Structural Estimation of Dynamic Stochastic Optimizing Models of Intertemporal Choice For Dummies!

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June 2012

- Efficient Solution Methods for Canonical C problem
  - CRRA utility
  - Plausible (microeconomically calibrated) uncertainty
  - Life cycle or infinite horizon
- How To Add a Second Choice Variable
- Method of Simulated Moments Estimation of Parameters

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The Basic Problem at Date t

$$\max \ \mathbb{E}_t \left[\sum_{n=0}^{T-t} \beta^n \mathrm{u}(\boldsymbol{c}_{t+n}) \right], \tag{1}$$

$$\boldsymbol{y}_t \ = \ \boldsymbol{p}_t \theta_t \tag{2}$$

$$\begin{aligned} \mathrm{R}_t &= \mathrm{R} && \text{- constant interest factor} = 1 + \mathrm{r} \\ \boldsymbol{p}_{t+1} &= \Gamma_{t+1} \boldsymbol{p}_t && \text{- permanent labor income dynamics} \\ \log \ \theta &\sim \mathcal{N}(-\sigma_\theta^2/2, \sigma_\theta^2) && \text{- lognormally distributed transitory shocks.} \end{aligned} \tag{3}$$

Bellman Equation

$$\mathbf{v}_t(\boldsymbol{m}_t, \boldsymbol{p}_t) \quad = \quad \max_{\boldsymbol{c}_t} \; \mathrm{u}(\boldsymbol{c}_t) + \mathbb{E}_t[\beta \mathbf{v}_{t+1}(\boldsymbol{m}_{t+1}, \boldsymbol{p}_{t+1})] \qquad (4)$$

$\boldsymbol{m}$ – 'market resources' (net worth plus current income)

$\boldsymbol{p}$ – permanent labor income

Trick: Normalize the Problem

$$
\begin{aligned}
\mathrm{v}_t(m_t) &= \max_{c_t} \quad \mathrm{u}(c_t) + \mathbb{E}_t[\beta \Gamma_{t+1}^{1-\rho} \mathrm{v}_{t+1}(m_{t+1})] \qquad (5) \\
&\text{s.t.} \\
a_t &= m_t - c_t \\
m_{t+1} &= \underbrace{(\mathrm{R}/\Gamma_{t+1})}_{\equiv \mathcal{R}_{t+1}} a_t + \theta_{t+1}
\end{aligned}
$$

where nonbold variables are bold ones normalized by $\boldsymbol{p}$:

$$
m_t = \boldsymbol{m}_t / \boldsymbol{p}_t \qquad (6)
$$

Yields $\mathrm{c}_t(m)$ from which we can obtain

$$
\boldsymbol{c}_t(\boldsymbol{m}_t, \boldsymbol{p}_t) = \mathrm{c}_t(\boldsymbol{m}_t / \boldsymbol{p}_t) \boldsymbol{p}_t \qquad (7)
$$

When Doesn't Normalization Work?

- Non-CRRA utility
- Non-Friedman income process (transitory/permanent)
  - ARMA(1,1)
  - But micro evidence is consistent with Friedman

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Trick: View Everything from End of Period

Define

$$\mathfrak{v}_t(a_t) = \mathbb{E}_t[\beta\Gamma_{t+1}^{1-\rho}\mathrm{v}_{t+1}(\mathcal{R}_{t+1}a_t + \theta_{t+1})] \tag{8}$$

so

$$\mathrm{v}_t(m_t) = \max_{c_t} \ \mathrm{u}(c_t) + \mathfrak{v}_t(m_t - c_t) \tag{9}$$

with FOC

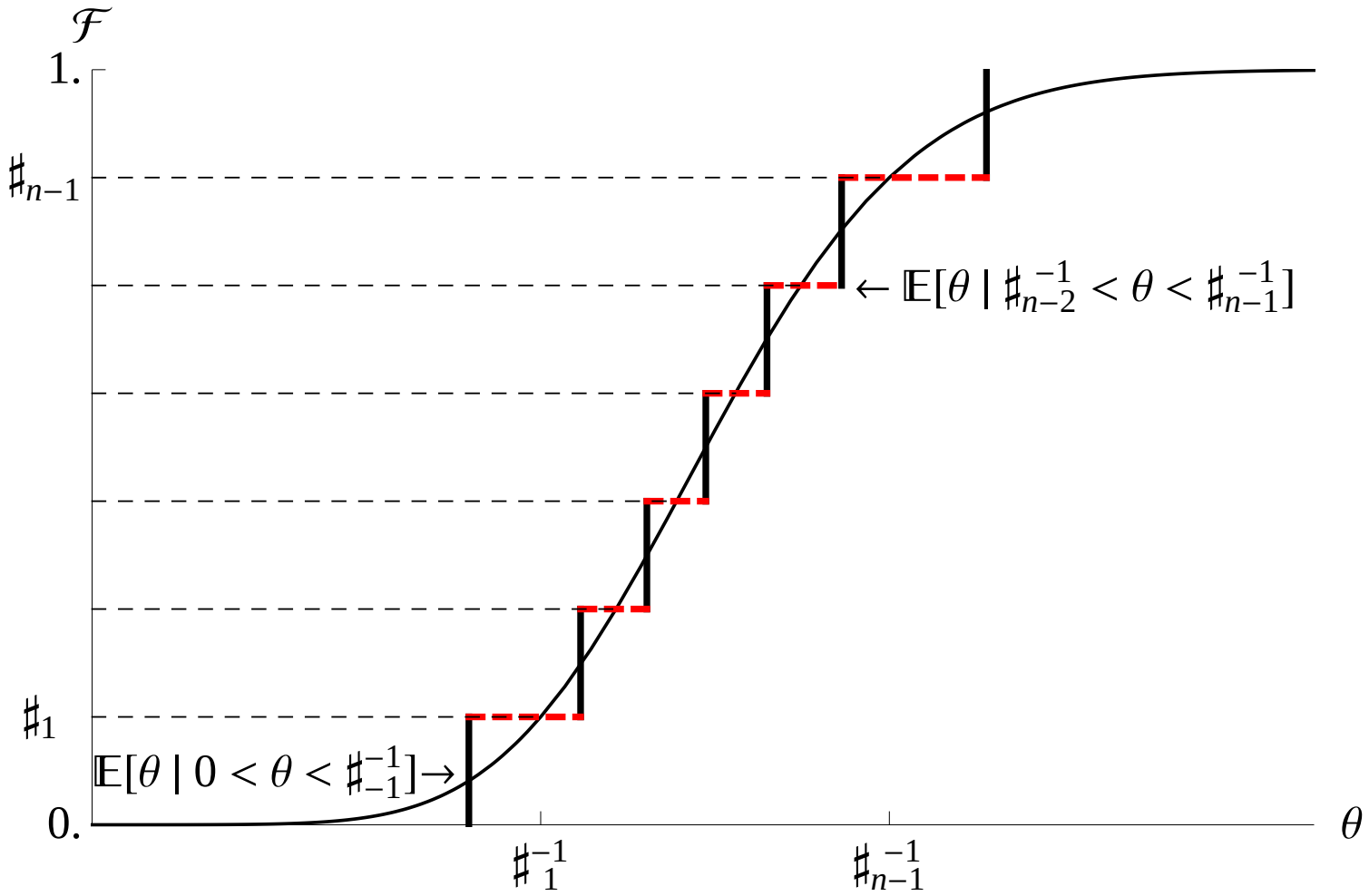
$$\mathrm{u}'(c_t) = \mathfrak{v}_t'(m_t - c_t). \tag{10}$$

and Envelope relation

$$\mathrm{u}'(c_t) = \mathrm{v}_t'(m_t) \tag{11}$$

Trick: Discretize the Risks

E.g. use an equiprobable 7-point distribution:



Trick: Discretize the Risks

$$\mathfrak{v}_t'(a_t) = \beta \mathrm{R} \Gamma_{t+1}^{-\rho} \left(\frac{1}{n}\right) \sum_{i=1}^{n} \mathrm{u}'\left(\mathrm{c}_{t+1}(\mathcal{R}_{t+1} a_t + \theta_i)\right) \quad (12)$$

So for any particular m_{T-1} the corresponding c_{T-1} can be found using the FOC:

$$\mathrm{u}'(c_t) = \mathfrak{v}_t'(m_t - c_t). \quad (13)$$

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Trick: Interpolate a Consumption Rule

1. Define a grid of points $\vec{m}$ (indexed $m[i]$)
2. Use numerical rootfinder to solve $\mathrm{u}'(c) = \mathfrak{v}_t'(m[i] - c)$
   - The c that solves this becomes $c[i]$
3. Construct interpolating function $\grave{\mathrm{c}}$ by linear interpolation
   - 'Connect-the-dots'

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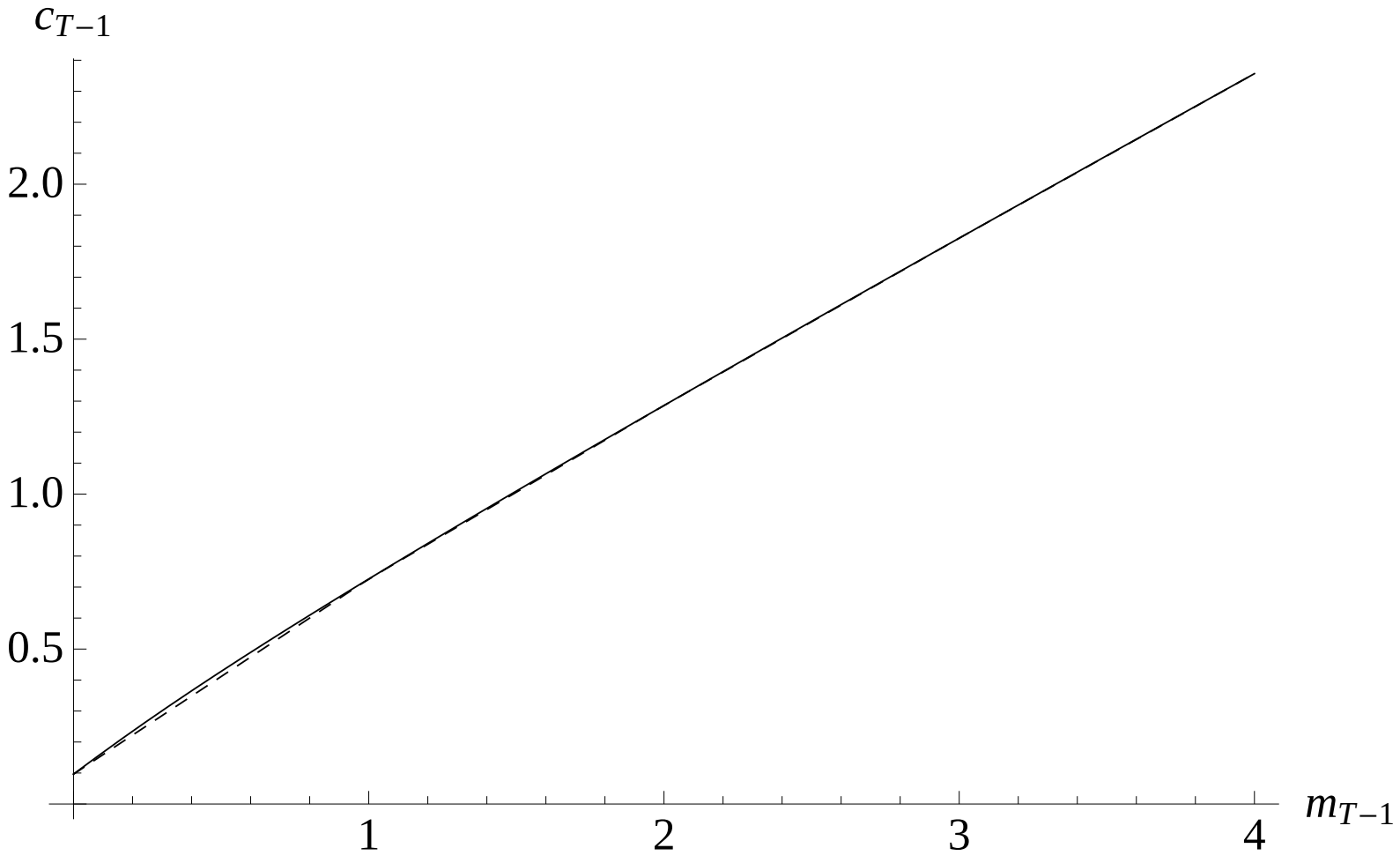
Trick: Interpolate a Consumption Rule

Example: $\vec{m}_{T-1} = \{0., 1., 2., 3., 4.\}$ (solid is 'correct' soln)



Problem: Numerical Rootfinding is *Slow*

Numerical search for optimal values of c_{T-1} corresponding to, say, 5 gridpoints of $\vec{m}_{T-1}$ may require *hundreds* of evaluations of

$$\mathfrak{v}'_{T-1}(\overbrace{m_{T-1} - c_{T-1}}^{a_{T-1}}) = \beta_T \Gamma_T^{1-\rho} \left(\frac{1}{n}\right) \sum_{i=1}^{n} \frac{(\mathcal{R}_T a_{T-1} + \theta_i)^{1-\rho}}{1-\rho}$$

Solution: The Method of Endogenous Gridpoints

- Define vector of *end-of-period* asset values $\vec{a}$
- For each $a[j]$ compute $\mathfrak{v}_t'(a[j])$

Each of these $\mathfrak{v}_t'[j]$ corresponds to a unique $c[j]$ via FOC:

$$c[j]^{-\rho} = \mathfrak{v}_t'(a[j]) \tag{14}$$

$$c[j] = \left(\mathfrak{v}_t'(a[j])\right)^{-1/\rho} \tag{15}$$

But the DBC says

$$a_t = m_t - c_t \tag{16}$$

$$m[j] = a[j] + c[j] \tag{17}$$

So computing $\mathfrak{v}_t'$ at a vector of $\vec{a}$ values has produced for us the corresponding $\vec{c}$ and $\vec{m}$ values at virtually no cost!

From these we can interpolate as before to construct $\grave{c}_t(m)$.

Solution: The Method of Endogenous Gridpoints

- Define vector of *end-of-period* asset values $\vec{a}$
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Why Approximating v_t is a Bad Idea

Principles of Approximation

- Hard to approximate things that approach ∞ for relevant m
  - Not a prob for Rep Agent models: 'relevant' m's are $\approx$ SS
- Hard to approximate things that are highly nonlinear

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Approximate Something That Would Be Linear in PF Case

Perfect Foresight Theory:

$$\mathrm{c}_t(m_t) \quad = \quad (b_t + h_t)\kappa_t \tag{18}$$

for bank balances b and human wealth h.

This is why it's a good idea to approximate c_t

Bonus: Easy to debug programs by setting $\sigma^2 = 0$ and testing whether numerical solution matches analytical!

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But What if You *Need* the Value Function?

Perfect foresight value function:

$$\bar{\mathrm{v}}_t(m_t) = \mathrm{u}(c_t)\mathbb{C}_t^T \tag{19}$$

which can be transformed as

$$\begin{aligned}\bar{\Lambda}_t(m_t) &= ((1-\rho)\bar{\mathrm{v}}_t)^{1/(1-\rho)} \\ &= \bar{\mathrm{c}}_t(m_t)(\mathbb{C}_t^T)^{1/(1-\rho)}\end{aligned}$$

which is linear.

If you need the value function, approximate the *inverted* value function to generate $\grave{\Lambda}_t$ and then obtain your approximation from

$$\grave{\mathrm{v}}_t = \mathrm{u}(\grave{\Lambda}_t) \tag{20}$$

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The Method of Moderation

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Imposing 'Artificial' Borrowing Constraints

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\begin{aligned}
\mathrm{v}_{T-1}(m_{T-1}) &= \max_{c_{T-1}} \ \mathrm{u}(c_{T-1}) + \mathbb{E}_{T-1}[\beta \Gamma_T^{1-\rho} \mathrm{v}_T(m_T)] \\
&\text{s.t.} \\
a_{T-1} &= m_{T-1} - c_{T-1} \\
m_T &= \mathcal{R}_T a_{T-1} + \theta_T \\
a_{T-1} &\geq 0.
\end{aligned}
$$

Define $\grave{\mathrm{c}}_t^*$ as soln to unconstrained problem. Then

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\grave{\mathrm{c}}_{T-1}(m_{T-1}) = \min[m_{T-1}, \grave{\mathrm{c}}_{T-1}^*(m_{T-1})]. \tag{21}
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Imposing 'Artificial' Borrowing Constraints

Point where constraint makes transition from binding to not is

$$\begin{aligned} \mathrm{u}'(m^{\#}_{T-1}) &= \mathfrak{v}'_{T-1}(0.) \\ m^{\#}_{T-1} &= \left(\mathfrak{v}'_{T-1}(0.)\right)^{-1/\rho} \end{aligned}$$

Procedure is very easy:

- Above $m^{\#}_{T-1}$, $\grave{\mathrm{c}}_{T-1}(m)$ obtained as before
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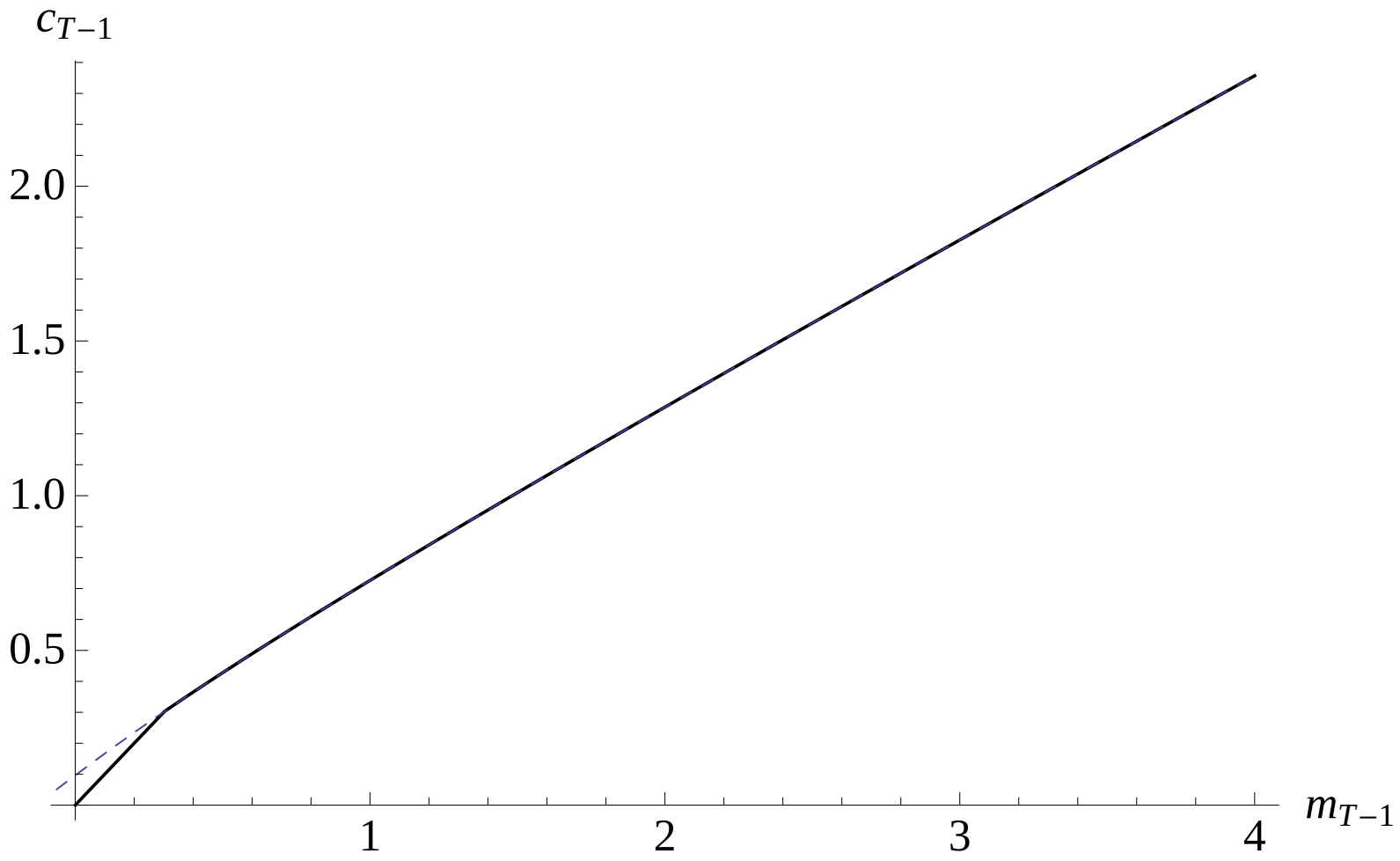

Figure: Constrained (solid) and Unconstrained (dashed) Consumption

Recursion: Period t Solution Given Period $t+1$

1. Construct

$$\begin{aligned} \mathfrak{c}_{t,i} &= \left(\mathfrak{v}_t'(a_{t,i})\right)^{-1/\rho}, && (22) \\ &= \left(\beta \mathbb{E}_t \left[\mathsf{R}\Gamma_{t+1}^{-\rho} (\grave{\mathrm{c}}_{t+1}(\mathcal{R}_{t+1} a_{t,i} + \theta_{t+1}))^{-\rho} \right]\right)^{-1/\rho}, && (23) \end{aligned}$$

2. Call the result $\vec{c}_t$ and generate the corresponding $\vec{m}_t = \vec{c}_t + \vec{a}_t$
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Consumption Rules $\grave{c}_{T-n}$ Converge

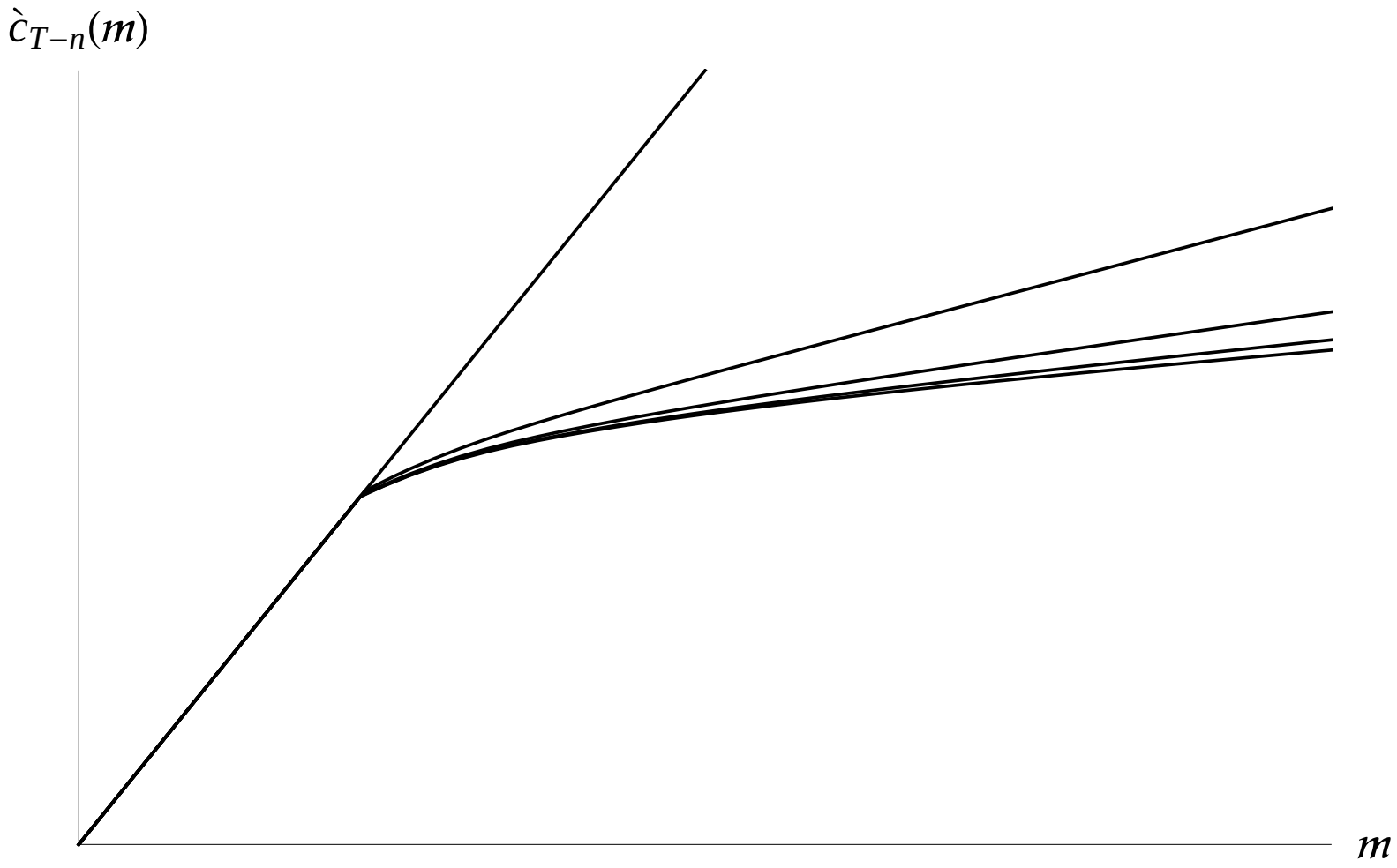

Figure: Converging $\grave{c}_{T-n}(m)$ Functions for $n = \{1, 5, 10, 15, 20\}$

Portfolio Choice

Now the consumer has a choice between a risky and a safe asset. The portfolio return is

$$\begin{aligned} \mathfrak{R}_{t+1} &= \mathrm{R}(1-\varsigma_t) + \mathbf{R}_{t+1}\varsigma_t & (24) \\ &= \mathrm{R} + (\mathbf{R}_{t+1} - \mathrm{R})\varsigma_t & (25) \end{aligned}$$

so (setting $\Gamma = 1$) the maximization problem is

$$\begin{aligned} \mathrm{v}_t(m_t) &= \max_{\{c_t, \varsigma_t\}} \quad \mathrm{u}(c_t) + \beta \mathbb{E}_t[\mathrm{v}_{t+1}(m_{t+1})] \\ &\text{s.t.} \\ \mathfrak{R}_{t+1} &= \mathrm{R} + (\mathbf{R}_{t+1} - \mathrm{R})\varsigma_t \\ m_{t+1} &= (m_t - c_t)\mathfrak{R}_{t+1} + \theta_{t+1} \\ 0 \leq \quad \varsigma_t &\leq 1, \end{aligned}$$

Portfolio Choice

The FOC with respect to c_t now yields an Euler equation

$$\mathrm{u}'(c_t) \quad = \quad \mathbb{E}_t[\beta \mathfrak{R}_{t+1} \mathrm{u}'(c_{t+1})]. \tag{26}$$

while the FOC with respect to the portfolio share yields

$$\begin{aligned} 0 &= \mathbb{E}_t[\mathrm{v}'_{t+1}(m_{t+1})(\mathbf{R}_{t+1} - \mathrm{R})a_t] \\ &= a_t \mathbb{E}_t \left[\mathrm{u}'\left(\mathrm{c}_{t+1}(m_{t+1})\right)(\mathbf{R}_{t+1} - \mathrm{R})\right]. \end{aligned} \tag{27}$$

Convergence

When the problem satisfies certain conditions (Carroll (2011)), it defines a 'converged' consumption rule with a 'target' ratio $\check{m}$ that satisfies:

$$\mathbb{E}_t[m_{t+1}/m_t] \quad = \quad 1 \text{ if } m_t = \check{m} \tag{28}$$

Define the target m implied by the consumption rule c_t as $\check{m}_t$.

Then a plausible metric for convergence is to define some value ϵ and to declare the solution to have converged when

$$|\check{m}_{t+1} - \check{m}_t| \quad < \quad \epsilon \tag{29}$$

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Life Cycle Maximization Problem

$$
\begin{aligned}
\mathrm{v}_t(m_t) &= \max_{c_t} \left\{ \mathrm{u}(c_t) + \beth \not{\mathcal{D}}_{t+1} \hat{\beta}_{t+1} \mathbb{E}_t[(\psi_{t+1}\Gamma_{t+1})^{1-\rho} \mathrm{v}_{t+1}(m_{t+1})] \right\} \\
&\text{s.t.} \\
a_t &= m_t - c_t \\
m_{t+1} &= a_t \underbrace{\left(\frac{\mathrm{R}}{\psi_{t+1}\Gamma_{t+1}} \right)}_{\equiv \mathcal{R}_{t+1}} + \theta_{t+1}
\end{aligned}
$$

$\not{\mathcal{D}}_s$	:	probability of being alive (not dead) until age s given being alive
$\hat{\beta}_s$	:	time-varying discount factor between age $s-1$ and s
Ψ_s	:	mean-one shock to permanent income
$\beth$	:	time-invariant discount factor

Details follow Cagetti (2003)

- Parameterization of Uncertainty
- Probability of Death
- Demographic Adjustments to β

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Empirical Wealth Profiles

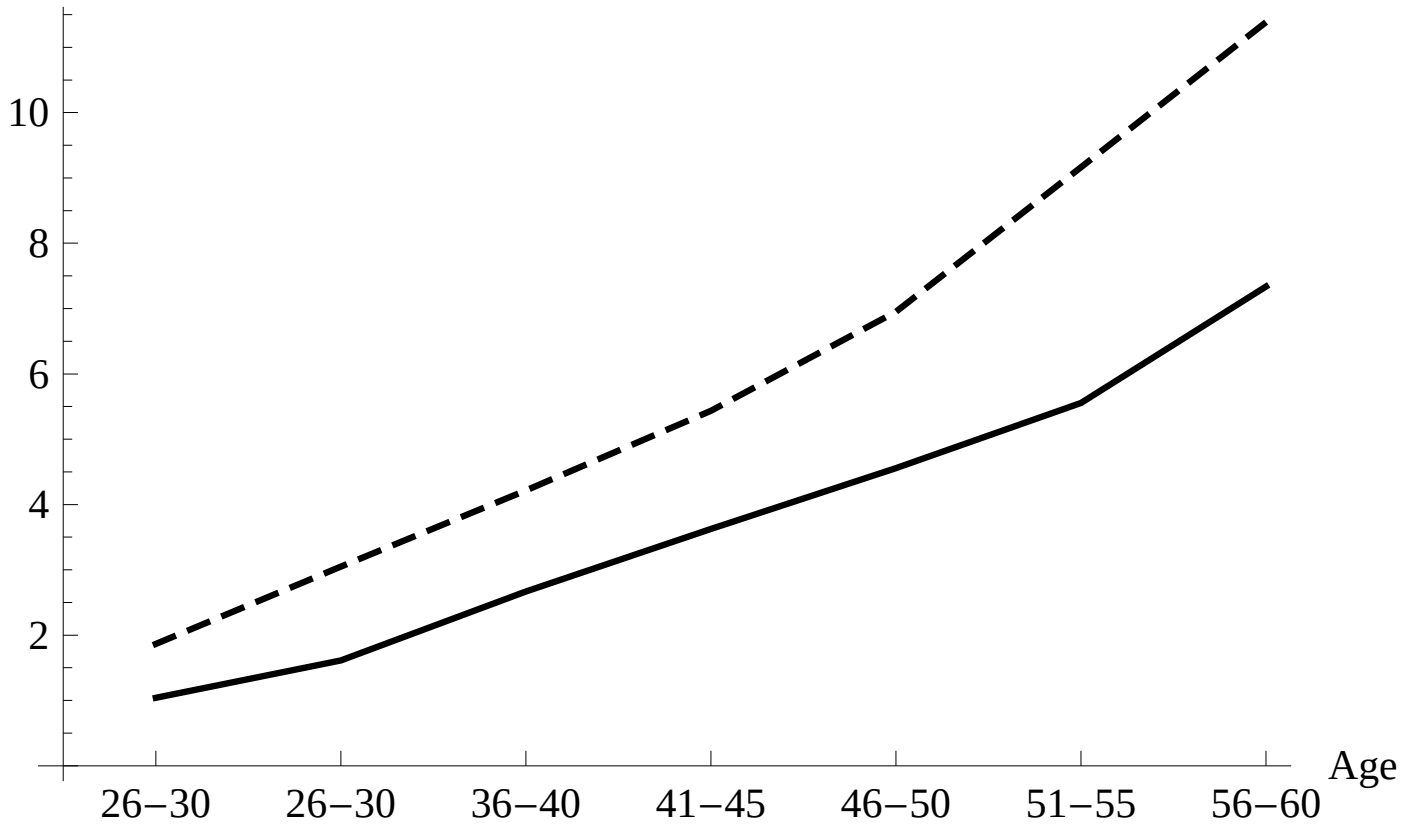

Figure: m from SCF (means (dashed) and medians (solid))

Simulated Moments

Given a set of parameter values $\{\rho, \beth\}$:

- Start at age 25 with empirical m data
- Draw shocks using calibrated $\sigma_\psi^2, \sigma_\theta^2$
- Consume according to solved c_t

$\Rightarrow$ m distribution by age

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$\Rightarrow$ m distribution by age

Choose What to Simulate

$$
\begin{aligned}
&\texttt{GapEmpiricalSimulatedMedians}[\rho, \beth] := \\
&\quad [\quad \texttt{ConstructcFuncLife}[\rho, \beth]; \\
&\qquad\quad \texttt{Simulate}; \\
&\qquad\quad \sum_{i}^{N} \omega_i \left| \varsigma_i^{\tau} - \mathbf{s}^{\tau}(\xi) \right| \\
&\quad];
\end{aligned}
$$

Calculate Match Between Theory and Data

$$\xi = \{\rho, \beth\} \tag{30}$$

solve

$$\min_{\xi} \sum_{i}^{N} \omega_i \left| \varsigma_i^{\tau} - \mathbf{s}^{\tau}(\xi) \right| \tag{31}$$

Bootstrap Standard Errors (Horowitz (2001))

Yields estimates of

Table: Estimation Results

ρ	$\beth$
4.68	1.00
(0.13)	(0.00)

Contour Plot

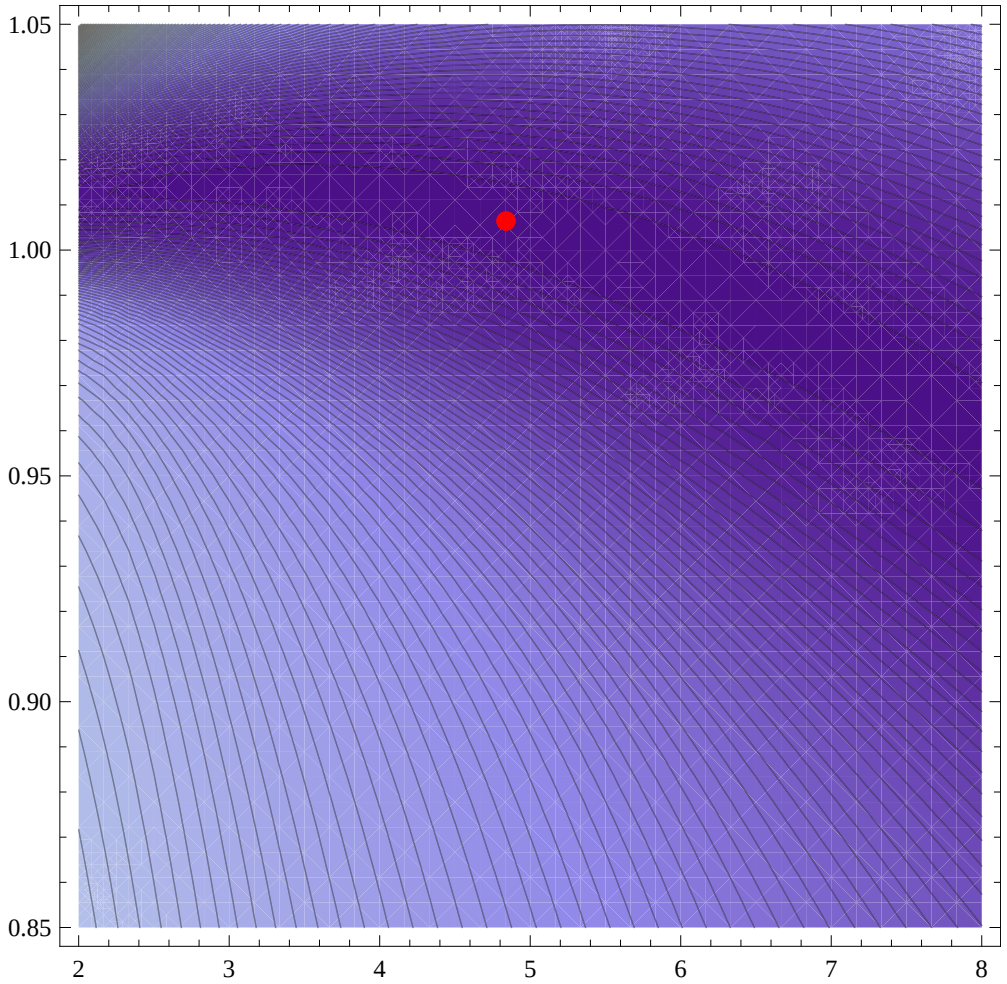

Figure: Point Estimate and Height of Minimized Function

References I

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