The Method of Moderation

June 24, 2012

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risky world, a pessimist assumes the worst will happen. Someone who ignores risk altogether is mist. Consumption decisions are mathematically simple for both the pessimist and the optimist e both behave as if they live in a riskless world. A consumer who is a realist (that is, who wants ond optimally to risk) faces a much more difficult problem, but (under standard conditions) oose a level of spending somewhere between that of the pessimist and the optimist. We use this redefine the space in which the realist searches for optimal consumption rules. The resulting n accurately represents the numerical consumption rule over the entire interval of feasible wealth with remarkably few computations.

Keywords Dynamic Stochastic Optimization
EL codes FillInLater

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, a consumption, investment, portfolio choice, or similar intertemporal optimization problem umerical methods generally requires the modeler to choose how to represent a policy or value n. In the stochastic case, where analytical solutions are generally not available, a common ch is to use low-order polynominal splines that exactly match the function (and maybe some ives) at a finite set of gridpoints, and then to assume that interpolated or extrapolated versions matching polynomial are a good representation elsewhere.

paper argues that, at least in the context of a standard consumption problem, a better approach able, which relies upon the fact that in the absence of uncertainty, the optimal consumption n has a simple analytical solution. The key insight is that, under standard assumptions, the ner who faces an uninsurable labor income risk will consume less than a consumer with the same r expected income but who does not perceive any uncertainty as being attached to that future . The 'realistic' consumer who *does perceive* the risks will engage in 'precautionary saving,' perfect foresight riskless solution provides an upper bound to the solution that will actually mal. A lower bound is provided by the behavior of a consumer who has the subjective belief e future level of income will be the worst that it can possibly be. This consumer, too, behaves ng to the analytical perfect foresight solution, but his certainty is that of a pessimist who is ely overconfident in his pessimism.

g results from Carroll (2011b), we show how to use these upper and lower bounds to tightly in the shape and characteristics of the solution to the 'realist's problem (that is, the solution to blem of a consumer who correctly perceives the risks to future income and behaves rationally in se).

r showing how to use the method in the baseline case, we show how refine the method to pass an even tighter theoretical bound, and how to extend it to solve a problem in which the ner faces both labor income risk and rate-of-return risk.

ne Realist's Problem

ume that the true reality facing the consumer is captured by the problem

$$\max\ \mathbb{E}_t\left[\sum_{n=0}^{T-t}\beta^n \mathrm{u}(\boldsymbol{c}_{t+n})\right], \tag{1}$$

t to

$$\begin{aligned}
\boldsymbol{a}_t &= \boldsymbol{m}_t - \boldsymbol{c}_t && (2)\\
\boldsymbol{b}_{t+1} &= \boldsymbol{a}_t \mathsf{R}_{t+1} && (3)\\
\boldsymbol{y}_{t+1} &= \boldsymbol{p}_{t+1}\theta_{t+1} && (4)\\
\boldsymbol{m}_{t+1} &= \boldsymbol{b}_{t+1} + \boldsymbol{y}_{t+1} && (5)
\end{aligned}$$

β	–	pure time discount factor
$\boldsymbol{a}_t$	–	assets after all actions have been accomplished in period t
$\boldsymbol{b}_{t+1}$	–	'bank balances' (nonhuman wealth) at the beginning of $t+1$
$\boldsymbol{c}_t$	–	consumption in period t
$\boldsymbol{m}_t$	–	'market resources' available for consumption ('cash-on-hand')
$\boldsymbol{p}_{t+1}$	–	'permanent labor income' in period $t+1$
R_{t+1}	–	interest factor $(1+\mathsf{r}_{t+1})$ from period t to $t+1$
$\boldsymbol{y}_{t+1}$	–	noncapital income in period $t+1$.

$$\begin{aligned} \boldsymbol{p}_{t+1} &= \Gamma_{t+1}\boldsymbol{p}_t && \text{- permanent labor income dynamics} \\ \log\ \theta_{t+n} &\sim \mathcal{N}(-\sigma_\theta^2/2, \sigma_\theta^2) && \text{- lognormal transitory shocks } \forall\ n > 0. \end{aligned} \tag{6}$$

rns out (see Carroll (2011a) for a proof) that this problem can be rewritten in a more convenient which choice and state variables are normalized by the level of permanent income, e.g., using d font for normalized variables, $m_t = \boldsymbol{c}_t/\boldsymbol{p}_t$. When that is done, the transformed version of the ner's problem is

$$\begin{aligned} \mathrm{v}_t(m_t) &= \max_{c_t}\ \mathrm{u}(c_t) + \mathbb{E}_t[\beta\Gamma_{t+1}^{1-\rho}\mathrm{v}_{t+1}(m_{t+1})] \\ &\text{s.t.} \\ a_t &= m_t - c_t \\ m_{t+1} &= \underbrace{(\mathsf{R}/\Gamma_{t+1})}_{\equiv\mathcal{R}_{t+1}} a_t + \theta_{t+1} \end{aligned} \tag{7}$$

ecause we have not imposed a liquidity constraint, the solution satisfies the Euler equation

$$\mathrm{u}'(c_t) = \mathbb{E}_t[\beta\mathsf{R}\Gamma_{t+1}^{-\rho}\mathrm{u}'(c_{t+1})]. \tag{8}$$

he remainder of the paper we will assume that permanent income $\boldsymbol{p}_t$ grows by a constant factor is not subject to stochastic shocks. (The generalizationto the case of permanent shocks is tforward.)

enchmark: The Method of Endogenous Gridpoints

nparison to our new solution method, we use the endogenous gridpoints solution to the microe- c problem presented in Carroll (2006). That method computes the level of consumption at a set points for market resources m that are determined endogenously using the Euler equation. The nption function is then constructed by linear interpolation among the gridpoints thus found.

oll (2011a) describes a specific calibration of the model and constructs a solution using five nts chosen to capture the structure of the consumption function reasonably well at values of m e infinite-horizon target value. (See those notes for details).

rtunately, the endogenous gridpoints solution is not very well-behaved outside the original range points targeted by the solution method. (Though other common solution methods are no better their own predefined ranges). Figure 1 demonstrates the point by plotting the amount of tionary saving implied by a linear extrapolation of our approximated consumption rule (the nption of the perfect foresight consumer $\bar{\mathrm{c}}_{T-1}$ minus our approximation to optimal consumption uncertainty, $\grave{\mathrm{c}}_{T-1}$). Although theory proves that precautionary saving is always positive, the extrapolated numerical approximation eventually predicts negative precautionary saving (at nt in the figure where the extrapolated locus crosses the horizontal axis).

error cannot be fixed by extending the upper gridpoint; in the presence of serious uncertainty, nsumption rule will need to be evaluated outside of *any* prespecified grid (because starting ne top gridpoint, a large enough realization of the uncertain variable will push next period's ion of assets above that top; a similar argument applies below the bottom gridpoint). While a us extrapolation technique can prevent this problem from being fatal (for example by carefully ng negative precautionary saving), the problem is often dealt with using inelegant methods whose tions for the accuracy of the solution are difficult to gauge.

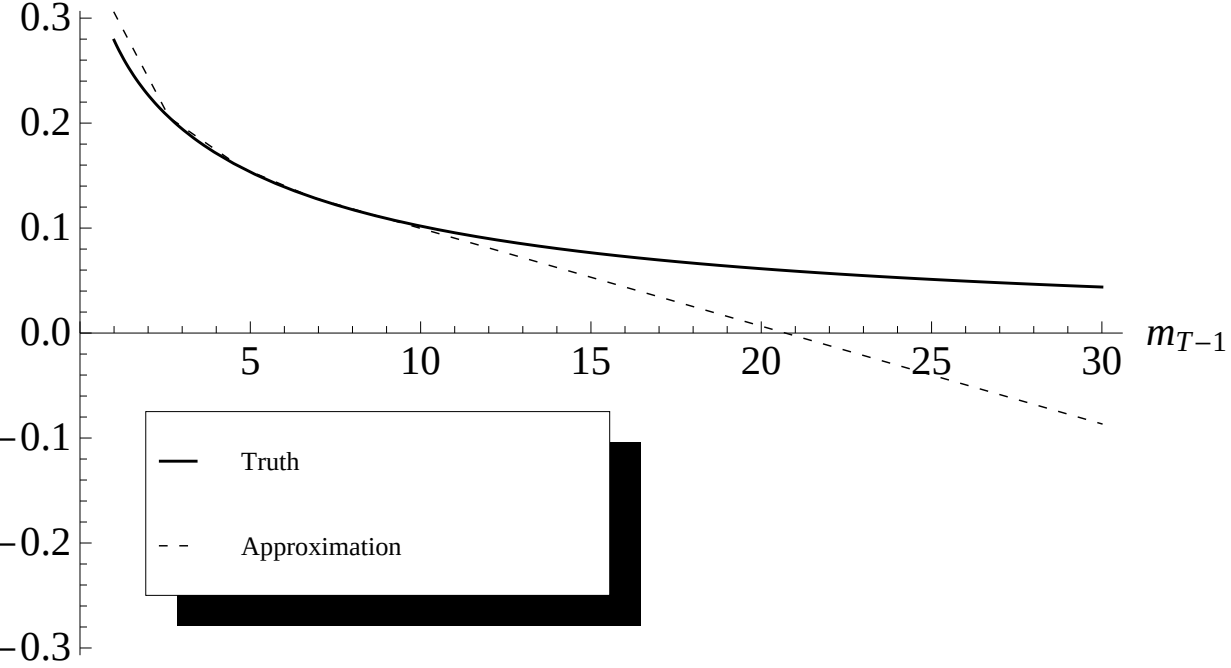


Figure 1 For Large Enough m_{T-1}, Predicted Precautionary Saving is Negative (Oops!)

he Method of Moderation

he Optimist, the Pessimist, and the Realist

The Consumption Function

reliminary to our solution, define $\mathfrak{h}_t$ as end-of-period human wealth (the present discounted value re labor income) for a perfect foresight version of the problem of a 'risk optimist:' a consumer who s with perfect confidence that the shocks will always take the value 1, $\theta_{t+n} = \mathbb{E}[\theta] = 1 \; \forall \; n > 0$. lution to a perfect foresight problem of this kind takes the form[1]

$$\bar{\mathrm{c}}_t(m_t) \;=\; (m_t + \mathfrak{h}_t)\underline{\kappa}_t \tag{9}$$

onstant minimal marginal propensity to consume $\underline{\kappa}_t$ given below. We similarly define $\underline{\mathfrak{h}}_t$ as al human wealth,' the present discounted value of labor income if the shocks were to take on orst possible value in every future period $\theta_{t+n} = \underline{\theta} \; \forall \; n > 0$ (which we define as corresponding to iefs of a 'pessimist').

st useful point is that, for the realist, a lower bound for the level of market resources is $\underline{m}_t = -\underline{\mathfrak{h}}_t$, e if m_t equalled this value then there would be a positive finite chance (however small) of receiving $\underline{\theta}$ in every future period, which would require the consumer to set c_t to zero in order to guarantee e intertemporal budget constraint holds. Since consumption of zero yields negative infinite utility, ution to realist consumer's problem is not well defined for values of $m_t < \underline{m}_t$, and the limiting f the realist's c_t is zero as $m_t \downarrow \underline{m}_t$.

n this result, it will be convenient to define 'excess' market resources as the amount by which resources exceed the lower bound, and 'excess' human wealth as the amount by which mean ed human wealth exceeds guaranteed minimum human wealth:

$$\begin{aligned} \blacktriangle m_t &= m_t + \overbrace{\underline{\mathfrak{h}}_t}^{=-\underline{m}_t} \\ \blacktriangle \mathfrak{h}_t &= \mathfrak{h}_t - \underline{\mathfrak{h}}_t. \end{aligned}$$

a derivation, see Carroll (2011b); $\underline{\kappa}_t$ is defined therein as the MPC of the perfect foresight consumer with horizon $T - t$.

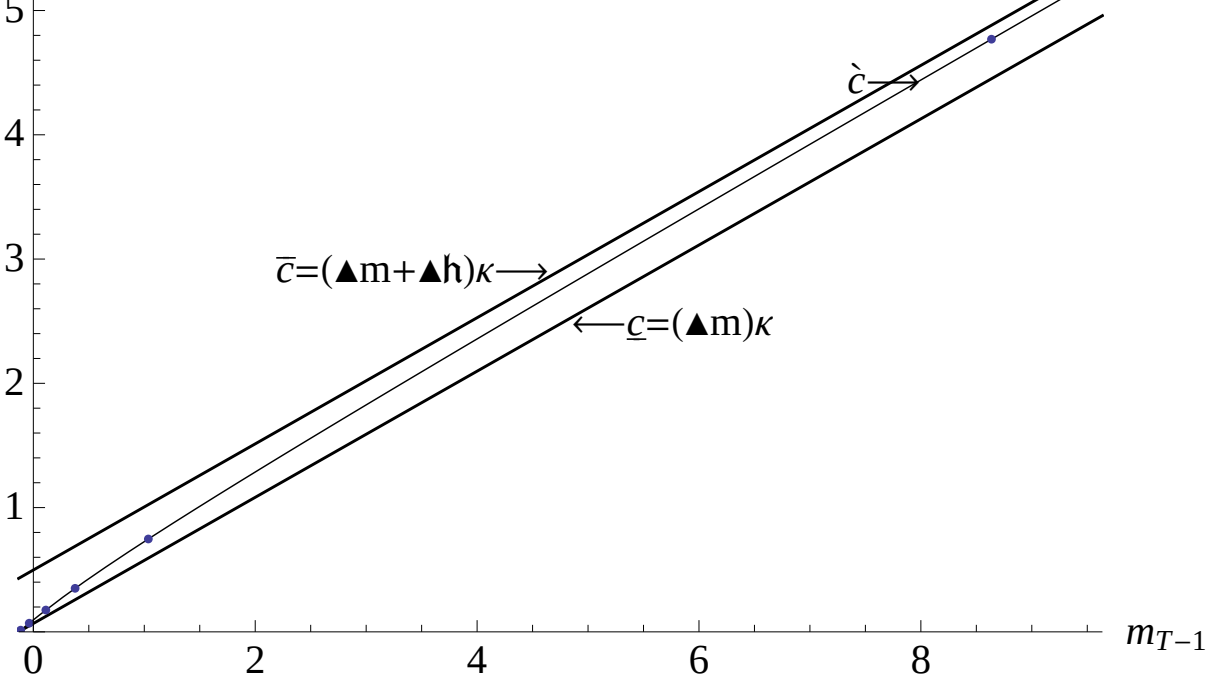


Figure 2 Moderation Illustrated: $\underline{\mathrm{c}}_{T-1} < \grave{\mathrm{c}}_{T-1} < \bar{\mathrm{c}}_{T-1}$

an now transparently define the optimal consumption rules for the two perfect foresight problems, f the 'optimist' and the 'pessimist.' The 'pessimist' perceives human wealth to be equal to its um feasible value $\underline{\mathfrak{h}}_t$ with certainty, so consumption is given by the perfect foresight solution

$$\begin{aligned}\underline{\mathrm{c}}_t(m_t) &= (m_t + \underline{\mathfrak{h}}_t)\underline{\kappa}_t \\ &= \blacktriangle m_t \underline{\kappa}_t.\end{aligned}$$

'optimist,' on the other hand, pretends that there is no uncertainty about future income, and re consumes

$$\begin{aligned}\bar{\mathrm{c}}_t(m_t) &= (m_t + \underline{\mathfrak{h}}_t - \underline{\mathfrak{h}}_t + \mathfrak{h}_t)\underline{\kappa}_t \\ &= (\blacktriangle m_t + \blacktriangle \mathfrak{h}_t)\underline{\kappa}_t \\ &= \underline{\mathrm{c}}_t(m_t) + \blacktriangle \mathfrak{h}_t \underline{\kappa}_t.\end{aligned}$$

ems obvious that the spending of the realist will be strictly greater than that of the pessimist ictly less than that of the optimist. Figure 2 illustrates the proposition for the consumption rule od $T-1$.

f is more difficult than might be imagined, but the necessary work is done in Carroll (2011b) so take the proposition a fact and proceed by manipulating the inequality:

$$\begin{array}{rcl}\blacktriangle m_t \underline{\kappa}_t < & \mathrm{c}_t(\underline{m}_t + \blacktriangle m_t) & < (\blacktriangle m_t + \blacktriangle \mathfrak{h}_t)\underline{\kappa}_t \\ -\blacktriangle m_t \underline{\kappa}_t > & -\mathrm{c}_t(\underline{m}_t + \blacktriangle m_t) & > -(\blacktriangle m_t + \blacktriangle \mathfrak{h}_t)\underline{\kappa}_t \\ \blacktriangle \mathfrak{h}_t \underline{\kappa}_t > & \bar{\mathrm{c}}_t(\underline{m}_t + \blacktriangle m_t) - \mathrm{c}_t(\underline{m}_t + \blacktriangle m_t) & > 0 \\ 1 > & \underbrace{\left(\dfrac{\bar{\mathrm{c}}_t(\underline{m}_t + \blacktriangle m_t) - \mathrm{c}_t(\underline{m}_t + \blacktriangle m_t)}{\blacktriangle \mathfrak{h}_t \underline{\kappa}_t}\right)}_{\equiv \hat{\gamma}_t} & > 0\end{array}$$

the fraction in the middle of the last inequality is the ratio of actual precautionary saving (the ator is the difference between perfect-foresight consumption and optimal consumption in the ce of uncertainty) to the maximum conceivable amount of precautionary saving (the amount that be undertaken by the pessimist who consumes nothing out of any future income beyond the ly certain component). Defining $\mu_t = \log \blacktriangle m_t$ (which can range from $-\infty$ to ∞), the object in

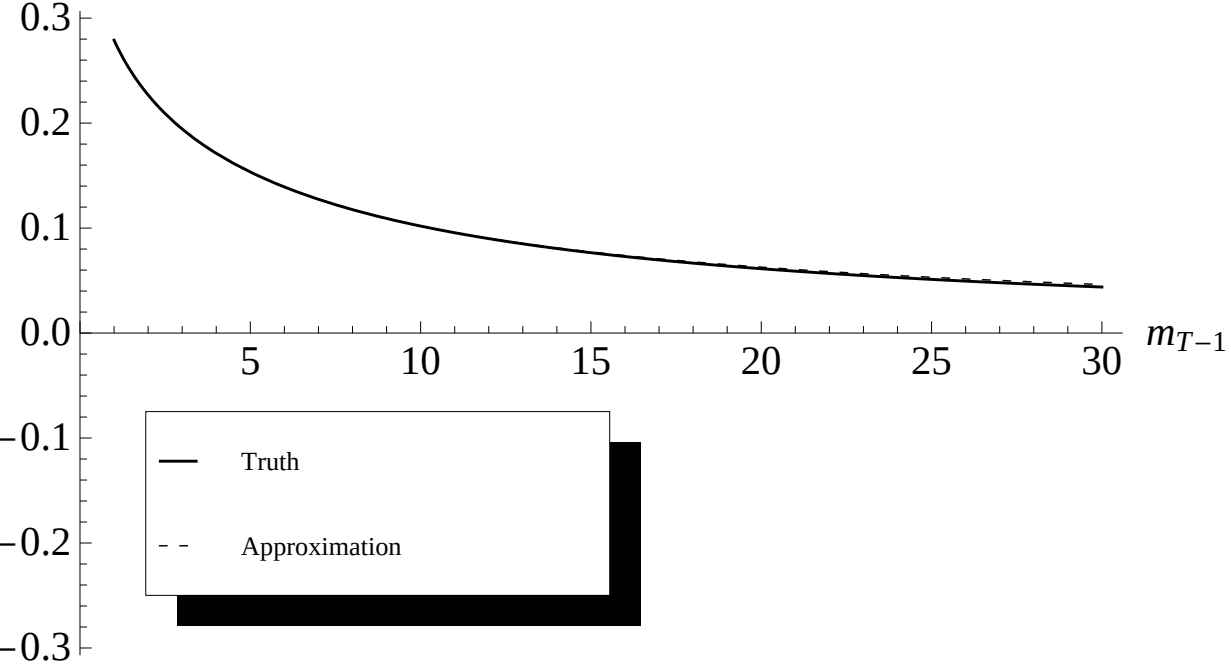


Figure 3 Extrapolated $\grave{\hat{\mathrm{c}}}_{T-1}$ Constructed Using the Method of Moderation

ddle of the last inequality is

$$\hat{\mathrm{o}}_t(\mu_t) \equiv \left(\frac{\bar{\mathrm{c}}_t(\underline{m}_t + e^{\mu_t}) - \mathrm{c}_t(\underline{m}_t + e^{\mu_t})}{\blacktriangle\mathfrak{h}_t\underline{\kappa}_t}\right), \tag{10}$$

now define

$$\hat{\boldsymbol{\chi}}_t(\mu_t) = \log\left(\frac{1-\hat{\mathrm{o}}_t(\mu_t)}{\hat{\mathrm{o}}_t(\mu_t)}\right) \tag{11}$$

$$= \log\left(1/\hat{\mathrm{o}}_t(\mu_t) - 1\right) \tag{12}$$

has the virtue that it is linear in the limit as μ_t approaches $+\infty$.
n $\hat{\boldsymbol{\chi}}$, the consumption function can be recovered from

$$\hat{\mathrm{c}}_t = \bar{\mathrm{c}}_t - \overbrace{\left(\frac{1}{1+\exp(\hat{\boldsymbol{\chi}}_t)}\right)}^{=\hat{\mathrm{o}}_t}\blacktriangle\mathfrak{h}_t\underline{\kappa}_t. \tag{13}$$

s, the procedure is to calculate $\hat{\boldsymbol{\chi}}_t$ at the points $\vec{\mu}_t$ corresponding to the log of the $\blacktriangle\vec{m}_t$ points above, and then using these to construct an interpolating approximation $\grave{\hat{\boldsymbol{\chi}}}_t$ from which we tly obtain our approximated consumption rule $\grave{\hat{\mathrm{c}}}_t$ by substituting $\grave{\hat{\boldsymbol{\chi}}}_t$ for $\hat{\boldsymbol{\chi}}$ in equation (13).
use this method relies upon the fact that the problem is easy to solve if the decision maker reasonable views (either in the optimistic or the pessimistic direction), and because the correct n is always between these immoderate extremes, we call our solution procedure the 'method of ation.'
lts are shown in Figure 3; a reader with very good eyesight might be able to detect the barest a discrepancy between the Truth and the Approximation at the far righthand edge of the figure.

ately, many of the tricks used when solving the consumption problem have a direct analogue in imation of the value function.

sider the perfect foresight (or 'optimist') case in period $T-1$:

$$\begin{aligned}
\mathrm{v}_{T-1}(m_{T-1}) &= \mathrm{u}(c_{T-1}) + \beta \mathrm{u}(c_T) \\
&= \mathrm{u}(c_{T-1})\left(1 + \beta((\beta_T \mathsf{R})^{1/\rho})^{1-\rho}\right) \\
&= \mathrm{u}(c_{T-1})\left(1 + \beta(\beta_T \mathsf{R})^{1/\rho-1}\right) \\
&= \mathrm{u}(c_{T-1})\left(1 + (\beta_T \mathsf{R})^{1/\rho}/\mathsf{R}\right) \\
&= \mathrm{u}(c_{T-1})\underbrace{\mathrm{PDV}_t^T(c)/c_{T-1}}_{\equiv \mathbb{C}_t^T}
\end{aligned}$$

$\mathrm{PDV}_t^T(c)$ is the present discounted value of consumption. A similar function can be constructed vely for earlier periods, yielding the general expression

$$\mathrm{v}_t(m_t) = \mathrm{u}(c_t)\mathbb{C}_t^T \tag{14}$$

can be transformed as

$$\begin{aligned}
\Lambda_t &\equiv ((1-\rho)\mathrm{v}_t)^{1/(1-\rho)} \\
&= c_t(\mathbb{C}_t^T)^{1/(1-\rho)}
\end{aligned}$$

ince $\mathbb{C}_t^T$ is a constant while the consumption function is linear, $\bar{\Lambda}_t$ will also be linear.

pply the same transformation to the value function for the problem with uncertainty (the realist's n):

$$\Lambda_t = ((1-\rho)\mathrm{v}_t(m_t))^{1/(1-\rho)}$$

excellent approximation to the value function can be obtained by calculating the values of Λ at ne gridpoints used by the consumption function approximation, and interpolating among those

ever, as with the consumption approximation, we can do even better if we realize that the $\bar{\Lambda}$ n for the optimist's problem is an upper bound for the Λ function in the presence of uncertainty, e value function for the pessimist is a lower bound. Analogously to (10), define an upper-case

$$\hat{\mathrm{Q}}_t(\mu_t) = \left(\frac{\bar{\Lambda}_t(\underline{m}_t + e^{\mu_t}) - \Lambda_t(\underline{m}_t + e^{\mu_t})}{\blacktriangle \mathfrak{h}_t \underline{\kappa}_t (\mathbb{C}_t^T)^{1/(1-\rho)}}\right) \tag{15}$$

n upper-case version of the $\boldsymbol{\chi}$ equation in (12):

$$\begin{aligned}
\hat{\mathrm{X}}_t(\mu_t) &= \log\left(\frac{1-\hat{\mathrm{Q}}_t(\mu_t)}{\hat{\mathrm{Q}}_t(\mu_t)}\right) && (16) \\
&= \log\left(1/\hat{\mathrm{Q}}_t(\mu_t) - 1\right) && (17)
\end{aligned}$$

we approximate these objects then invert them (as above with the $\hat{\mathsf{o}}$ and $\hat{\boldsymbol{\chi}}$ functions) we obtain high-quality approximation to our inverted value function at the same points for which we have proximated value function:

$$\hat{\Lambda}_t = \bar{\Lambda}_t - \overbrace{\left(\frac{1}{1+\exp(\hat{\mathrm{X}}_t)}\right)}^{=\hat{\mathrm{Q}}_t} \blacktriangle \mathfrak{h}_t \underline{\kappa}_t (\mathbb{C}_t^T)^{1/(1-\rho)} \tag{18}$$

hich we obtain our approximation to the value function as

$$\begin{aligned}
\hat{\mathrm{v}}_t &= \mathrm{u}(\hat{\Lambda}_t) \\
\hat{\mathrm{v}}_t^m &= \mathrm{u}'(\hat{\Lambda}_t)\hat{\Lambda}^m.
\end{aligned}$$

rable virtue that the v_t derived from it numerically satisfies the envelope theorem at each of the nts for which the problem has been solved.

xtensions

Tighter Upper Bound

(2011b) derives an upper limit $\bar{\kappa}_t$ for the MPC as m_t approaches its lower bound. Using this us the strict concavity of the consumption function yields the proposition that

$$\mathrm{c}_t(\underline{m}_t + \blacktriangle m_t) \quad < \bar{\kappa}_t \blacktriangle m_t. \tag{19}$$

solution method described above does not guarantee that approximated consumption will respect nstraint between gridpoints, and a failure to respect the constraint can occasionally cause tational problems in solving or simulating the model. Here, we describe a method for constructing roximation that always satisfies the constraint.

ning $m_t^{\#}$ as the 'cusp' point where the two upper bounds intersect:

$$\begin{aligned}
\left(\blacktriangle m_t^{\#} + \blacktriangle \mathfrak{h}_t\right) \underline{\kappa}_t &= \bar{\kappa}_t \blacktriangle m_t^{\#} \\
\blacktriangle m_t^{\#} &= \frac{\underline{\kappa}_t \blacktriangle \mathfrak{h}_t}{(1-\underline{\kappa}_t)\bar{\kappa}_t} \\
m_t^{\#} &= \frac{\underline{\kappa}_t \mathfrak{h}_t - \underline{\mathfrak{h}}_t}{(1-\underline{\kappa}_t)\bar{\kappa}_t},
\end{aligned}$$

nt to construct a consumption function for $m_t \in (\underline{m}_t, m_t^{\#}]$ that respects the tighter upper bound:

$$\begin{array}{rcl}
\blacktriangle m_t \underline{\kappa}_t < & \mathrm{c}_t(\underline{m}_t + \blacktriangle m_t) & < \bar{\kappa}_t \blacktriangle m_t \\
\blacktriangle m_t(\bar{\kappa}_t - \underline{\kappa}_t) > & \bar{\kappa}_t \blacktriangle m_t - \mathrm{c}_t(\underline{m}_t + \blacktriangle m_t) & > 0 \\
1 > & \left(\frac{\bar{\kappa}_t \blacktriangle m_t - \mathrm{c}_t(\underline{m}_t + \blacktriangle m_t)}{\blacktriangle m_t(\bar{\kappa}_t - \underline{\kappa}_t)}\right) & > 0.
\end{array}$$

defining $\mu_t = \log \blacktriangle m_t$, the object in the middle of the inequality is

$$\check{\mathrm{c}}_t(\mu_t) \quad \equiv \quad \frac{\bar{\kappa}_t - \mathrm{c}_t(\underline{m}_t + e^{\mu_t})e^{-\mu_t}}{\bar{\kappa}_t - \underline{\kappa}_t}.$$

approaches $-\underline{m}_t$, $\check{\mathrm{c}}_t(\mu_t)$ converges to zero, while as m_t approaches $+\infty$, $\check{\mathrm{c}}_t(\mu_t)$ approaches 1.

efore, we can derive an approximated consumption function; call it $\check{\mathrm{c}}_t$. This function will do a better job approximating the consumption function for low values of m_t while the previous imation will perform better for high values of m_t.

middling values of m it is not clear which of these functions will perform better. However, an tive is available which performs well. Define the highest gridpoint below $m_t^{\#}$ as $\bar{\check{m}}_t^{\#}$ and the gridpoint above $m_t^{\#}$ as $\underline{\hat{m}}_t^{\#}$. Then there will be a unique interpolating polynomial that matches el and slope of the consumption function at these two points. Call this function $\tilde{\mathrm{c}}_t(m)$.

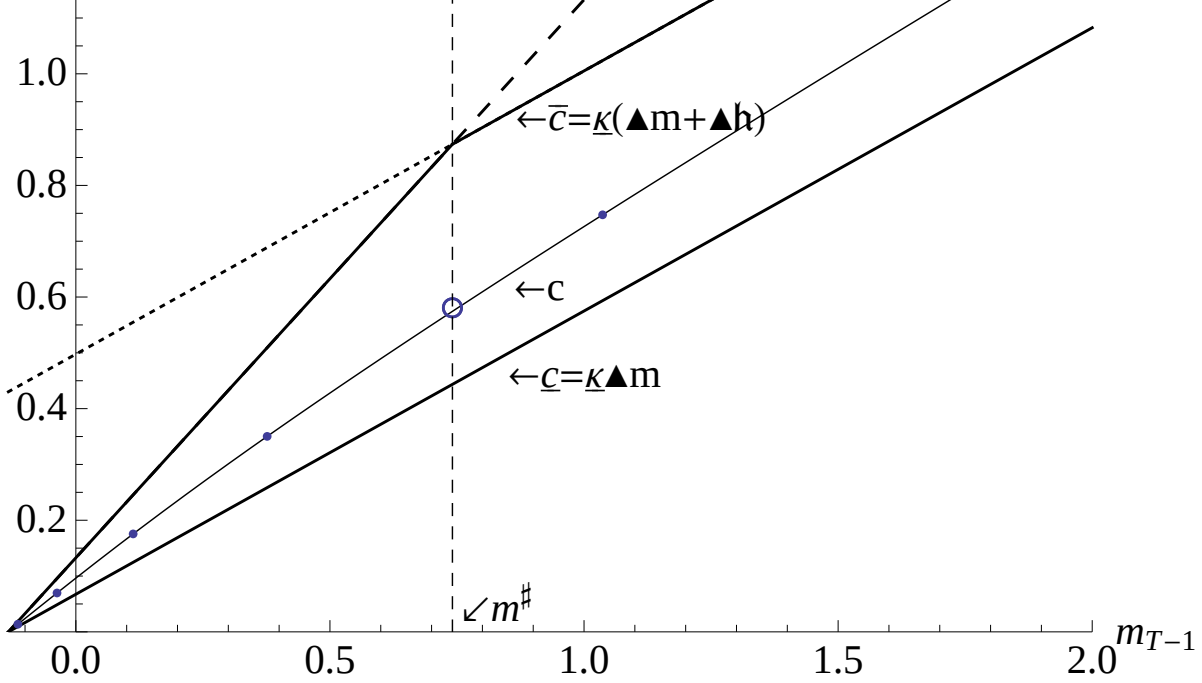
g indicator functions that are zero everywhere except for specified intervals,

$$\begin{aligned}
\mathbf{1}_{\mathrm{Lo}}(m) &= 1 \text{ if } m \leq \bar{\check{m}}_t^{\#} \\
\mathbf{1}_{\mathrm{Mid}}(m) &= 1 \text{ if } \quad \bar{\check{m}}_t^{\#} < m < \underline{\hat{m}}_t^{\#} \\
\mathbf{1}_{\mathrm{Hi}}(m) &= 1 \text{ if } \quad \underline{\hat{m}}_t^{\#} \leq m
\end{aligned}$$

define a well-behaved approximating consumption function

$$\grave{\mathrm{c}}_t \quad = \quad \mathbf{1}_{\mathrm{Lo}}\grave{\check{\mathrm{c}}}_t + \mathbf{1}_{\mathrm{Mid}}\grave{\tilde{\mathrm{c}}}_t + \mathbf{1}_{\mathrm{Hi}}\grave{\hat{\mathrm{c}}}_t. \tag{20}$$

ust says that, for each interval, we use the approximation that is most appropriate. The function inuous and once-differentiable everywhere, and is therefore well behaved for computational es.



onstruct the corresponding refined representation of the value function we must first clarify one The upper-bound value function that we are constructing will be the one implied by a consumer spending behavior is consistent with the refined upper-bound consumption rule.

$m_t \geq m_t^{\#}$, this consumption rule is the same as before, so the constructed upper-bound value n is also the same. However, for values $m_t < m_t^{\#}$ matters are slightly more complicated.

with the fact that at the cusp point,

$$\begin{aligned} \bar{\mathrm{v}}_t(m_t^{\#}) &= \mathrm{u}(\bar{c}_t(m_t^{\#}))\mathbb{C}_t^T \\ &= \mathrm{u}(\blacktriangle m_t^{\#}\bar{\kappa}_t)\mathbb{C}_t^T. \end{aligned}$$

for *all* m_t,

$$\bar{\mathrm{v}}_t(m) = \mathrm{u}(\bar{c}_t(m)) + \bar{\mathfrak{v}}_t(m - \bar{c}_t(m)),$$

e assume that for the consumer below the cusp point consumption is given by $\bar{\kappa}\blacktriangle m_t$ so for $m_t^{\#}$

$$\bar{\mathrm{v}}_t(m) = \mathrm{u}(\bar{\kappa}_t\blacktriangle m) + \bar{\mathfrak{v}}_t((1-\bar{\kappa}_t)\blacktriangle m),$$

is easy to compute because $\mathfrak{v}_t(a_t) = \beta\bar{\mathrm{v}}_{t+1}(a_t\mathcal{R}+1)$ where $\bar{\mathrm{v}}_t$ is as defined above because a ner who ends the current period with assets exceeding the lower bound will not expect to be ained next period. (Recall again that we are merely constructing an object that is guaranteed to *upper bound* for the value that the 'realist' consumer will experience.) At the gridpoints defined solution of the consumption problem can then construct

$$\bar{\Lambda}_t(m) = ((1-\rho)\bar{\mathrm{v}}_t(m))^{1/(1-\rho)}$$

yields the appropriate vector for constructing $\check{\mathrm{X}}$ and $\check{\Upsilon}$. The rest of the procedure is analogous performed for the consumption rule and is thus omitted for brevity.

ochastic Rate of Return

ar we have assumed that the interest factor is constant at R. Extending the previous derivations w for a perfectly forecastable time-varying interest factor R_t would be trivial. Allowing for a stic interest factor is less trivial.

se in this case Merton (1969) and Samuelson (1969) showed that for a consumer without labor (or with perfectly forecastable labor income) the consumption function is linear, with an MPC[2]

$$\kappa = 1 - \left(\beta\, \mathbb{E}_t[\mathbf{R}_{t+1}^{1-\rho}]\right)^{1/\rho} \tag{22}$$

this case the previous analysis applies once we substitute this MPC for the one that characterizes rfect foresight problem without rate-of-return risk. The more realistic case where the interest has some serial correlation is more complex. We consider the simplest case that captures the eatures of empirical interest rate dynamics: An AR(1) process. Thus the specification is

$$\mathbf{r}_{t+1} - \mathbf{r} = (\mathbf{r}_t - \mathbf{r})\gamma + \epsilon_{t+1} \tag{23}$$

r is the long-run mean log interest factor, $0 < \gamma < 1$ is the AR(1) serial correlation coefficient, $_{+1}$ is the stochastic shock.

consumer's problem in this case now has two state variables, m_t and $\mathbf{r}_t$, and is described by

$$\begin{aligned} \mathrm{v}_t(m_t, \mathbf{r}_t) &= \max_{c_t}\ \mathrm{u}(c_t) + \mathbb{E}_t[\beta_{t+1}\Gamma_{t+1}^{1-\rho}\mathrm{v}_{t+1}(m_{t+1}, \mathbf{r}_{t+1})] \\ &\text{s.t.} \\ a_t &= m_t - c_t \\ \mathbf{r}_{t+1} - \mathbf{r} &= (\mathbf{r}_t - \mathbf{r})\gamma + \epsilon_{t+1} \\ \mathbf{R}_{t+1} &= \exp(\mathbf{r}_{t+1}) \\ m_{t+1} &= \underbrace{(\mathbf{R}_{t+1}/\Gamma_{t+1})}_{\equiv \mathfrak{R}_{t+1}} a_t + \theta_{t+1}. \end{aligned} \tag{24}$$

pproximate the AR(1) process by a Markov transition matrix using standard techniques. The tic interest factor is allowed to take on 11 values centered around the steady-state value **r** osen [how?]. Given this Markov transition matrix, *conditional* on the Markov AR(1) state the ption functions for the 'optimist' and the 'pessimist' will still be linear, with identical MPC's e computed numerically. Given these MPC's, the (conditional) realist's consumption function computed for each Markov state, and the converged consumption rules constitute the solution ent on the dynamics of the stochastic interest rate process.

rinciple, this refinement should be combined with the previous one; further exposition of this ation is omitted here because no new insights spring from the combination of the two techniques.

onclusion

ethod proposed here is not universally applicable. For example, the method cannot be used for ns for which upper and lower bounds to the 'true' solution are not known. But many problems e obvious upper and lower bounds, and in those cases (as in the consumption example used in per), the method may result in substantial improvements in accuracy and stability of solutions.

CRRA-RateRisk for a derivation.

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