The Method of Moderation For Solving Dynamic Stochastic Optimization Problems

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http://econ.jhu.edu/people/ccarroll/papers/ctwMoM/

The Basic Problem at Date t

$$\max \ \mathbb{E}_t \left[\sum_{n=0}^{T-t} \beta^n \mathrm{u}(\boldsymbol{c}_{t+n}) \right], \tag{1}$$

$$\boldsymbol{y}_t = \boldsymbol{p}_t \theta_t \tag{2}$$

$$\begin{aligned} \mathsf{R}_t &= \mathsf{R} && \text{- constant interest factor} = 1 + \mathsf{r} \\ \boldsymbol{p}_{t+1} &= \Gamma_{t+1} \boldsymbol{p}_t && \text{- permanent labor income dynamics} \\ \log \ \theta_{t+n} &\sim \mathcal{N}(-\sigma_\theta^2/2, \sigma_\theta^2) && \text{- lognormal transitory shocks } \forall \ n > 0. \end{aligned} \tag{3}$$

- "Bewley" problem if liquidity constraint, Γ_{t+1} nonstochastic
- "Friedman/Buffer Stock" problem if also permanent shocks

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Bellman Equation

$$\mathbf{v}_t(\boldsymbol{m}_t, \boldsymbol{p}_t) = \max_{\boldsymbol{c}_t} \mathrm{u}(\boldsymbol{c}_t) + \mathbb{E}_t[\beta \mathbf{v}_{t+1}(\boldsymbol{m}_{t+1}, \boldsymbol{p}_{t+1})] \quad (4)$$

$\boldsymbol{m}$ – 'market resources' (net worth plus current income)

$\boldsymbol{p}$ – permanent labor income

where

$$\mathrm{u}(c) = \left(\frac{c^{1-\rho}}{1-\rho}\right) \quad (5)$$

Normalize the Problem

$$
\begin{aligned}
\mathrm{v}_t(m_t) &= \max_{c_t} \quad \mathrm{u}(c_t) + \mathbb{E}_t[\beta \Gamma_{t+1}^{1-\rho} \mathrm{v}_{t+1}(m_{t+1})] \qquad (6) \\
&\text{s.t.} \\
a_t &= m_t - c_t \\
m_{t+1} &= \underbrace{(\mathsf{R}/\Gamma_{t+1})}_{\equiv \mathcal{R}_{t+1}} a_t + \theta_{t+1}
\end{aligned}
$$

where nonbold variables are bold ones normalized by $\boldsymbol{p}$:

$$
m_t = \boldsymbol{m}_t / \boldsymbol{p}_t \qquad (7)
$$

$\Rightarrow \mathrm{c}_t(m)$ from which we can obtain

$$
\boldsymbol{c}_t(\boldsymbol{m}_t, \boldsymbol{p}_t) = \mathrm{c}_t(\boldsymbol{m}_t / \boldsymbol{p}_t) \boldsymbol{p}_t \qquad (8)
$$

My First Notational Atrocity

... in this paper.

$$\mathfrak{v}_t(a_t) = \mathbb{E}_t[\beta\Gamma_{t+1}^{1-\rho}\mathrm{v}_{t+1}(\mathcal{R}_{t+1}a_t + \theta_{t+1})] \tag{9}$$

so FOC is

$$\mathrm{u}'(c_t) = \mathfrak{v}_t'(m_t - c_t). \tag{10}$$

or

$$c_t^{-\rho} = \mathfrak{v}_t'(a_t) \tag{11}$$

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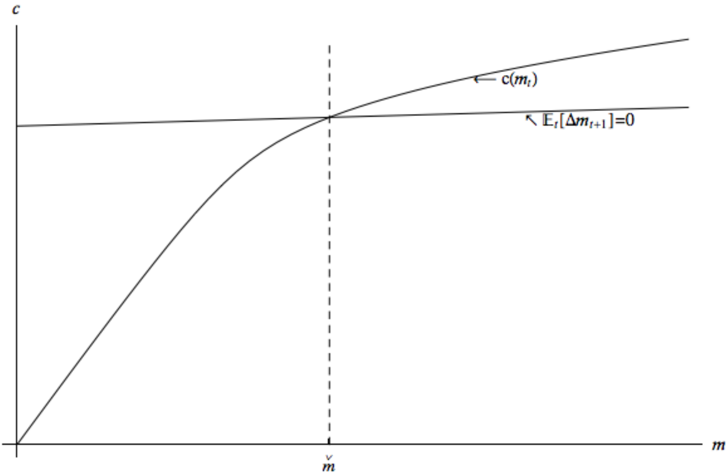
Concave Consumption Function, Target Wealth Ratio



The Method of Endogenous Gridpoints

- Define vector of *end-of-period* asset values $\vec{a}$
- For each $a[j]$ compute $\mathfrak{v}'_t(a[j])$

Each of these $\mathfrak{v}'_t[j]$ corresponds to a unique $c[j]$ via FOC:

$$c[j]^{-\rho} = \mathfrak{v}'_t(a[j]) \tag{12}$$

$$c[j] = \left(\mathfrak{v}'_t(a[j])\right)^{-1/\rho} \tag{13}$$

But the DBC says

$$a_t = m_t - c_t \tag{14}$$

$$m[j] = a[j] + c[j] \tag{15}$$

So computing $\mathfrak{v}'_t$ at a vector of $\vec{a}$ values (easy) has produced for us the corresponding $\vec{c}$ and $\vec{m}$ values at virtually no cost!

From these we can interpolate to construct $\grave{c}_t(m)$

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- Define vector of *end-of-period* asset values $\vec{a}$
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Problem: Extrapolating Outside the Grid (6 Gridpoints)

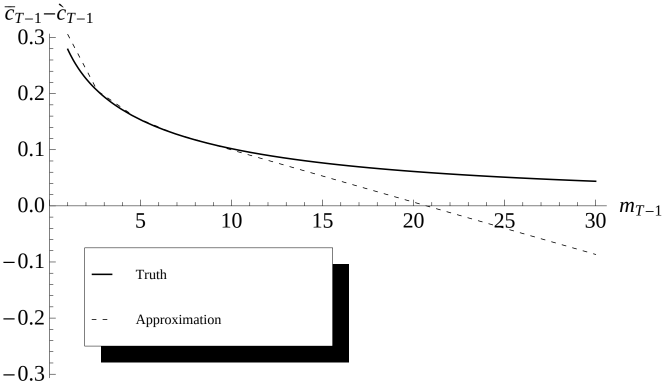

Figure: Oops

Call the 'true' problem that of a 'realist'

Contrasts with:

$$\begin{aligned} \text{Optimist} &: \text{Believes } \theta_{t+n} = \overbrace{1}^{=\mathbb{E}_t[\theta_{t+n}]} \ \forall\ n > 0, \text{ consumes } \bar{c}_t \\ \text{Pessimist} &: \text{Believes } \theta_{t+n} = \underline{\theta}\ \forall\ n > 0, \text{ consumes } \underline{c}_t \end{aligned}$$

Useful because:

1. Both of these are perfect foresight problems
   - Linear analytical solution (easy!)
2. $\underline{c}_t < c_t < \bar{c}_t$

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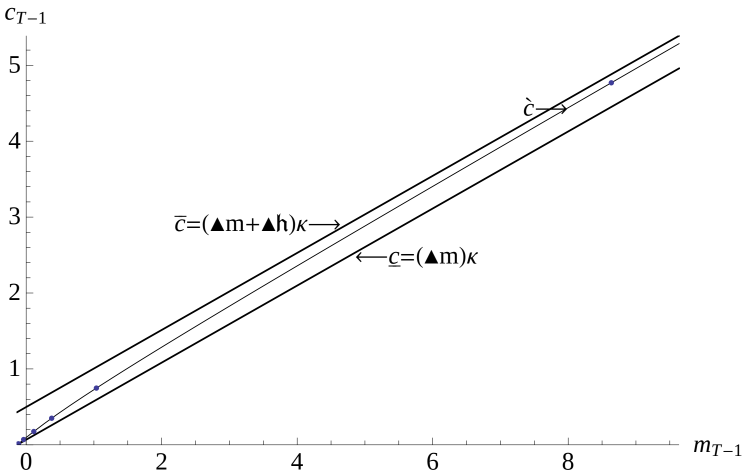
Illustration (6 Gridpoints)



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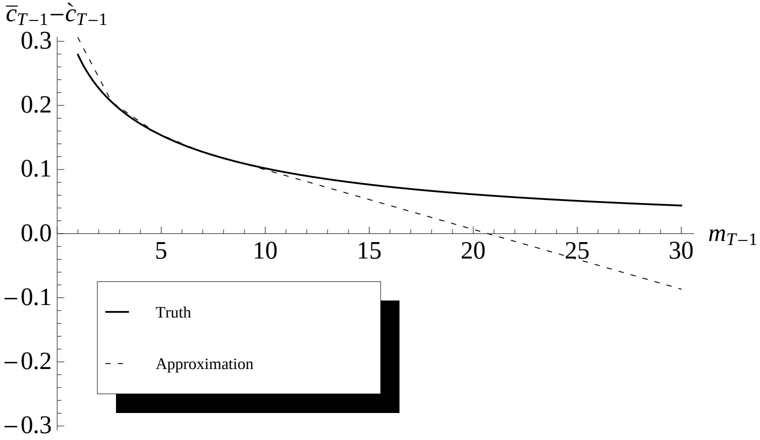
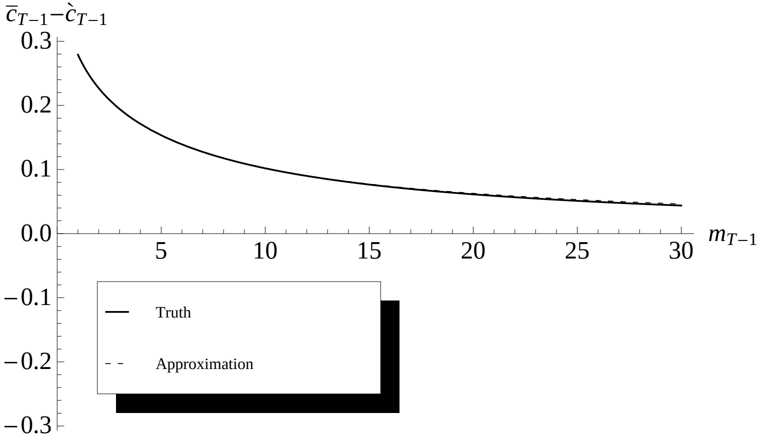


Solution (Same 6 Gridpoints!)



Definitions and Solutions for Perfect Foresight Problem

$$\underline{\kappa}_t \; : \; \text{MPC for perfect foresight problem in period } t \tag{16}$$

$$\bar{\mathfrak{h}}_t \; : \; \text{end-of-period-}t \text{ human wealth for `optimist'} \tag{17}$$

$$\underline{\mathfrak{h}}_t \; : \; \text{end-of-period-}t \text{ human wealth for `pessimist'} \tag{18}$$

- Pessimist's solution:

$$\underline{c}_t(m) = \underbrace{(m + \underline{\mathfrak{h}}_t)}_{\equiv \blacktriangle m} \underline{\kappa}_t \tag{19}$$

- Optimist's solution:

$$\bar{c}_t(m) = (m + \bar{\mathfrak{h}}_t)\underline{\kappa}_t \tag{20}$$

$$= \underline{c}_t(m) + \underbrace{(\bar{\mathfrak{h}}_t - \underline{\mathfrak{h}}_t)}_{\equiv \blacktriangle \mathfrak{h}} \underline{\kappa}_t \tag{21}$$

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$$
\begin{array}{rcl}
\blacktriangle m_t \underline{\kappa}_t < & \mathrm{c}_t(\underline{m}_t + \blacktriangle m_t) & < (\blacktriangle m_t + \blacktriangle \mathfrak{h}_t)\underline{\kappa}_t \\
-\blacktriangle m_t \underline{\kappa}_t > & -\mathrm{c}_t(\underline{m}_t + \blacktriangle m_t) & > -(\blacktriangle m_t + \blacktriangle \mathfrak{h}_t)\underline{\kappa}_t \\
\blacktriangle \mathfrak{h}_t \underline{\kappa}_t > & \bar{\mathrm{c}}_t(\underline{m}_t + \blacktriangle m_t) - \mathrm{c}_t(\underline{m}_t + \blacktriangle m_t) & > 0 \\
1 > & \underbrace{\left(\frac{\bar{\mathrm{c}}_t(\underline{m}_t + \blacktriangle m_t) - \mathrm{c}_t(\underline{m}_t + \blacktriangle m_t)}{\blacktriangle \mathfrak{h}_t \underline{\kappa}_t} \right)}_{\equiv \hat{\koppa}_t} & > 0
\end{array}
$$

Defining $\mu_t = \log \blacktriangle m_t$ (which can range from $-\infty$ to ∞), the object in the middle of the last inequality is

$$
\hat{\koppa}_t(\mu_t) \equiv \left(\frac{\bar{\mathrm{c}}_t(\underline{m}_t + e^{\mu_t}) - \mathrm{c}_t(\underline{m}_t + e^{\mu_t})}{\blacktriangle \mathfrak{h}_t \underline{\kappa}_t} \right), \tag{22}
$$

and we now define

$$
\hat{\chi}_t(\mu_t) = \log \left(\frac{1 - \hat{\koppa}_t(\mu_t)}{\hat{\koppa}_t(\mu_t)} \right) \tag{23}
$$

Approximating Precautionary Saving

$$\hat{c}_t = \bar{c}_t - \overbrace{\left(\frac{1}{1+\exp(\hat{\chi}_t)}\right)}^{=\hat{\varrho}_t} \blacktriangle \mathfrak{h}_t \underline{\kappa}_t. \tag{25}$$

In the limit as μ approaches either ∞ or $-\infty$ this gives the 'right' answer.

Solving for the Value Function

Given analytical object

$$\mathbb{C}_t^T \equiv \mathrm{PDV}_t^T(c)/c_t$$

The perfect foresight value function is:

$$\mathrm{v}_t(m_t) = \mathrm{u}(c_t)\mathbb{C}_t^T \tag{26}$$

which can be transformed as

$$\begin{aligned} \Lambda_t &\equiv ((1-\rho)\mathrm{v}_t)^{1/(1-\rho)} \\ &= c_t(\mathbb{C}_t^T)^{1/(1-\rho)} \end{aligned}$$

In the limit as μ approaches either ∞ or $-\infty$ this gives the 'right' answer.

Approximate Inverted Value Function

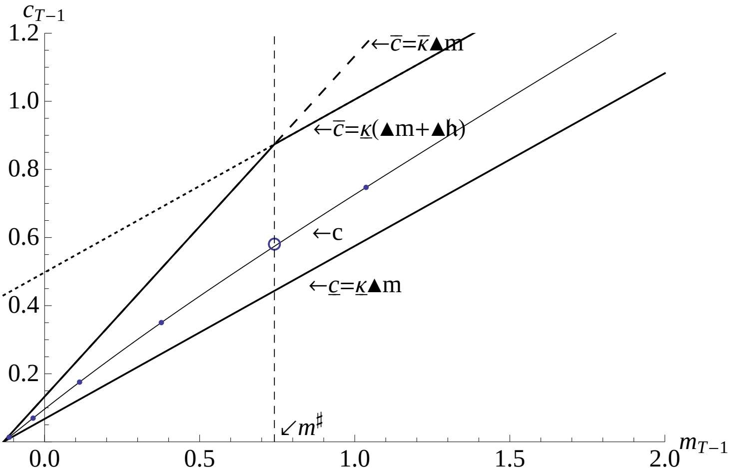
Same steps as for consumption function, because

$$\underline{\mathrm{v}}_t < \mathrm{v}_t < \bar{\mathrm{v}}_t \tag{27}$$

Produce

$$\grave{\mathrm{v}}_t(m) \quad = \quad \mathrm{u}(\grave{\Lambda}_t(m)) \tag{28}$$

A Tighter Limit: $c_t(m) < \min[\bar{\kappa}_t \blacktriangle m_t, \bar{c}_t(m_t)]$



The Tighter Upper Bound

we want to construct a consumption function for $m_t \in (\underline{m}_t, m_t^{\#}]$ that respects the tighter upper bound:

$$
\begin{array}{rcl}
\blacktriangle m_t \underline{\kappa}_t < & \mathrm{c}_t(\underline{m}_t + \blacktriangle m_t) & < \bar{\kappa}_t \blacktriangle m_t \\
\blacktriangle m_t(\bar{\kappa}_t - \underline{\kappa}_t) > & \bar{\kappa}_t \blacktriangle m_t - \mathrm{c}_t(\underline{m}_t + \blacktriangle m_t) & > 0 \\
1 > & \left(\frac{\bar{\kappa}_t \blacktriangle m_t - \mathrm{c}_t(\underline{m}_t + \blacktriangle m_t)}{\blacktriangle m_t(\bar{\kappa}_t - \underline{\kappa}_t)}\right) & > 0.
\end{array}
$$

Again defining $\mu_t = \log \blacktriangle m_t$, the object in the middle of the inequality is

$$
\check{\varphi}_t(\mu_t) \equiv \frac{\bar{\kappa}_t - \mathrm{c}_t(\underline{m}_t + e^{\mu_t})e^{-\mu_t}}{\bar{\kappa}_t - \underline{\kappa}_t}.
$$

Combining Old and New Functions

$$\begin{aligned}
\mathbf{1}_{\text{Lo}}(m) &= 1 \text{ if } m \leq \bar{\check{m}}_t^{\#} \\
\mathbf{1}_{\text{Mid}}(m) &= 1 \text{ if } \quad \bar{\check{m}}_t^{\#} < m < \underline{\hat{m}}_t^{\#} \\
\mathbf{1}_{\text{Hi}}(m) &= 1 \text{ if } \qquad \underline{\hat{m}}_t^{\#} \leq m
\end{aligned}$$

we can define a well-behaved approximating consumption function

$$\grave{\mathrm{c}}_t = \mathbf{1}_{\text{Lo}}\grave{\check{\mathrm{c}}}_t + \mathbf{1}_{\text{Mid}}\grave{\tilde{\mathrm{c}}}_t + \mathbf{1}_{\text{Hi}}\grave{\hat{\mathrm{c}}}_t. \tag{29}$$

Extension To Case With Financial Risk Too

$$\log \mathbf{R}_{t+n} \sim \mathcal{N}(\mathsf{r} + \phi - \sigma_{\mathbf{r}}^2/2, \sigma_{\mathbf{r}}^2) \; \forall \; n > 0 \tag{30}$$

Merton (1969) and Samuelson (1969): Without labor income,

$$\kappa = 1 - \left(\beta \mathbb{E}_t[\mathbf{R}_{t+1}^{1-\rho}]\right)^{1/\rho} \tag{31}$$

and in this case the previous analysis applies once we substitute this MPC for the one that characterizes the perfect foresight problem without rate-of-return risk.

How Useful Is It?

When it works:

- Can approximate solution *globally* with very few gridpoints
- Can speed solution substantially

When Might It Not Work?

- Consumption With Nonconvexities
- Problems With No Perfect Foresight Solution
  - How Many Such Problems Have A Solution With Shocks?
- *Should* Work For Value Functions Fairly Generally

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  - How Many Such Problems Have A Solution With Shocks?
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When it works:

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