SED Version

The Method of Moderation

June 24, 2012

Christopher D. Carroll[1] — JHU
Kiichi Tokuoka[2] — ECB
Weifeng Wu[3] — Fannie Mae

Abstract

In a risky world, a pessimist assumes the worst will happen. Someone who ignores risk altogether is an optimist. Consumption decisions are mathematically simple for both the pessimist and the optimist because both behave as if they live in a riskless world. A consumer who is a realist (that is, who wants to respond optimally to risk) faces a much more difficult problem, but (under standard conditions) will choose a level of spending somewhere between that of the pessimist and the optimist. We use this fact to redefine the space in which the realist searches for optimal consumption rules. The resulting solution accurately represents the numerical consumption rule over the entire interval of feasible wealth values with remarkably few computations.

Keywords Dynamic Stochastic Optimization

JEL codes FillInLater

PDF: http://econ.jhu.edu/people/ccarroll/papers/ctwMoM.pdf
Slides: http://econ.jhu.edu/people/ccarroll/papers/ctwMoM-Slides.pdf
Web: http://econ.jhu.edu/people/ccarroll/papers/ctwMoM/
Archive: http://econ.jhu.edu/people/ccarroll/papers/ctwMoM.zip

[1]Carroll: Department of Economics, Johns Hopkins University, Baltimore, MD, http://econ.jhu.edu/people/ccarroll/, ccarroll@jhu.edu [2]Tokuoka: International Monetary Fund, Washington, DC, ktokuoka@imf.org [3]Wu: Weifeng Wu, Fanie Mae, Washington DC, .

The views presented in this paper are those of the authors, and should not be attributed to the International Monetary Fund, its Executive Board, or management, or to the European Central Bank.

1 Introduction

Solving a consumption, investment, portfolio choice, or similar intertemporal optimization problem using numerical methods generally requires the modeler to choose how to represent a policy or value function. In the stochastic case, where analytical solutions are generally not available, a common approach is to use low-order polynominal splines that exactly match the function (and maybe some derivatives) at a finite set of gridpoints, and then to assume that interpolated or extrapolated versions of the matching polynomial are a good representation elsewhere.

This paper argues that, at least in the context of a standard consumption problem, a better approach is available, which relies upon the fact that in the absence of uncertainty, the optimal consumption function has a simple analytical solution. The key insight is that, under standard assumptions, the consumer who faces an uninsurable labor income risk will consume less than a consumer with the same path for expected income but who does not perceive any uncertainty as being attached to that future income. The 'realistic' consumer who *does perceive* the risks will engage in 'precautionary saving,' so the perfect foresight riskless solution provides an upper bound to the solution that will actually be optimal. A lower bound is provided by the behavior of a consumer who has the subjective belief that the future level of income will be the worst that it can possibly be. This consumer, too, behaves according to the analytical perfect foresight solution, but his certainty is that of a pessimist who is extremely overconfident in his pessimism.

Using results from Carroll (2011b), we show how to use these upper and lower bounds to tightly constrain the shape and characteristics of the solution to the 'realist's problem (that is, the solution to the problem of a consumer who correctly perceives the risks to future income and behaves rationally in response).

After showing how to use the method in the baseline case, we show how refine the method to encompass an even tighter theoretical bound, and how to extend it to solve a problem in which the consumer faces both labor income risk and rate-of-return risk.

2 The Realist's Problem

We assume that the true reality facing the consumer is captured by the problem

$$\max \ \mathbb{E}_t \left[\sum_{n=0}^{T-t} \beta^n \mathrm{u}(\boldsymbol{c}_{t+n}) \right], \tag{1}$$

subject to

$$\begin{aligned} \boldsymbol{a}_t &= \boldsymbol{m}_t - \boldsymbol{c}_t && (2) \\ \boldsymbol{b}_{t+1} &= \boldsymbol{a}_t \mathsf{R}_{t+1} && (3) \\ \boldsymbol{y}_{t+1} &= \boldsymbol{p}_{t+1} \theta_{t+1} && (4) \\ \boldsymbol{m}_{t+1} &= \boldsymbol{b}_{t+1} + \boldsymbol{y}_{t+1} && (5) \end{aligned}$$

where

β – pure time discount factor

$$
\begin{aligned}
\boldsymbol{a}_t &- \text{assets after all actions have been accomplished in period } t \\
\boldsymbol{b}_{t+1} &- \text{'bank balances' (nonhuman wealth) at the beginning of } t+1 \\
\boldsymbol{c}_t &- \text{consumption in period } t \\
\boldsymbol{m}_t &- \text{'market resources' available for consumption ('cash-on-hand')} \\
\boldsymbol{p}_{t+1} &- \text{'permanent labor income' in period } t+1 \\
\mathsf{R}_{t+1} &- \text{interest factor } (1+\mathsf{r}_{t+1}) \text{ from period } t \text{ to } t+1 \\
\boldsymbol{y}_{t+1} &- \text{noncapital income in period } t+1.
\end{aligned}
$$

and the exogenous variables evolve according to

$$
\begin{aligned}
\mathsf{R}_t &= \mathsf{R} && \text{- constant interest factor} = 1+\mathsf{r} \\
\boldsymbol{p}_{t+1} &= \Gamma_{t+1}\boldsymbol{p}_t && \text{- permanent labor income dynamics} \\
\log\ \theta_{t+n} &\sim \mathcal{N}(-\sigma_\theta^2/2, \sigma_\theta^2) && \text{- lognormal transitory shocks } \forall\ n>0.
\end{aligned} \tag{6}
$$

It turns out (see Carroll (2011a) for a proof) that this problem can be rewritten in a more convenient form in which choice and state variables are normalized by the level of permanent income, e.g., using nonbold font for normalized variables, $m_t = \boldsymbol{c}_t/\boldsymbol{p}_t$. When that is done, the transformed version of the consumer's problem is

$$
\begin{aligned}
\mathrm{v}_t(m_t) &= \max_{c_t}\ \mathrm{u}(c_t) + \mathbb{E}_t[\beta\Gamma_{t+1}^{1-\rho}\mathrm{v}_{t+1}(m_{t+1})] \\
&\text{s.t.} \\
a_t &= m_t - c_t \\
m_{t+1} &= \underbrace{(\mathsf{R}/\Gamma_{t+1})}_{\equiv\mathcal{R}_{t+1}} a_t + \theta_{t+1}
\end{aligned} \tag{7}
$$

and because we have not imposed a liquidity constraint, the solution satisfies the Euler equation

$$
\mathrm{u}'(c_t) = \mathbb{E}_t[\beta\mathsf{R}\Gamma_{t+1}^{-\rho}\mathrm{u}'(c_{t+1})]. \tag{8}
$$

For the remainder of the paper we will assume that permanent income $\boldsymbol{p}_t$ grows by a constant factor Γ and is not subject to stochastic shocks. (The generalizationto the case of permanent shocks is straightforward.)

3 Benchmark: The Method of Endogenous Gridpoints

For comparison to our new solution method, we use the endogenous gridpoints solution to the microeconomic problem presented in Carroll (2006). That method computes the level of consumption at a set of gridpoints for market resources m that are determined endogenously using the Euler equation. The consumption function is then constructed by linear interpolation among the gridpoints thus found.

Carroll (2011a) describes a specific calibration of the model and constructs a solution using five gridpoints chosen to capture the structure of the consumption function reasonably well at values of m near the infinite-horizon target value. (See those notes for details).

Unfortunately, the endogenous gridpoints solution is not very well-behaved outside the original range of gridpoints targeted by the solution method. (Though other common solution methods are no better outside their own predefined ranges). Figure 1 demonstrates the point by plotting the amount of precautionary saving implied by a linear extrapolation of our approximated consumption rule (the consumption of the perfect foresight consumer $\bar{\mathrm{c}}_{T-1}$ minus our approximation to optimal consumption under uncertainty, $\grave{\mathrm{c}}_{T-1}$). Although theory proves that precautionary saving is always positive, the linearly extrapolated numerical approximation eventually predicts negative precautionary saving (at the point in the figure where the extrapolated locus crosses the horizontal axis).

This error cannot be fixed by extending the upper gridpoint; in the presence of serious uncertainty, the consumption rule will need to be evaluated outside of *any* prespecified grid (because starting from the top gridpoint, a large enough realization of the uncertain variable will push next period's realization of assets above that top; a similar argument applies below the bottom gridpoint). While a judicious extrapolation technique can prevent this problem from being fatal (for example by carefully excluding negative precautionary saving), the problem is often dealt with using inelegant methods whose implications for the accuracy of the solution are difficult to gauge.

4 The Method of Moderation

4.1 The Optimist, the Pessimist, and the Realist

4.1.1 The Consumption Function

As a preliminary to our solution, define $\mathfrak{h}_t$ as end-of-period human wealth (the present discounted value of future labor income) for a perfect foresight version of the problem of a 'risk optimist:' a consumer who believes with perfect confidence that the shocks will always take the value 1, $\theta_{t+n} = \mathbb{E}[\theta] = 1 \; \forall \; n > 0$. The solution to a perfect foresight problem of this kind takes the form[1]

$$\bar{\mathrm{c}}_t(m_t) \quad = \quad (m_t + \mathfrak{h}_t)\underline{\kappa}_t \tag{9}$$

for a constant minimal marginal propensity to consume $\underline{\kappa}_t$ given below. We similarly define $\underline{\mathfrak{h}}_t$ as 'minimal human wealth,' the present discounted value of labor income if the shocks were to take on their worst possible value in every future period $\theta_{t+n} = \underline{\theta} \; \forall \; n > 0$ (which we define as corresponding to the beliefs of a 'pessimist').

A first useful point is that, for the realist, a lower bound for the level of market resources is $\underline{m}_t = -\underline{\mathfrak{h}}_t$, because if m_t equalled this value then there would be a positive finite chance (however small) of receiving $\theta_{t+n} = \underline{\theta}$ in every future period,

[1] For a derivation, see Carroll (2011b); $\underline{\kappa}_t$ is defined therein as the MPC of the perfect foresight consumer with horizon $T - t$.

which would require the consumer to set c_t to zero in order to guarantee that the intertemporal budget constraint holds. Since consumption of zero yields negative infinite utility, the solution to realist consumer's problem is not well defined for values of $m_t < \underline{m}_t$, and the limiting value of the realist's c_t is zero as $m_t \downarrow \underline{m}_t$.

Given this result, it will be convenient to define 'excess' market resources as the amount by which actual resources exceed the lower bound, and 'excess' human wealth as the amount by which mean expected human wealth exceeds guaranteed minimum human wealth:

$$\begin{aligned} \blacktriangle m_t &= m_t + \overbrace{\underline{\mathfrak{h}}_t}^{=-\underline{m}_t} \\ \blacktriangle \mathfrak{h}_t &= \mathfrak{h}_t - \underline{\mathfrak{h}}_t. \end{aligned}$$

We can now transparently define the optimal consumption rules for the two perfect foresight problems, those of the 'optimist' and the 'pessimist.' The 'pessimist' perceives human wealth to be equal to its minimum feasible value $\underline{\mathfrak{h}}_t$ with certainty, so consumption is given by the perfect foresight solution

$$\begin{aligned} \underline{\mathrm{c}}_t(m_t) &= (m_t + \underline{\mathfrak{h}}_t)\underline{\kappa}_t \\ &= \blacktriangle m_t \underline{\kappa}_t. \end{aligned}$$

The 'optimist,' on the other hand, pretends that there is no uncertainty about future income, and therefore consumes

$$\begin{aligned} \bar{\mathrm{c}}_t(m_t) &= (m_t + \underline{\mathfrak{h}}_t - \underline{\mathfrak{h}}_t + \mathfrak{h}_t)\underline{\kappa}_t \\ &= (\blacktriangle m_t + \blacktriangle \mathfrak{h}_t)\underline{\kappa}_t \\ &= \underline{\mathrm{c}}_t(m_t) + \blacktriangle \mathfrak{h}_t \underline{\kappa}_t. \end{aligned}$$

It seems obvious that the spending of the realist will be strictly greater than that of the pessimist and strictly less than that of the optimist. Figure 2 illustrates the proposition for the consumption rule in period $T-1$.

Proof is more difficult than might be imagined, but the necessary work is done in Carroll (2011b) so we will take the proposition a fact and proceed by manipulating the inequality:

$$\begin{array}{rcl} \blacktriangle m_t \underline{\kappa}_t < & \mathrm{c}_t(\underline{m}_t + \blacktriangle m_t) & < (\blacktriangle m_t + \blacktriangle \mathfrak{h}_t)\underline{\kappa}_t \\ -\blacktriangle m_t \underline{\kappa}_t > & -\mathrm{c}_t(\underline{m}_t + \blacktriangle m_t) & > -(\blacktriangle m_t + \blacktriangle \mathfrak{h}_t)\underline{\kappa}_t \\ \blacktriangle \mathfrak{h}_t \underline{\kappa}_t > & \bar{\mathrm{c}}_t(\underline{m}_t + \blacktriangle m_t) - \mathrm{c}_t(\underline{m}_t + \blacktriangle m_t) & > 0 \\ 1 > & \underbrace{\left(\dfrac{\bar{\mathrm{c}}_t(\underline{m}_t + \blacktriangle m_t) - \mathrm{c}_t(\underline{m}_t + \blacktriangle m_t)}{\blacktriangle \mathfrak{h}_t \underline{\kappa}_t}\right)}_{\equiv \hat{\phi}_t} & > 0 \end{array}$$

where the fraction in the middle of the last inequality is the ratio of actual precautionary saving (the numerator is the difference between perfect-foresight consumption and optimal consumption in the presence of uncertainty) to the maximum conceivable amount of precautionary saving (the amount that would be undertaken by the pessimist who consumes nothing out of any future income beyond the perfectly certain component). Defining $\mu_t = \log \blacktriangle m_t$ (which can range from $-\infty$ to

∞), the object in the middle of the last inequality is

$$\hat{\mathsf{o}}_t(\mu_t) \equiv \left(\frac{\bar{\mathsf{c}}_t(\underline{m}_t + e^{\mu_t}) - \mathsf{c}_t(\underline{m}_t + e^{\mu_t})}{\blacktriangle\mathfrak{h}_t\underline{\kappa}_t}\right), \tag{10}$$

and we now define

$$\hat{\boldsymbol{\chi}}_t(\mu_t) = \log\left(\frac{1-\hat{\mathsf{o}}_t(\mu_t)}{\hat{\mathsf{o}}_t(\mu_t)}\right) \tag{11}$$

$$= \log\left(1/\hat{\mathsf{o}}_t(\mu_t) - 1\right) \tag{12}$$

which has the virtue that it is linear in the limit as μ_t approaches $+\infty$.

Given $\hat{\boldsymbol{\chi}}$, the consumption function can be recovered from

$$\hat{\mathsf{c}}_t = \bar{\mathsf{c}}_t - \overbrace{\left(\frac{1}{1+\exp(\hat{\boldsymbol{\chi}}_t)}\right)}^{=\hat{\mathsf{o}}_t}\blacktriangle\mathfrak{h}_t\underline{\kappa}_t. \tag{13}$$

Thus, the procedure is to calculate $\hat{\boldsymbol{\chi}}_t$ at the points $\vec{\mu}_t$ corresponding to the log of the $\blacktriangle\vec{m}_t$ points defined above, and then using these to construct an interpolating approximation $\grave{\hat{\boldsymbol{\chi}}}_t$ from which we indirectly obtain our approximated consumption rule $\grave{\hat{\mathsf{c}}}_t$ by substituting $\grave{\hat{\boldsymbol{\chi}}}_t$ for $\hat{\boldsymbol{\chi}}$ in equation (13).

Because this method relies upon the fact that the problem is easy to solve if the decision maker has unreasonable views (either in the optimistic or the pessimistic direction), and because the correct solution is always between these immoderate extremes, we call our solution procedure the 'method of moderation.'

Results are shown in Figure 3; a reader with very good eyesight might be able to detect the barest hint of a discrepancy between the Truth and the Approximation at the far righthand edge of the figure.

4.1.2 The Value Function

Often it is useful to know the value function as well as the consumption rule associated with a problem. Fortunately, many of the tricks used when solving the consumption problem have a direct analogue in approximation of the value function.

Consider the perfect foresight (or 'optimist') case in period $T-1$:

$$\begin{aligned}
\mathsf{v}_{T-1}(m_{T-1}) &= \mathsf{u}(c_{T-1}) + \beta\mathsf{u}(c_T) \\
&= \mathsf{u}(c_{T-1})\left(1+\beta((\beta_T\mathsf{R})^{1/\rho})^{1-\rho}\right) \\
&= \mathsf{u}(c_{T-1})\left(1+\beta(\beta_T\mathsf{R})^{1/\rho-1}\right) \\
&= \mathsf{u}(c_{T-1})\left(1+(\beta_T\mathsf{R})^{1/\rho}/\mathsf{R}\right) \\
&= \mathsf{u}(c_{T-1})\underbrace{\mathrm{PDV}_t^T(c)/c_{T-1}}_{\equiv\mathbb{C}_t^T}
\end{aligned}$$

where $\mathrm{PDV}_t^T(c)$ is the present discounted value of consumption. A similar function can be constructed recursively for earlier periods, yielding the general expression

$$\mathsf{v}_t(m_t) = \mathsf{u}(c_t)\mathbb{C}_t^T \tag{14}$$

which can be transformed as

$$\begin{aligned}\Lambda_t &\equiv ((1-\rho)\mathrm{v}_t)^{1/(1-\rho)} \\ &= c_t(\mathbb{C}_t^T)^{1/(1-\rho)}\end{aligned}$$

and since $\mathbb{C}_t^T$ is a constant while the consumption function is linear, $\bar{\Lambda}_t$ will also be linear.

We apply the same transformation to the value function for the problem with uncertainty (the realist's problem):

$$\Lambda_t = ((1-\rho)\mathrm{v}_t(m_t))^{1/(1-\rho)}$$

and an excellent approximation to the value function can be obtained by calculating the values of Λ at the same gridpoints used by the consumption function approximation, and interpolating among those points.

However, as with the consumption approximation, we can do even better if we realize that the $\bar{\Lambda}$ function for the optimist's problem is an upper bound for the Λ function in the presence of uncertainty, and the value function for the pessimist is a lower bound. Analogously to (10), define an upper-case

$$\hat{\Upsilon}_t(\mu_t) = \left(\frac{\bar{\Lambda}_t(\underline{m}_t + e^{\mu_t}) - \Lambda_t(\underline{m}_t + e^{\mu_t})}{\blacktriangle\mathfrak{h}_t\underline{\kappa}_t(\mathbb{C}_t^T)^{1/(1-\rho)}}\right) \tag{15}$$

and an upper-case version of the $\boldsymbol{\chi}$ equation in (12):

$$\hat{\mathrm{X}}_t(\mu_t) = \log\left(\frac{1-\hat{\Upsilon}_t(\mu_t)}{\hat{\Upsilon}_t(\mu_t)}\right) \tag{16}$$

$$= \log\left(1/\hat{\Upsilon}_t(\mu_t) - 1\right) \tag{17}$$

and if we approximate these objects then invert them (as above with the $\hat{\varphi}$ and $\hat{\boldsymbol{\chi}}$ functions) we obtain a very high-quality approximation to our inverted value function at the same points for which we have our approximated value function:

$$\hat{\Lambda}_t = \bar{\Lambda}_t - \overbrace{\left(\frac{1}{1+\exp(\hat{\mathrm{X}}_t)}\right)}^{=\hat{\Upsilon}_t}\blacktriangle\mathfrak{h}_t\underline{\kappa}_t(\mathbb{C}_t^T)^{1/(1-\rho)} \tag{18}$$

from which we obtain our approximation to the value function as

$$\begin{aligned}\hat{\mathrm{v}}_t &= \mathrm{u}(\hat{\Lambda}_t) \\ \hat{\mathrm{v}}_t^m &= \mathrm{u}'(\hat{\Lambda}_t)\hat{\Lambda}^m.\end{aligned}$$

Although a linear interpolation that matches the level of Λ at the gridpoints is simple, a Hermite interpolation that matches both the level and the derivative of the $\hat{\Lambda}_t$ function at the gridpoints has the considerable virtue that the $\bar{\mathrm{v}}_t$ derived from it numerically satisfies the envelope theorem at each of the gridpoints for which the problem has been solved.

5 Extensions

5.1 A Tighter Upper Bound

Carroll (2011b) derives an upper limit $\bar{\kappa}_t$ for the MPC as m_t approaches its lower bound. Using this fact plus the strict concavity of the consumption function yields the proposition that

$$\mathrm{c}_t(\underline{m}_t + \blacktriangle m_t) \quad < \bar{\kappa}_t \blacktriangle m_t. \tag{19}$$

The solution method described above does not guarantee that approximated consumption will respect this constraint between gridpoints, and a failure to respect the constraint can occasionally cause computational problems in solving or simulating the model. Here, we describe a method for constructing an approximation that always satisfies the constraint.

Defining $m_t^{\#}$ as the 'cusp' point where the two upper bounds intersect:

$$\begin{aligned}
\left(\blacktriangle m_t^{\#} + \blacktriangle \mathfrak{h}_t\right) \underline{\kappa}_t &= \bar{\kappa}_t \blacktriangle m_t^{\#} \\
\blacktriangle m_t^{\#} &= \frac{\underline{\kappa}_t \blacktriangle \mathfrak{h}_t}{(1-\underline{\kappa}_t)\bar{\kappa}_t} \\
m_t^{\#} &= \frac{\underline{\kappa}_t \mathfrak{h}_t - \underline{\mathfrak{h}}_t}{(1-\underline{\kappa}_t)\bar{\kappa}_t},
\end{aligned}$$

we want to construct a consumption function for $m_t \in (\underline{m}_t, m_t^{\#}]$ that respects the tighter upper bound:

$$\begin{array}{rcl}
\blacktriangle m_t \underline{\kappa}_t < & \mathrm{c}_t(\underline{m}_t + \blacktriangle m_t) & < \bar{\kappa}_t \blacktriangle m_t \\
\blacktriangle m_t(\bar{\kappa}_t - \underline{\kappa}_t) > & \bar{\kappa}_t \blacktriangle m_t - \mathrm{c}_t(\underline{m}_t + \blacktriangle m_t) & > 0 \\
1 > & \left(\frac{\bar{\kappa}_t \blacktriangle m_t - \mathrm{c}_t(\underline{m}_t + \blacktriangle m_t)}{\blacktriangle m_t(\bar{\kappa}_t - \underline{\kappa}_t)}\right) & > 0.
\end{array}$$

Again defining $\mu_t = \log \blacktriangle m_t$, the object in the middle of the inequality is

$$\check{\varrho}_t(\mu_t) \quad \equiv \quad \frac{\bar{\kappa}_t - \mathrm{c}_t(\underline{m}_t + e^{\mu_t})e^{-\mu_t}}{\bar{\kappa}_t - \underline{\kappa}_t}.$$

As m_t approaches $-\underline{m}_t$, $\check{\varrho}_t(\mu_t)$ converges to zero, while as m_t approaches $+\infty$, $\check{\varrho}_t(\mu_t)$ approaches 1.

As before, we can derive an approximated consumption function; call it $\check{\mathrm{c}}_t$. This function will clearly do a better job approximating the consumption function for low values of m_t while the previous approximation will perform better for high values of m_t.

For middling values of m it is not clear which of these functions will perform better. However, an alternative is available which performs well. Define the highest gridpoint below $m_t^{\#}$ as $\bar{\check{m}}_t^{\#}$ and the lowest gridpoint above $m_t^{\#}$ as $\underline{\hat{m}}_t^{\#}$. Then there will be a unique interpolating polynomial that matches the level and slope of the consumption function at these two points. Call this function $\tilde{\mathrm{c}}_t(m)$.

Using indicator functions that are zero everywhere except for specified intervals,

$$\begin{aligned}
\mathbf{1}_{\mathrm{Lo}}(m) &= 1 \text{ if } m \leq \bar{\check{m}}_t^{\#} \\
\mathbf{1}_{\mathrm{Mid}}(m) &= 1 \text{ if } \qquad \bar{\check{m}}_t^{\#} < m < \underline{\hat{m}}_t^{\#}
\end{aligned}$$

$$\mathbf{1}_{\text{Hi}}(m) \;=\; 1 \text{ if } \qquad\qquad \underline{\hat{m}}_t^{\#} \leq m$$

we can define a well-behaved approximating consumption function

$$\grave{\mathrm{c}}_t \;=\; \mathbf{1}_{\text{Lo}}\grave{\bar{\mathrm{c}}}_t + \mathbf{1}_{\text{Mid}}\grave{\tilde{\mathrm{c}}}_t + \mathbf{1}_{\text{Hi}}\grave{\hat{\mathrm{c}}}_t. \tag{20}$$

This just says that, for each interval, we use the approximation that is most appropriate. The function is continuous and once-differentiable everywhere, and is therefore well behaved for computational purposes.

To construct the corresponding refined representation of the value function we must first clarify one point: The upper-bound value function that we are constructing will be the one implied by a consumer whose spending behavior is consistent with the refined upper-bound consumption rule.

For $m_t \geq m_t^{\#}$, this consumption rule is the same as before, so the constructed upper-bound value function is also the same. However, for values $m_t < m_t^{\#}$ matters are slightly more complicated.

Start with the fact that at the cusp point,

$$\begin{aligned} \bar{\mathrm{v}}_t(m_t^{\#}) &= \mathrm{u}(\bar{c}_t(m_t^{\#}))\mathbb{C}_t^T \\ &= \mathrm{u}(\blacktriangle m_t^{\#}\bar{\kappa}_t)\mathbb{C}_t^T. \end{aligned}$$

But for *all* m_t,

$$\bar{\mathrm{v}}_t(m) \;=\; \mathrm{u}(\bar{c}_t(m)) + \bar{\mathfrak{v}}_t(m - \bar{c}_t(m)),$$

and we assume that for the consumer below the cusp point consumption is given by $\bar{\kappa}\blacktriangle m_t$ so for $m_t < m_t^{\#}$

$$\bar{\mathrm{v}}_t(m) \;=\; \mathrm{u}(\bar{\kappa}_t\blacktriangle m) + \bar{\mathfrak{v}}_t((1-\bar{\kappa}_t)\blacktriangle m),$$

which is easy to compute because $\mathfrak{v}_t(a_t) = \beta\bar{\mathrm{v}}_{t+1}(a_t\mathcal{R}+1)$ where $\bar{\mathrm{v}}_t$ is as defined above because a consumer who ends the current period with assets exceeding the lower bound will not expect to be constrained next period. (Recall again that we are merely constructing an object that is guaranteed to be an *upper bound* for the value that the 'realist' consumer will experience.) At the gridpoints defined by the solution of the consumption problem can then construct

$$\bar{\Lambda}_t(m) \;=\; ((1-\rho)\bar{\mathrm{v}}_t(m))^{1/(1-\rho)}$$

which yields the appropriate vector for constructing $\check{\mathrm{X}}$ and $\check{\Upsilon}$. The rest of the procedure is analogous to that performed for the consumption rule and is thus omitted for brevity.

5.2 Stochastic Rate of Return

Thus far we have assumed that the interest factor is constant at R. Extending the previous derivations to allow for a perfectly forecastable time-varying interest factor R_t would be trivial. Allowing for a stochastic interest factor is less trivial.

The easiest case is where the interest factor is i.i.d.,

$$\log \mathbf{R}_{t+n} \;\sim\; \mathcal{N}(\mathsf{r} + \phi - \sigma_{\mathbf{r}}^2/2, \sigma_{\mathbf{r}}^2) \;\forall\; n > 0 \tag{21}$$

because in this case Merton (1969) and Samuelson (1969) showed that for a consumer without labor income (or with perfectly forecastable labor income) the con-

sumption function is linear, with an MPC[2]

$$\kappa \;=\; 1 - \left(\beta\, \mathbb{E}_t[\mathbf{R}_{t+1}^{1-\rho}]\right)^{1/\rho} \tag{22}$$

and in this case the previous analysis applies once we substitute this MPC for the one that characterizes the perfect foresight problem without rate-of-return risk. The more realistic case where the interest factor has some serial correlation is more complex. We consider the simplest case that captures the main features of empirical interest rate dynamics: An AR(1) process. Thus the specification is

$$\mathbf{r}_{t+1} - \mathbf{r} \;=\; (\mathbf{r}_t - \mathbf{r})\gamma + \epsilon_{t+1} \tag{23}$$

where $\mathbf{r}$ is the long-run mean log interest factor, $0 < \gamma < 1$ is the AR(1) serial correlation coefficient, and ϵ_{t+1} is the stochastic shock.

The consumer's problem in this case now has two state variables, m_t and $\mathbf{r}_t$, and is described by

$$\begin{aligned}
\mathrm{v}_t(m_t, \mathbf{r}_t) &= \max_{c_t}\; \mathrm{u}(c_t) + \mathbb{E}_t[\beta_{t+1}\Gamma_{t+1}^{1-\rho}\mathrm{v}_{t+1}(m_{t+1}, \mathbf{r}_{t+1})] \qquad (24)\\
&\text{s.t.}\\
a_t &= m_t - c_t\\
\mathbf{r}_{t+1} - \mathbf{r} &= (\mathbf{r}_t - \mathbf{r})\gamma + \epsilon_{t+1}\\
\mathbf{R}_{t+1} &= \exp(\mathbf{r}_{t+1})\\
m_{t+1} &= \underbrace{(\mathbf{R}_{t+1}/\Gamma_{t+1})}_{\equiv \mathcal{R}_{t+1}}\, a_t + \theta_{t+1}.
\end{aligned}$$

We approximate the AR(1) process by a Markov transition matrix using standard techniques. The stochastic interest factor is allowed to take on 11 values centered around the steady-state value $\mathbf{r}$ and chosen [how?]. Given this Markov transition matrix, *conditional* on the Markov AR(1) state the consumption functions for the 'optimist' and the 'pessimist' will still be linear, with identical MPC's that are computed numerically. Given these MPC's, the (conditional) realist's consumption function can be computed for each Markov state, and the converged consumption rules constitute the solution contingent on the dynamics of the stochastic interest rate process.

In principle, this refinement should be combined with the previous one; further exposition of this combination is omitted here because no new insights spring from the combination of the two techniques.

6 Conclusion

The method proposed here is not universally applicable. For example, the method cannot be used for problems for which upper and lower bounds to the 'true' solution are not known. But many problems do have obvious upper and lower bounds, and in those cases (as in the consumption example used in the paper), the method may result in substantial improvements in accuracy and stability of solutions.

[2]See `CRRA-RateRisk` for a derivation.

References

Carroll, Christopher D. (2006): "The Method of Endogenous Gridpoints for Solving Dynamic Stochastic Optimization Problems," *Economics Letters*, pp. 312–320, http://econ.jhu.edu/people/ccarroll/EndogenousGridpoints.pdf.

——— (2011a): "Solving Microeconomic Dynamic Stochastic Optimization Problems," *Archive, Johns Hopkins University*.

——— (2011b): "Theoretical Foundations of Buffer Stock Saving," *Manuscript, Department of Economics, Johns Hopkins University*, http://econ.jhu.edu/people/ccarroll/papers/BufferStockTheory.

Merton, Robert C. (1969): "Lifetime Portfolio Selection under Uncertainty: The Continuous Time Case," *Review of Economics and Statistics*, 50, 247–257.

Samuelson, Paul A. (1969): "Lifetime Portfolio Selection by Dynamic Stochastic Programming," *Review of Economics and Statistics*, 51, 239–46.

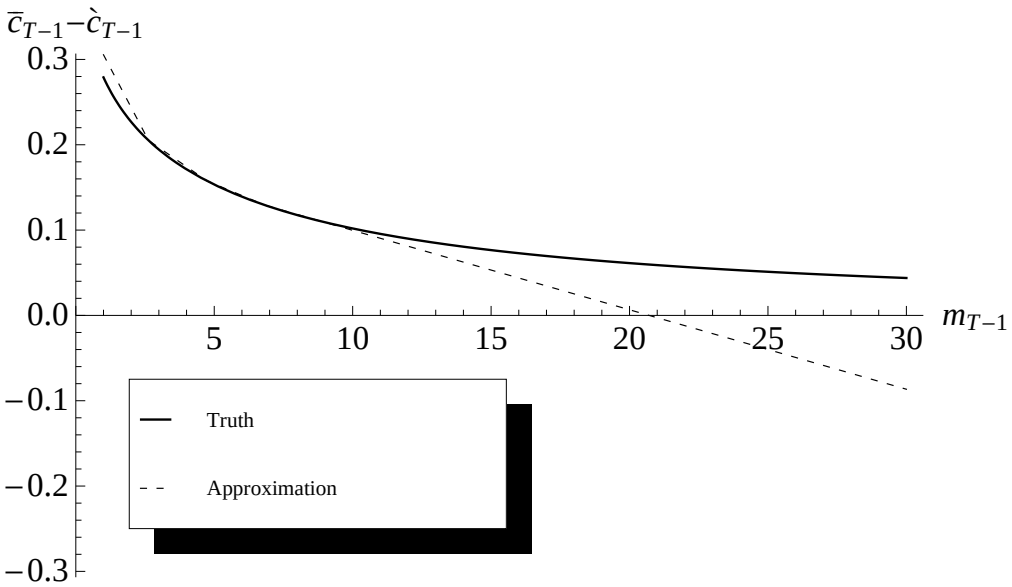


Figure 1 For Large Enough m_{T-1}, Predicted Precautionary Saving is Negative (Oops!)

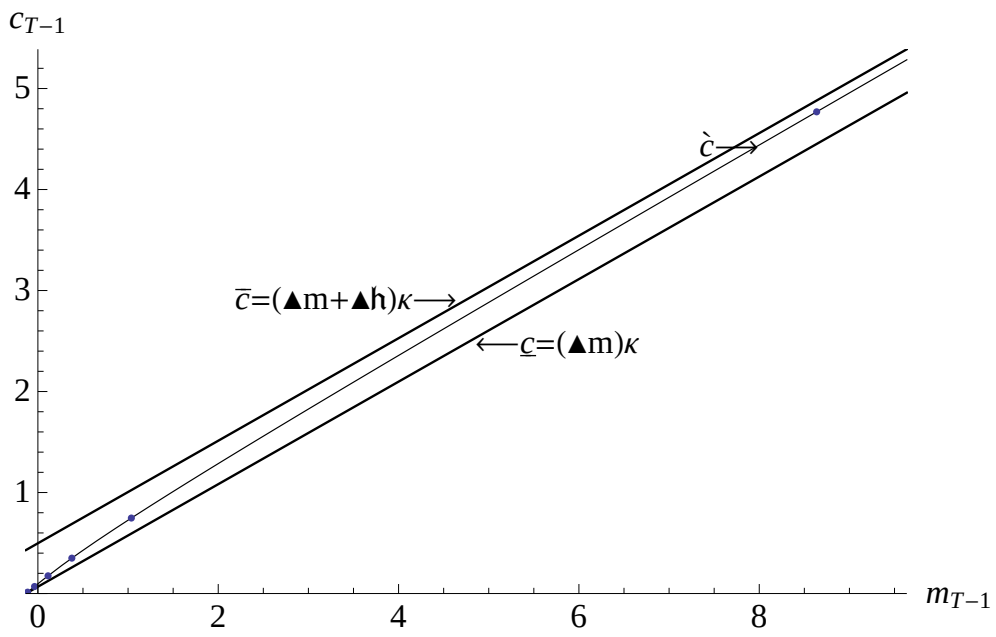


Figure 2 Moderation Illustrated: $\underline{c}_{T-1} < \grave{c}_{T-1} < \bar{c}_{T-1}$

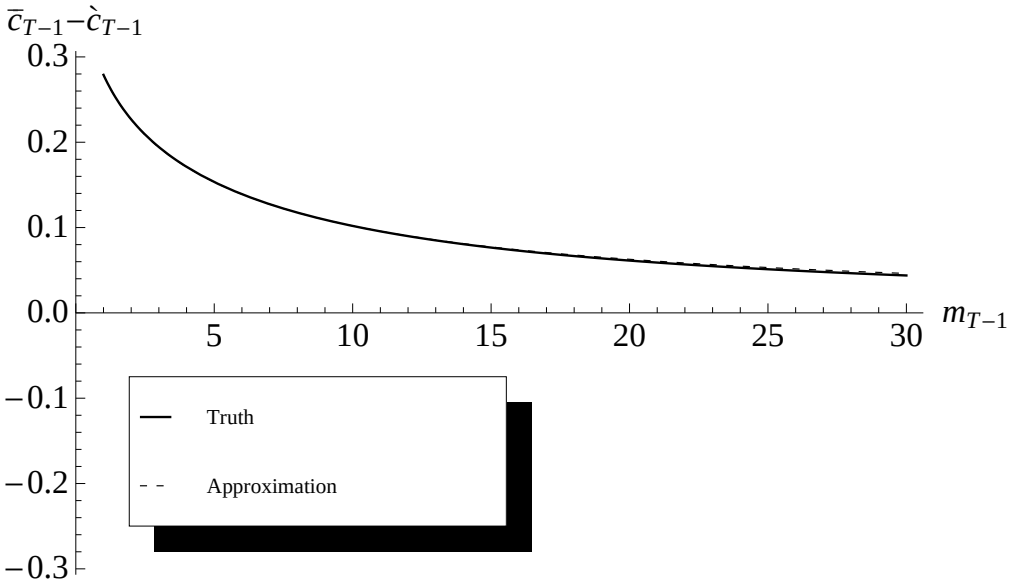
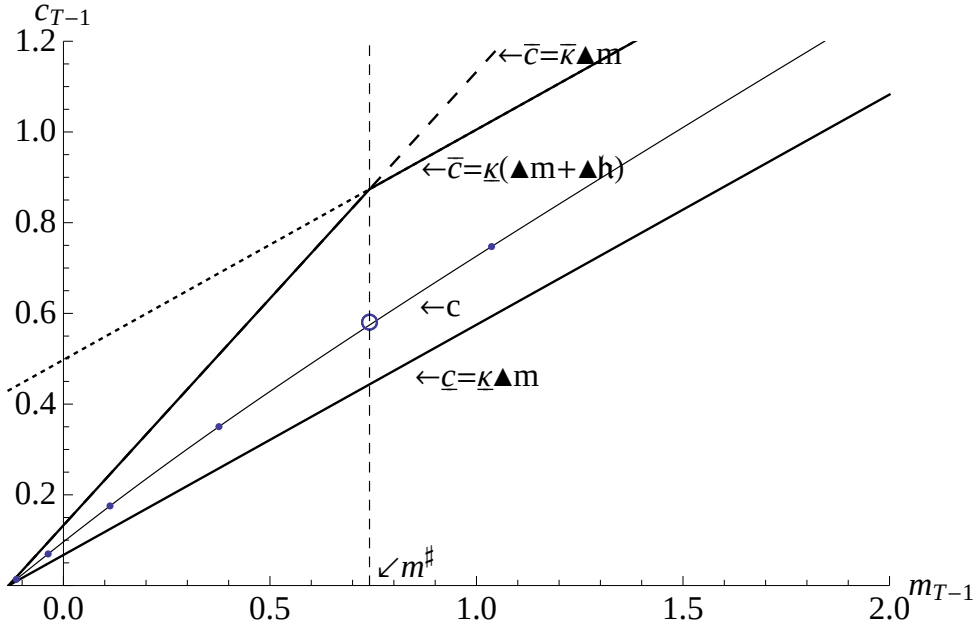


Figure 3 Extrapolated $\grave{c}_{T-1}$ Constructed Using the Method of Moderation


c_{T-1}
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0.5
1.0
1.5
2.0
m_{T-1}
←$\bar{c}$=$\bar{\kappa}$▲m
←$\bar{c}$=$\underline{\kappa}$(▲m+▲h)
←c
←$\underline{c}$=$\underline{\kappa}$▲m
↙$m^{\sharp}$