels of wealth $\underline{\tilde{\omega}}_{s,n+1} < w < \tilde{\tilde{\omega}}_{s,n+1}$, where $\underline{\tilde{\omega}}_{s,n+1}$ (resp. $\tilde{\tilde{\omega}}_{s,n+1}$) is the level of wealth such that a consumer in period $s$ with wealth $w_s = \underline{\tilde{\omega}}_{s,n+1}$ (resp. $w_s = \tilde{\tilde{\omega}}_{s,n+1}$) will arrive at the beginning of period $t$ with wealth $\underline{\tilde{\omega}}_{t,n+1}$ (resp. $\tilde{\tilde{\omega}}_{t,n+1}$). For the exponential and power utility cases, the inequality is also strict for $w_s \leq \underline{\tilde{\omega}}_{s,n+1}$, while in the quadratic utility case* (193) *holds with equality for $w_s \leq \underline{\tilde{\omega}}_{s,n+1}$.*

In intuitive terms, this theorem says that the precautionary saving induced by the introduction of the risk $\xi_{t+1}$ is greater for the consumer facing $n+1$ constraints than for the consumer facing $n$ constraints at levels of wealth at which there is some probability that the $n+1$'th constraint will actually bind.

The easiest way to see that the theorem holds (given the lemma) is to realize that in an unconstrained perfect foresight model behavior in period $s < t$ can be reformulated to combine multiple periods into one. Thus, for example, for CRRA utility, consumption two periods earlier can be obtained via the Euler equation from consumption one period earlier, and so on. This means that the foregoing logic can be directly reinterpreted with appropriate substitutions of $s$ for $t$ and appropriate reformulation of the dynamic budget constraint.

### 6.3.9 Earlier Risks and Constraints

The form of theorem 3 is suggestive of a stronger result that one might hope would hold if earlier risks and constraints were imposed. To state then examine this result, we need to develop a last bit of notation. We define

$$c^m_{\tau,n}$$

as the consumption function in period $\tau$ assuming that the first $n$ constraints and the first $m$ risks have been imposed, counting risks, like constraints, backwards from period $T$. Thus, relating our new notation to our previous usage, $c^0_{\tau,n} = c_{\tau,n}$ because 0 risks have been imposed. All other functions are defined correspondingly, e.g. $\Omega^p_{\tau,n}$ is the end-of-period-$\tau$ value function assuming the first $n$ constraints and $m$ risks have been imposed. We will continue to use the notation $\tilde{c}_{\tau,n}$ to designate the effects of imposition of a single risk that will be realized between periods $\tau$ and $\tau+1$. Finally, we need to introduce a counter that keeps track of how many future risks exist beyond a given date, analogous to our $k_t$ counter that tells how many constraints exist after period $t$. Call the risk-counter $q_t$.

Suppose now the presence of multiple risks that will be realized between $t$ and $T$. One might hope to show that the precautionary effect of imposing all risks in the presence of all constraints would be greater than the effect of imposing all risks in the absence of any constraints:

$$c^0_{t,k_t}(w) - c^{q_t}_{t,k_t}(w) \geq c^0_{t,1}(w) - c^{q_t}_{t,1}(w). \tag{194}$$

Such a hope, however, would be in vain. In fact, we will now show that even the considerably weaker condition, involving only the single risk $\xi_{t+1}$ and all constraints, $c^0_{t,k_t}(w) - c^1_{t,k_t}(w) \geq c^0_{t,1}(w) - c^1_{t,1}(w)$, can fail to hold for some $w$.

### 6.3.10 An Immediate Constraint

Consider a situation in which $\mathcal{T}_t[k_t] = T-t$; that is, the chronologically earliest constraint in $\mathcal{T}_t$ applies at the end of period $t$. Since $c^1_{t,k_{t+1}}$ designates the consumption rule that will be optimal prior to imposing the period-$t$ constraint, the consumption rule imposing all constraints will be

$$c^1_{t,k_t}(w) \quad = \quad \min[c^1_{t,k_{t+1}}(w), w]. \tag{195}$$

Now define the level of wealth below which the period $t$ constraint binds for a consumer facing the risk as $\underline{\omega}^1_{t,k_t}$. For values of wealth $w \geq \underline{\omega}^1_{t,k_t}$, analysis of the effects of the risk is identical to analysis in the previous subsection where the first $k_{t+1}$ constraints were imposed. For levels of wealth $w < \underline{\omega}^1_{t,k_t}$, we have $c^1_{t,k_t}(w) = c_{t,k_t}(w) = w$ (for the simple $c \geq w$ constraint; a corresponding point applies to the more sophisticated form of constraint); that is, for consumers with wealth below $\underline{\omega}^1_{t,k_t}$, the introduction of the risk in period $t+1$ has no effect on consumption in $t$, because for these levels of wealth the constraint at the end of $t$ has the effect of 'hiding' the risk from view (they were constrained before the risk was imposed and remain constrained afterwards). Thus for exponential and CRRA consumers, for whom the inequality (192) holds strictly in the absence of the constraint at $t$, at levels of wealth below $\underline{\omega}^1_{t,k_t}$ the precautionary effect of the risk is wiped out. (For quadratic consumers, precautionary saving is zero for wealth below $\underline{\omega}^1_{t,k_t}$ both before and after the risk is imposed).

### 6.3.11 An Earlier Risk

Consider now the question of how the addition of a risk $\xi_t$ that will be realized between periods $t-1$ and $t$ affects the consumption function at the beginning of period $t-1$, in the absence of any constraint at the beginning of period $t$.

The question at hand is then whether we can say that

$$c^1_{t-1,1}(w) - c^2_{t-1,1}(w) \quad \geq \quad c_{t-1,1}(w) - c^1_{t-1,1}(w); \tag{196}$$

that is, does the introduction of the risk $\xi_t$ have a greater precautionary effect on consumption in the presence of the subsequent risk $\xi_{t+1}$ than in its absence?

The answer again is "not necessarily." To see why, recall our earlier example of a CRRA utility problem in which in a certain limit the introduction of a risk produced an effect on the consumption function that was indistinguishable from the effect of a liquidity constraint. If the risk $\xi_t$ is of this liquidity-constraint-indistinguishable form, then the logic of the previous subsection clearly applies: For some levels of wealth, the introduction of the risk at $t$ can 'hide' the precautionary effect of any risks at $t+1$ or later.

### 6.3.12 What Can Be Said?

It might seem that the previous two subsections imply that little useful can be said about the precautionary effects of introducing a new risk in the presence of preexisting constraints and risks. It turns out, however, that there is at least one useful proposition.

Suppose we are interested in the effect, from the perspective of some period $\tau$, of introducing a risk that will be realized between $\tau$ and $\tau+1$. Our goal is to find a set of values of wealth $\mathcal{P}_{\tau,k_\tau}$ for which

$$c^{q_\tau+1}_{\tau,k_\tau}(w) - c^{q_\tau}_{\tau,k_\tau}(w) \quad > \quad c_{\tau,1}(w) - c^1_{\tau,1}(w). \tag{197}$$

That is, if we call the consumer subject to the complete set of future risks and constraints (but not the immediate risk) the 'blighted' consumer (corresponding to $c^{q_\tau+1}_{\tau,k_\tau}$) and the consumer subject to no future risks or constraints the 'unblighted' consumer, we are looking for levels of beginning-of-period-$\tau$ wealth such that the introduction of the immediate risk has a greater (precautionary) effect on the blighted than on the unblighted consumer.

Again assuming that the last risk is $\xi_{t+1}$, start by defining $\mathcal{W}^{q_{t+1}}_{t+1,k_{t+1}}$ as the set of points at which the perfect foresight value function $V_{t+1,k_{t+1}} = V^{q_{t+1}}_{t+1,k_{t+1}}$ exhibits property strict CC, which will be the set of kink points in the perfect foresight problem $\vec{\omega}_{t+1,k_{t+1}}$ because $q_{t+1} = 0$ (there are no risks beyond period $t+1$). Now define $\mathcal{S}^{q_t}_{t,k_{t+1}}$ as the set of values of end-of-period-$t$ savings $s$ at which the end-of-period value function imposing all future risks $\Omega^{q_t}_{t,k_{t+1}}$ exhibits property strict CC.

For the quadratic case, define $\zeta^+$ as the set of values of $\zeta$ with positive probability or positive probability density. Then the results in section 5.2 tell us that

$$\mathcal{S}^{q_t}_{t,k_{t+1}} = \{s | s + \bar{y} + \zeta \in \mathcal{W}^{q_{t+1}}_{t+1,k_{t+1}}\} \tag{198}$$

for some $\zeta \in \zeta^+$. In words, for the quadratic case $\mathcal{S}^{q_t}_{t,k_{t+1}}$ is the set of values of $s$ from which the probability that some constraint will bind weakly is affected by the outcome of the risk.[23]

For the exponential case, we have from 5.2

$$\mathcal{S}^{q_t}_{t,k_{t+1}} = \{s | s + \bar{y} + \zeta \in \mathcal{W}^{q_{t+1}}_{t+1,k_{t+1}}\} \tag{199}$$

for some $\zeta \in [\underline{\zeta}, \bar{\zeta}]$. In words, $\mathcal{S}^{q_t}_{t,k_{t+1}}$ is the set of values of $s$ for which there is either a positive probability that a constraint will bind next period, or from which the probability that some constraint will bind weakly is affected by the outcome of the risk.

For the CRRA case, we know from the results in section 5.2 that if $\xi_{t+1}$ is nondegenerate then $\mathcal{S}^{q_t}_{t,k_{t+1}}$ is the set of all feasible values of $s$.

Now define $\mathcal{W}^{q_t}_{t,k_{t+1}}$ as the set of values of $w$ at which $V^{q_t}_{t,k_{t+1}}$ exhibits property strict CC. $\mathcal{W}^{q_t}_{t,k_{t+1}}$ is easy to construct: Our theorems on horizontal aggregation in 5.1 tell us that $V^{q_t}_{t,k_{t+1}}(w)$ has the same kind of concavity (strict or borderline) as does $\Omega^{q_t}_{t,k_{t+1}}$ at the level of savings that is optimal for initial wealth $w$.

Finally define $\mathcal{W}^{q_t}_{t,k_t}$ as the set of values of wealth at which the value function exhibits property strict CC once the constraint (if any) in period $t$ has been imposed. Then since the consumption function is linear below the point at which the constraint binds, the formal definition is

$$\mathcal{W}^{q_t}_{t,k_t} \quad = \quad \mathcal{W}^{q_t}_{t,k_{t+1}} \cap \{w | w \geq c^{q_t}_{t,k_{t+1}}(w)\}. \tag{200}$$

This completes a set of steps by which $\mathcal{W}^{q_t}_{t,k_t}$ can be constructed from $\mathcal{W}^{q_{t+1}}_{t+1,k_{t+1}}$. The same steps can be iterated to any earlier period to generate $\mathcal{W}^{q_{\tau+1}}_{\tau+1,k_{\tau+1}}$ for $\tau < t-1$.

Consider now the case where there is no constraint that could bind at the beginning of period $\tau$, and define $\bar{\mathcal{W}}^{q_\tau}_{\tau,k_{\tau+1}}$ as the maximum value in $\mathcal{W}^{q_\tau}_{\tau,k_{\tau+1}}$. Then our claim is that the set

$$\mathcal{P}_{\tau,k_\tau} = \mathcal{P}_{\tau,k_{\tau+1}} = \begin{cases} \mathcal{W}^{q_\tau}_{\tau,k_{\tau+1}} & \text{for quadratic} \\ \{w | w < \bar{\mathcal{W}}^{q_\tau}_{\tau,k_{\tau+1}}\} & \text{for exponential and CRRA} \end{cases} \tag{201}$$

will satisfy condition (197). In words: In the quadratic case, precautionary saving induced by $\xi_{\tau+1}$ is higher for the blighted consumer at levels of wealth $w_\tau$ such that

[23] By 'bind weakly' we mean to include points on the cusp, in the sense that for an arbitrarily small $\delta$ some constraint makes a transition from binding strictly at $s - \delta$ to not binding strictly at $s + \delta$.

there is at least one constraint that will bind with probability $0 < p < 1$ in some future period for some realizations of $\zeta$ but bind with a strictly different probability for other realizations of $\zeta$. In the exponential case, precautionary saving is higher if there is a positive probability that some future constraint could be binding. In the CRRA case, precautionary saving is higher if there is a positive probability that some future constraint could be binding, or if there is any future risk (because in this case $\bar{\mathcal{W}}^{q_\tau}_{\tau,k_{\tau+1}} = \infty$).

The proof of the proposition for the quadratic and exponential cases is identical to the proofs presented above. Intuitively, in the quadratic case these are the conditions under which the risk $\xi_{\tau+1}$ interacts with some point at which $c^{q_{\tau+1}}_{\tau+1,k_{\tau+1}}$ is strictly concave, while in the exponential case these are the conditions where there is either an interaction with a point of strict concavity, or the marginal propensity to consume is higher.

The CRRA case is again the difficult one. Given some $\mathbf{w} \in \mathcal{P}_{\tau,k_\tau}$ define

$$
\begin{aligned}
c(z) &= c_{\tau+1,1}(\mathbf{s}_{\tau,1} + \bar{y} + z) && (202)\\
\grave{c}(z) &= c^{q_{\tau+1}}_{\tau+1,k_{\tau+1}}(\mathbf{s}^{q_{\tau+1}}_{\tau,k_{\tau+1}} + \bar{y} + z)(\mathbf{c}_{\tau,1}/\mathbf{c}^{q_{\tau+1}}_{\tau,k_{\tau+1}}) && (203)
\end{aligned}
$$

with associated CPP's $\kappa$ and $\grave{\kappa}$.

Our method again involves constructing a $\check{c}(z)$ function for which $\grave{\kappa} > \check{\kappa} > \kappa$. Begin by noting that $\lim_{w\to\infty} c^{q_{\tau+1}}_{\tau+1,k_{\tau+1}}(w) - c_{\tau+1,1}(w) = 0$, along with concavity of $c^{q_{\tau+1}}_{\tau+1,k_{\tau+1}}(w)$ and the fact that $\mathbf{s}^{q_{\tau+1}}_{\tau,k_{\tau+1}} > \mathbf{s}_{\tau,k_{\tau+1}}$ imply that, while for low enough $z$ absolute prudence of $\grave{c}(z)$ must exceed that of $c(z)$, as $z$ rises there must come a $z$ where absolute prudence of $\grave{c}(z)$ falls below absolute prudence of $c(z)$. Assume that we are considering a value of $\mathbf{w}$ for which $\grave{A}(z) < A(z)$ for some $z \in (\kappa+\underline{\zeta}, \grave{\kappa}+\bar{\zeta})$ and for which $\grave{A}(z') > A(z')$ for some other $z' \in (\kappa+\underline{\zeta}, \grave{\kappa}+\bar{\zeta})$; this is the only interesting case, as lemma 6 tells us that otherwise that the ranking of $\grave{\kappa}$ and $\kappa$ corresponds to the (unambiguous) ranking of $\grave{A}(z)$ and $A(z)$. (This range restriction plays the role in this more general case of the restriction $\underline{\omega}_{t,n} < \mathbf{w} < \omega_{t,n}$ in the case with only constraints and a single risk).

Define $\not{z}_1$ as the value of $z$ above which absolute prudence of $\grave{c}(z)$ first falls below absolute prudence of $c(z)$ (assume here that $\grave{c}(z)$ is differentiable; below we relax this assumption).

Define $\check{c}(z) = \grave{c}(z)$ for all $z < \not{z}_1$. Then making absolute prudence of $\grave{c}(z)$ and $c(z)$ be identical for $z = \not{z}_1$ requires

$$
\begin{aligned}
\left(\frac{\check{c}'(\not{z}_1)}{\check{c}(\not{z}_1)}\right) &= \left(\frac{c'(\not{z}_1)}{c(\not{z}_1)}\right) && (204)\\
\check{c}'(\not{z}_1) &= c'(\not{z}_1)\left(\frac{\check{c}(\not{z}_1)}{c(\not{z}_1)}\right), && (205)
\end{aligned}
$$

where equality of derivatives here and throughout is taken to apply to both left and right derivatives.

From this point recursively upward, we can define $\check{c}'(z)$ as the value that produces identical prudence of $\check{c}(z)$ and $c(z)$ for all $z > \not{z}_1$, with $\check{c}(z)$ implicitly defined by its derivative. This is similar to the definition of $\check{c}$ from section 6.3.8, and as before, since prudence of $\check{c}(z)$ is greater than prudence of $c(z)$ below $\not{z}_1$ and equal above, it is clear that $\breve{\kappa} > \kappa$ so long as we are considering a level of wealth for which the relevant $z \in (\kappa + \underline{\zeta}, \kappa + \bar{\zeta})$.

The remaining question is whether $\grave{\kappa} > \breve{\kappa}$, which we again show through a contradiction argument. Since $\grave{c}(\not{z}_1) > c(\not{z}_1)$ it is clear that $\check{c}'(\not{z}_1) > \grave{c}'(\not{z}_1)$. Furthermore, since for $z > \not{z}_1$ the function $\grave{c}(z)$ is concave but $\check{c}(z)$ is linear, the ratio of $\check{c}'(z)$ to $\grave{c}'(z)$ can only increase as $z$ rises. Thus it is clear that $\check{c}(z) > \grave{c}(z)$ for all $z > \not{z}_1$. Now defining $\pi$ as the probability of drawing a $\zeta$ such that $\grave{\kappa} + \zeta < \not{z}_1$ and $E[\bullet| <]$ and $E[\bullet| \geq]$ correspondingly, the CPP's are defined implicitly from

$$
\begin{aligned}
u'(\mathbf{c}_{t,2}) &= \pi E[u'(\grave{c}(\grave{\kappa} + \zeta))| <] + (1 - \pi) E[u'(\grave{c}(\grave{\kappa} + \zeta))| \geq] \qquad (206) \\
u'(\mathbf{c}_{t,2}) &= \pi E[u'(\check{c}(\breve{\kappa} + \zeta))| <] + (1 - \pi) E[u'(\check{c}(\breve{\kappa} + \zeta))| \geq] \qquad (207)
\end{aligned}
$$

and as before the assumption $\breve{\kappa} = \grave{\kappa}$ generates a contradiction because the first terms on the LHS are identical but the second terms are not, and this contradiction would arise with even greater force for $\breve{\kappa} > \grave{\kappa}$, leaving $\breve{\kappa} < \grave{\kappa}$ as the only possibility.

Finally, we turn to the case where there is a constraint that could bind at the beginning of period $\tau$. Here, the set

$$
\mathcal{P}_{\tau,k_\tau} = \mathcal{P}_{\tau,k_{\tau+1}} \cap \{w | w \geq \underline{\mathcal{W}}^{q_{\tau+1}}_{\tau,k_\tau}\} \qquad (208)
$$

will be a set of values at which (197) will hold, where $\underline{\mathcal{W}}^{q_{\tau+1}}_{\tau,k_\tau}$ is the level of period-$\tau$ wealth below which the constraint would be binding in the absence of the risk $\xi_{\tau+1}$. If the current constraint would not have been binding in the absence of the risk then it will surely not be binding in the presence of $\xi_{\tau+1}$, so for these values of wealth the addition of the risk does not interact with the period-$\tau$ constraint.

This is not the complete set of points where (197) holds. In particular, note that at levels of wealth below $\underline{\mathcal{W}}^{q_\tau}_{\tau,k_\tau}$ the introduction of $\xi_{\tau+1}$ has no effect for the blighted consumer (who sets $c = w$ in the presence or the absence of the risk) but can have an effect for the unblighted consumer, while at wealth equal to $\underline{\mathcal{W}}^{q_\tau}_{\tau,k_{\tau+1}} > \underline{\mathcal{W}}^{q_\tau}_{\tau,k_\tau}$ equation (197) holds as a strict inequality. Since all involved consumption functions are continuous, the intermediate value theorem tells us that there will be some set of values of $w$ between $\underline{\mathcal{W}}^{q_\tau}_{\tau,k_{\tau+1}}$ and $\underline{\mathcal{W}}^{q_\tau}_{\tau,k_\tau}$ for which (197) will hold. But there does not appear to be any way of characterizing these points that is simpler than (197) itself; thus we stick with the more limited claim embodied in (208).

All of this is summarized in our final theorem:

**Theorem 4** *Introduction of the risk $\xi_{\tau+1}$ has a larger (precautionary) effect on the level of period-$\tau$ consumption in the presence of all future risks and constraints than in absence of any future risks and constraints (i.e., equation* (197) *holds) at levels of wealth period-$\tau$ wealth $w$ such that, in the absence of the new risk the consumer would not be constrained in the current period ($c^{q_{\tau+1}}_{\tau,k_\tau}(w) < w$), and in the presence of the risk: 1) if utility is quadratic, there is some future constraint such that the probability that it binds depends on the value of $\xi_{\tau+1}$ that is realized; 2) if utility is exponential, there is a positive probability that some future constraint will bind; 3) if* *utility is CRRA, there is a positive probability that some future constraint will bind, or the initial situation is one in which there is some nondegenerate future risk.*

The theorem was established by the foregoing arguments.

It seems to us that a fair summary of this theorem is that in most circumstances the presence of future constraints and risks does increase the amount of precautionary saving induced by the introduction of a given new risk. The primary circumstance under which this should not be expected is for levels of wealth at which the consumer was severely constrained even in the absence of the new risk. There is no guararantee that the new risk will produce a sufficiently intense precautionary saving motive to move the initially-constrained consumer off his constraint. If it does, the effect will be precautionary, but it is possible that no effect will occur.

# 7 Conclusion

The central message of this paper is that the effects of precautionary saving and liquidity constraints are very similar to each other, because both spring from the concavity of the consumption function. The paper provides an explanation of the apparently contradictory results that have emerged from simulation studies, which have sometimes seemed to indicate that constraints intensify precautionary saving motives, and sometimes have found constraints and precautionary behavior to be substitutes.

Our results may have important applications even beyond the traditional consumption/saving problem in which the results were derived. The precautionary-saving effect of liquidity constraints may apply in many circumstances where a decision-maker faces the possibility of future liquidity constraints which raise the marginal value of an extra dollar of cash. Thus, firms that are not currently liquidity constrained may engage in precautionary saving if they believe there is some risk that constraints may bind in the future. Governments that worry about whether they will always be able to borrow on international markets may engage in precautionary saving even in periods when they are unconstrained. The logic could even apply to central banks charged with the responsibility of maintaining stable exchange rate regimes; the possibility of a run on the currency might induce 'precautionary' holdings of international reserves that are larger than a risk-neutral central bank would hold. Of course, these are all ideas that have appeared, at least informally and sometimes formally, in the relevant literatures. But this paper provides a general logic which can be applied to clarify precisely when and why one should expect such effects to emerge.

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