A Tractable Model of Buffer Stock Saving

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Abstract

In the spirit of Merton (1969) and Samuelson (1969), we present an analytically tractable model of the effects of nonfinancial risk on intertemporal choice. Our simple framework can be adopted in contexts where modelers have, until now, chosen not to incorporate serious nonfinancial risk because available methods did not readily yield transparent insights. Our model produces an intuitive formula for target assets, and we show how to analyze transition dynamics using a familiar Ramsey-style phase diagram. Despite its starkness, we argue that our model captures the key implications of nonfinancial risk for intertemporal choice.

Keywords risk, uncertainty, precautionary saving, buffer stock saving

JEL codes C61, D11, E24

PDF: http://econ.jhu.edu/people/ccarroll/papers/ctDiscrete.pdf
Web: http://econ.jhu.edu/people/ccarroll/papers/ctDiscrete/
Archive: http://econ.jhu.edu/people/ccarroll/papers/ctDiscrete.zip
(Contains Mathematica and Matlab code solving the model)

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1 Introduction

The Merton-Samuelson model of portfolio choice is the foundation for the vast literature analyzing financial risk,[1] not because it offers conclusions that cannot be obtained from other frameworks,[2] but because it is easy to use and its key insights emerge in a way that is natural, transparent, and intuitive — in a word, the Merton-Samuelson model is tractable.

Unfortunately, nonfinancial risk[3] (which is much more important than financial risk for most households)[4] has proven more difficult to analyze. Of course, a large and impressive numerical literature has carefully computed the theoretical effects of realistically calibrated nonfinancial risks in a variety of contexts.[6] But because a formidable investment in human capital is required to learn the computational methods necessary to solve such models, much of the economic literature (and much graduate-level instruction) has dodged the question of how nonfinancial risk influences choices, by assuming perfect insurance markets or perfect foresight or risk neutrality or quadratic utility or Constant Absolute Risk Aversion, or by attempting only to match aggregate risks (which are orders of magnitude smaller than idiosyncratic risks). These approaches rob the question of its essence, either by assuming that markets transform nonfinancial risk into financial risk or by making implausible assumptions in order to reach the implausible conclusion that decisions are largely or entirely unaffected by such risk.[7]

Our paper's contribution is to offer a tractable model that captures the key qualitative features (and many quantitative features) of realistic models of the optimal response to nonfinancial risk, but without the customary technical difficulties. The model is a natural extension of the no-risk perfect foresight framework. Its solution is characterized by simple, intuitive equations and we show how the model's results can be analyzed using a phase diagram analysis that will be familiar to every student of the canonical Ramsey growth model taught in graduate school.

The trick that yields tractability is to distill all nonreturn risk into a stark and simple possibility: A one-time uninsurable permanent loss in nonfinancial income. For an individual's decision problem, this may be directly interpreted as the risk of a permanent transition into unemployment (or disability, or retirement). Our view is that the consumer's response to this single, tractable risk can capture most of the substantive essence (that is, the qualitative

[1]Merton (1969); Samuelson (1969); see Sethi and Thompson (2000) for an overview and extensions.

[2]Merton and Samuelson cite the pioneering work of Markowitz (1959), Tobin (1958), and Phelps (1960) among others whose work had already contained the key qualitative insights.

[3]By which we mean risk that is both imperfectly insurable and imperfectly correlated with financial wealth.

[4]Nonfinancial income typically accounts for the great majority of most households' total income, while risky assets like stocks represent a relatively small percentage of total wealth. Furthermore, stock returns are poorly correlated with the return on the index portfolio of all the assets in the economy.[5] Idiosyncratic risk is even more poorly spanned by market risk and is several orders of magnitude greater than aggregate risk.

[6]The literature on in heterogeneous-agents macroeconomic models includes, among many others, Carroll (1992), Aiyagari (1994), and Krusell and Smith (1998), with roots that go back to Schechtman and Escudero (1977) and Bewley (1977), with other important contributions by Clarida (1987), Zeldes (1989), and Chamberlain and Wilson (2000).

[7]The case of CARA utility with only labor income risk is included here because Carroll and Kimball (1996) show that it is a knife-edge case that is unrepresentative of the broader effects of uncertainty (notably, it fails to exhibit the consumption concavity that holds for virtually every other combination of assumptions); indeed, the addition of rate-of-return risk renders the optimal consumption function concave even under CARA utility. (The other traditional objection is that the optimal consumption plan under CARA utility generally involves setting consumption to a negative value in some states of the world; it is hard to think of a plausible economic interpretation of negative consumption.)

and quantitative implications) of the results obtained in the numerical literature under more realistic but more complex assumptions about income dynamics.

The same framework could be interpreted to apply in other contexts as well. For instance, the risk faced by a country whose exports are dominated by a commodity whose price might collapse permanently (e.g., oil exporters, if cold fusion had worked).[8]

The optimal response to such a risk is to aim to accumulate a buffer stock of precautionary assets, as a form of "self-insurance." The existing literature has found the numerical value of the target under specific parametric assumptions, but has struggled to present an intuitive picture of the determinants of that target. In contrast, we are able to derive an analytical formula for the target level of wealth, and show transparently how the precautionary motive interacts with the other saving motives that have been well understood since Irving Fisher (1930)'s work: The income, substitution, and human wealth effects.

The literature's principal alternative to numerical methods for analysis of precautionary behavior has been to attempt to approximate the nonlinear part of the logarithmic consumption Euler equation (the part that drives precautionary saving and the target level of assets). The Euler equation approach, however, has foundered because the higher-level (beyond first-order) components of the Euler equation are *endogenous* in a way that has proven difficult to master. Thanks to our model's tractability, we are able to derive a simple expression that shows how the familiar perfect-foresight consumption Euler equation is modified by the presence of a one-shot risk. Specifically, our equation shows that the effect of the risk on consumption growth is related to the probability of the risky event, to its magnitude, to the consumer's degree of risk aversion, and to the consumer's wealth. We obtain an exact analytical expression (not a log-linearized one) for the combined value of the higher-order terms at the target. With this expression in hand, the intuition comes into clear focus, and the problems that have bedeviled the literature can be plainly articulated and understood.

Our hope is that tractability of this kind will eventually allow a model like ours to become the standard reference point to which more specialized models can be compared in much of economics, replacing the perfect foresight, certainty equivalent, or perfect markets models that are currently so widely used because of their own tractability. A further ambition is that even the specialist literatures like heterogeneous-agents macroeconomics may find ours a useful 'toy model' with which to exposit more clearly some of the subtle points that authors in those literatures find difficult to communicate simply by appealing to results from numerical simulations.[9]

[8] The model could even be interpreted as applying to the behavior of a firm controlled by a risk-neutral manager, so long as the collapse of a line of business could have the effect of reducing the firm's collateral value and therefore increasing its cost of external finance *a la* Bernanke, Gertler, and Gilchrist (1996). In this case, the convex increase in borrowing rates when cash drops plays the same role as the convexity of the marginal utility function for a consumer; see also Berk, Stanton, and Zechner (2009) for an argument that senior firm managers are not risk neutral even if shareholders are, because poor performance under their tenure will reduce their own future employment opportunities. A firm controlled by such managers may behave very much like a risk-averse household.

[9] In order to assist authors in modifying our model for other purposes, we have constructed a public archive that contains Matlab and Mathematica programs that produce all the results and figures reported in this paper, along with some other examples of uses to which the model could be put. The archive is available on the first author's website.

2 The Decision Problem

For concreteness, we analyze the problem of an individual consumer facing a labor income risk. Other interpretations (like the ones mentioned in the introduction) are left for future work or other authors. We couch the problem in discrete time, but in most cases we provide the logarithmic approximations that will correspond to the exact solution to the corresponding problem in continuous time.[10]

The aggregate wage rate, W_t, grows by a constant factor G from the current time period to the next, reflecting exogenous productivity growth:

$$\mathsf{W}_{t+1} = \mathsf{G}\mathsf{W}_t. \tag{1}$$

The interest rate is exogenous and constant (the economy is small and open); the interest factor is denoted R.[11] Define $\boldsymbol{m}$ as market resources (financial wealth plus current income), $\boldsymbol{a}$ as end-of-period assets after all actions have been accomplished (specifically, after the consumption decision), and $\boldsymbol{b}$ as bank balances before receipt of labor income. Individuals are subject to a dynamic budget constraint (DBC) that can be decomposed into the following elements:

$$\boldsymbol{a}_t = \boldsymbol{m}_t - \boldsymbol{c}_t \tag{2}$$

$$\boldsymbol{b}_{t+1} = \mathsf{R}\boldsymbol{a}_t \tag{3}$$

$$\boldsymbol{m}_{t+1} = \boldsymbol{b}_{t+1} + \ell_{t+1}\mathsf{W}_{t+1}\xi_{t+1} \tag{4}$$

where ℓ measures the consumer's labor productivity (hours of work for an employed consumer are assumed to be exogenous and fixed) and ξ is a dummy variable indicating the consumer's employment state: Everyone in this economy is either employed ($\xi = 1$, a state indicated by the letter 'e') or unemployed ($\xi = 0$, a state indicated by 'u'). Thus, labor income is zero for unemployed consumers.[12]

2.1 The Unemployed Consumer's Problem

There is no way out of unemployment; once an individual becomes unemployed, that individual remains unemployed forever, $\xi_t = 0 \Longrightarrow \xi_{t+1} = 0\ \forall\ t$. Consumers have a CRRA utility function $\mathrm{u}(\bullet) = \bullet^{1-\rho}/(1-\rho)$, with $\rho > 1$, and they discount future utility geometrically by β per period. We show below that the simplicity of the unemployed consumer's behavior is what makes employed consumer's problem tractable. The solution to the unemployed consumer's optimization problem is simply:[13]

$$\boldsymbol{c}_t^u = \kappa^u \boldsymbol{b}_t, \tag{5}$$

[10] See Toche (2005) for an explicit but brief treatment of a closely related model in continuous time.

[11] General equilibrium is not much more difficult; it requires specifying a production function and finding the level of capital for which the optimal level of saving is zero (net of depreciation). Little further insight is obtained, while many potential extra sources of confusion are added.

[12] This is without loss of generality. We could allow for unemployment insurance by modifying the value of ξ associated with unemployment. On this, see also footnote 4 in Toche (2005).

[13] This is a standard result, which follows from the first-order condition and the budget constraint:

$$\mathrm{u}'(\boldsymbol{c}_t^u) = \mathsf{R}\beta\mathrm{u}'(\boldsymbol{c}_{t+1}^u) \Rightarrow (\boldsymbol{c}_t^u)^{-\rho} = \mathsf{R}\beta\left(\boldsymbol{c}_{t+1}^u\right)^{-\rho} \Rightarrow \boldsymbol{c}_{t+1}^u = (\mathsf{R}\beta)^{1/\rho}\boldsymbol{c}_t^u.$$

where κ^u is the 'marginal propensity to consume' out of total wealth (MPC), which for the unemployed consists in balances $\boldsymbol{b}$ only.

2.1.1 Parameter Restrictions

Table 1 summarizes our notation and should serve as a useful guide to the reader. We follow the terminology in Carroll (2011), where a detailed discussion of the concepts is provided. The marginal propensity to consume out of total wealth κ^u is related to the 'return patience factor' Þ_{R} as follows[14]

$$\kappa^u = 1 - \text{Þ}_{\mathsf{R}}, \quad \text{Þ}_{\mathsf{R}} \equiv \mathsf{R}^{-1}(\mathsf{R}\beta)^{1/\rho}. \tag{6}$$

The MPC for the problem without risk is strictly below the MPC for the problem with risk (Carroll and Kimball, 1996). We impose a 'return impatience condition' (RIC),

$$\text{Þ}_{\mathsf{R}} < 1 \Rightarrow \kappa^u > 0, \tag{7}$$

The interpretation is that the consumer must not be so patient that a boost to total wealth would fail to boost current consumption.[15] An alternative (equally correct) interpretation is that the condition guarantees that the present discounted value (PDV) of consumption for the unemployed consumer remains finite. Þ_{R} is the 'return patience factor' because it defines desired perfect-foresight consumption growth *relative* to the rate of return R. We define the 'return patience rate' as the lower-case version:

$$\text{þ}_{\mathsf{r}} \equiv \log \text{Þ}_{\mathsf{R}} \approx \text{Þ}_{\mathsf{R}} - 1 = -\kappa^u.$$

For short, we will sometimes say that a consumer is 'return impatient' (or, 'the RIC holds') if $\text{Þ}_{\mathsf{R}} < 1$ or if $\text{þ}_{\mathsf{r}} < 0$ or if $\kappa^u > 0$, all three conditions being equivalent.[16] A consumer who is return impatient is someone who will be spending enough to make the ratio of consumption to total wealth decline over time.

The return patience factor can be compared to the 'absolute patience factor'

$$\text{Þ} \;=\; (\mathsf{R}\beta)^{1/\rho} \tag{8}$$

which is the growth factor for consumption in the no-risk perfect foresight model. We say that a consumer is 'absolutely impatient' if

$$\text{Þ} \;<\; 1, \tag{9}$$

in which case the consumer will choose to spend more than the amount that would permit constant consumption; such a consumer's absolute level of wealth declines over time, and

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Consumption grows at the geometric rate $(\mathsf{R}\beta)^{1/\rho}$. The present discounted value of consumption at time t must equal total wealth, so that

$$\sum_{i=0}^{\infty} R^{-i} \boldsymbol{c}^u_{t+i} = \sum_{i=0}^{\infty} R^{-i} (\mathsf{R}\beta)^{i/\rho} \boldsymbol{c}^u_t = \sum_{i=0}^{\infty} (1-\kappa^u)^i \boldsymbol{c}^u_t = \boldsymbol{c}^u_t/\kappa^u = \boldsymbol{b}_t.$$

[14] Þ is the Old English letter 'thorn'; its modern equivalent is the digraph 'th.'

[15] 'Pathologically patient' consumers who do not satisfy this condition would hoard any increase in wealth in order to enable even more extra consumption in the distant future.

[16] Throughout, we casually treat logs of factors like Þ_{R} as equivalent to the level minus 1; that is, we treat expressions like $\log \text{Þ}_{\mathsf{R}}$ and $\text{Þ}_{\mathsf{R}} - 1$ as interchangeable, which is an appropriate approximation so long as the factor is 'close' to 1. In practice, the approximation is very good.

therefore consumption itself declines, since consumption is proportional to total wealth. Analogously to (8), we define the absolute patience rate as

$$þ \equiv \log Þ \approx \rho^{-1}(\mathsf{r} - \vartheta). \tag{10}$$

2.2 The Employed Consumer's Problem

The consumer's preferences are the same in the employment and unemployment states; only exposure to risk differs.

2.2.1 A Human-Wealth-Preserving Spread in Unemployment Risk

A consumer who is *employed* in the current period has $\xi_t = 1$; if this person is still employed next period ($\xi_{t+1} = 1$), market resources will be:

$$\boldsymbol{m}^e_{t+1} = (\boldsymbol{m}^e_t - \boldsymbol{c}^e_t)\mathsf{R} + \mathsf{W}_{t+1}\ell_{t+1}. \tag{11}$$

However, there is no guarantee that the consumer will remain employed: Employed consumers face a constant risk $\mho$ of becoming unemployed. It is convenient to define $\not{\mho} \equiv 1 - \mho$, the complementary probability that a consumer does *not* become unemployed. We assume that ℓ grows by a factor $\not{\mho}^{-1}$ every period,

$$\ell_{t+1} = \ell_t/\not{\mho}, \tag{12}$$

because under this assumption, for a consumer who remains employed, labor income will grow by factor $\Gamma = \mathsf{G}/\not{\mho}$, so that the *expected* labor income growth factor for employed consumers is the same G as in the no-risk perfect foresight case:

$$\begin{aligned} \mathbb{E}_t[\mathsf{W}_{t+1}\ell_{t+1}\xi_{t+1}] &= \left(\frac{\ell_t \mathsf{G} \mathsf{W}_t}{\not{\mho}}\right)\left((\mho \times 0 + \not{\mho} \times 1)\right) \\ \Rightarrow \frac{\mathbb{E}_t[\mathsf{W}_{t+1}\ell_{t+1}\xi_{t+1}]}{\mathsf{W}_t\ell_t} &= \mathsf{G} \end{aligned}$$

implying that an increase in $\mho$ is a pure increase in risk with no effect on the PDV of expected labor income – a mean-preserving spread in the intertemporal sense. Thus, any change in behavior that results from a change in $\mho$ will be cleanly interpretable as reflecting an effect of uncertainty rather than the effect of a change in human wealth.

2.2.2 First Order Optimality Condition

The usual steps lead to the standard consumption Euler equation. Using $i \in \{e, u\}$ to stand for the two possible states,

$$\begin{aligned} \mathrm{u}'(\boldsymbol{c}^e_t) &= \mathsf{R}\beta\,\mathbb{E}_t\left[\mathrm{u}'(\boldsymbol{c}^i_{t+1})\right] \\ \Rightarrow 1 &= \mathsf{R}\beta\,\mathbb{E}_t\left[\left(\frac{\boldsymbol{c}^i_{t+1}}{\boldsymbol{c}^e_t}\right)^{-\rho}\right]. \end{aligned} \tag{13}$$

Henceforth nonbold variables will be used to represent the bold equivalent divided by the level of permanent labor income for an employed consumer, e.g. $c^e_t = \boldsymbol{c}^e_t/(\mathsf{W}_t\ell_t)$; thus we can rewrite the consumption Euler equation as:

$$
\begin{aligned}
1 &= \mathsf{R}\beta\,\mathbb{E}_t\left[\left(\frac{c^i_{t+1}\mathsf{W}_{t+1}\ell_{t+1}}{c^e_t\mathsf{W}_t\ell_t}\right)^{-\rho}\right] \\
\Rightarrow &= \mathsf{R}\beta\,\mathbb{E}_t\left[\left(\frac{c^i_{t+1}}{c^e_t}\Gamma\right)^{-\rho}\right] \\
\Rightarrow &= \Gamma^{-\rho}\mathsf{R}\beta\left\{(1-\mho)\left(\frac{c^e_{t+1}}{c^e_t}\right)^{-\rho} + \mho\left(\frac{c^u_{t+1}}{c^e_t}\right)^{-\rho}\right\},
\end{aligned}
\tag{14}
$$

where the term in braces is a probability-weighted average of the growth rates of marginal utility in the case where the consumer remains employed (the first term) and the case where the consumer becomes unemployed (the second term).

2.2.3 Analysis and Intuition of Consumption Growth Path

It will be useful now to define a 'growth patience factor' $\text{Þ}_\Gamma = (\mathsf{R}\beta)^{1/\rho}/\Gamma$ which is the factor by which the consumption-income ratio c^e would grow in the absence of labor income risk. With this notation, (14) can be written as:

$$
\begin{aligned}
1 &= \text{Þ}_\Gamma^{\rho}\left(\frac{c^e_{t+1}}{c^e_t}\right)^{-\rho}\left\{1-\mho+\mho\left[\left(\frac{c^u_{t+1}}{c^e_t}\right)\left(\frac{c^e_t}{c^e_{t+1}}\right)\right]^{-\rho}\right\} \\
\Rightarrow \left(\frac{c^e_{t+1}}{c^e_t}\right) &= \text{Þ}_\Gamma\left\{1+\mho\left[\left(\frac{c^e_{t+1}}{c^u_{t+1}}\right)^{\rho}-1\right]\right\}^{1/\rho}.
\end{aligned}
\tag{15}
$$

To understand (15), it is useful to consider an approximation. Define $\nabla_{t+1} \equiv \left(\frac{c^e_{t+1}-c^u_{t+1}}{c^u_{t+1}}\right)$, the proportion by which consumption next period would drop in the event of a transition into unemployment; we refer to this loosely as the size of the 'consumption risk.' Define ω, the 'excess prudence' factor, as $\omega = (\rho-1)/2$.[17] Applying a Taylor approximation to (15) (see appendix A) yields:

$$
\left(\frac{c^e_{t+1}}{c^e_t}\right) \approx (1+\mho(1+\omega\nabla_{t+1})\nabla_{t+1})\,\text{Þ}_\Gamma \tag{16}
$$

which simplifies further in the logarithmic utility case (since $\rho = 1$ and thus $\omega = 0$),

$$
\left(\frac{c^e_{t+1}}{c^e_t}\right) \approx (1+\mho\nabla_{t+1})\,\text{Þ}_\Gamma. \tag{17}
$$

The approximations in (16) or (17) capture the essence of equation (15). As a consequence of missing insurance markets, consumption growth depends on the employment outcome;[18] consumption if employed next period c^e_{t+1} is greater than consumption if unemployed next period c^u_{t+1}, so that ∇_{t+1} is positive. In the limit case, as unemployment risk $\mho$ vanishes, so

[17] It is 'excess' in the sense of exceeding the benchmark case of logarithmic utility which corresponds to $\rho = 1$. Logarithmic utility is often viewed as a lower bound on the possible degree of risk aversion.

[18] Markets are incomplete by assumption. The no-slavery provisions of the U.S. Constitution prohibit even indentured servitude, providing a moral hazard explanation for why this insurance market should be missing. Adverse selection arguments provide an even better explanation.

does consumption risk ∇,[19] and thus c^e_{t+1}/c^e_t approaches Þ_Γ. Equation (16) thus shows that the presence of unemployment risk boosts consumption growth by an amount proportional to the probability of becoming unemployed $\mho$ multiplied by a factor that is increasing in the amount of consumption risk ∇. In the logarithmic case, equation (17) shows that the precautionary boost to consumption growth is directly proportional to the size of the consumption risk.

The effect of risk on saving is transparent. For a given value of m^e_t, risk has no effect on the PDV of future labor income and human wealth, but the larger is $\mho$, the faster consumption growth must be, as equation (16) shows. For consumption growth to be faster while keeping the PDV constant, the *level* of *current* c^e must be lower. Thus, the introduction of a risk of unemployment $\mho$ induces a (precautionary) increase in saving.

In the (persuasive) case where $\rho > 1$, (16) implies that a consumer with a higher degree of prudence (larger ρ and therefore larger ω) will anticipate greater consumption growth. This reflects the greater precautionary saving motive induced by a higher degree of prudence.

To compute the steady state of the model, we must find the $\Delta c^e_{t+1} = 0$ and $\Delta m^e_{t+1} = 0$ loci. Consider a consumer who is unemployed in period $t+1$. Dividing both sides of (4) by $\mathsf{W}_{t+1}\ell_{t+1}$ yields $m^u_{t+1} = b^u_{t+1} = (m^e_t - c^e_t)\mathcal{R}$ (where $\mathcal{R} \equiv \mathsf{R}/\Gamma$). Substituting $c^e_{t+1} = c^e_t$ and $c^u_{t+1} = \kappa^u m^u_{t+1}$ into (15) yields:

$$\begin{aligned} 1 &= \left\{1 + \mho\left[\left(\frac{c^e_{t+1}}{\kappa^u m^u_{t+1}}\right)^\rho - 1\right]\right\}\text{Þ}^\rho_\Gamma \\ \Rightarrow \frac{c^e_{t+1}}{(m^e_t - c^e_t)\mathcal{R}\kappa^u} &= \left(\frac{\text{Þ}^{-\rho}_\Gamma - \cancel{\mho}}{\mho}\right)^{1/\rho} \equiv \Pi \\ \Rightarrow c^e_{t+1} &= (m^e_t - c^e_t)\mathcal{R}\kappa^u\Pi \end{aligned} \tag{18}$$

where the expression $m^u_{t+1} = (m^e_t - c^e_t)\mathcal{R}$ has been used in the second line.

We now turn to parameter restrictions necessary to guarantee a positive steady state.

2.2.4 *Parameter Restrictions (Continued)*

Consider equation (18). We know that $m^e_t > c^e_t > 0$ because, with CRRA preferences, zero consumption carries an infinite penalty, implying that a consumer facing the risk of perpetual unemployment will never borrow. Since we have assumed $\kappa^u > 0$ (the RIC), steady-state consumption is positive only if Π is positive; so we impose the condition

$$\text{Þ}_\Gamma < (1 - \mho)^{-1/\rho}. \tag{19}$$

In the limit as $\mho$ approaches zero, (19) therefore reduces to a requirement that the growth patience factor Þ_Γ be less than one,

$$\text{Þ}_\Gamma < 1. \tag{20}$$

Following Carroll (2011), we call condition (20) the 'perfect foresight growth impatience condition' (PF-GIC), by analogy with the 'return impatience condition' (7) imposed earlier. If $\mho = 0$, the consumer knows with perfect certainty what will happen in the future; the PF-

[19] While unemployment risk $\mho$ is exogenous, consumption risk ∇ is endogenous.

GIC ensures that such a consumer facing no risk would be sufficiently impatient to choose a wealth-to-permanent-income ratio that would be falling over time.[20,21]

Using $\gamma \equiv \log \Gamma$, we similarly define the corresponding 'growth patience rate'

$$þ_\gamma \quad \equiv \quad \log \text{Þ}_\Gamma \tag{21}$$

so that the PF-GIC can also be written

$$þ_\gamma \approx \rho^{-1}(\mathsf{r} - \vartheta) - \gamma \quad < \quad 0. \tag{22}$$

2.2.5 *Why Increased Unemployment Risk Increases Growth Impatience*

Under the maintained assumption that the RIC holds, the (generalized) GIC in (19) slackens (becomes easier to satisfy) as unemployment risk rises because, with relative risk aversion $\rho > 1$, an increase in $\mho$ reduces the right-hand side of (19). This occurs for two reasons. First, an increase in $\mho$ is like a reduction in the future downweighting factor (that is, a decrease in patience), conditional on the consumer remaining employed.[22] Second, an increase in $\mho$ slackens the GIC because our mean-preserving-spread assumption requires that labor productivity growth be adjusted so that the value of human wealth is independent of $\mho$ – see (12). The higher $\mho$ is, the faster growth is *conditional on remaining employed*. As income growth (conditional on employment) increases, the continuously-employed (lucky) consumer is effectively more 'impatient' in the sense of desiring consumption growth below employment-conditional income growth.[23]

The fact that the GIC is easier to satisfy as $\mho$ increases means that if the PF-GIC (20) is satisfied, then (19) must be satisfied.

2.2.6 *The Target Level of m^e*

We first characterize the steady state. Setting $\Delta c^e_{t+1} = 0$ and $\Delta m^e_{t+1} = 0$ yields, respectively

$$c^e_t \quad = \quad \left(\frac{\mathcal{R}\kappa^u \Pi}{1 + \mathcal{R}\kappa^u \Pi}\right) m^e_t \tag{23}$$

$$\quad = \quad (1 - \mathcal{R}^{-1}) m^e_t + \mathcal{R}^{-1}. \tag{24}$$

Equation (23) is obtained by imposing the RIC and the GIC on (18). Equation (24) follows from normalizing the DBC in (11).

The steady-state levels of m^e and c^e are the values for which both (24) and (23) hold. This system of two equations in two unknowns can be solved explicitly (see the appendix). For illustration, consider the special case of logarithmic utility ($\rho = 1$). The appendix shows that

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[20] The PF-GIC is a slightly stronger condition than is strictly necessary; the necessary condition is (19). However, the PF-GIC guarantees that the solution is well behaved as the risk vanishes, which lends itself to a more intuitive interpretation.

[21] In addition, the condition $m^e_t > 0$ suggests the condition $\mathcal{R}\kappa^u\Pi > 1 - \mathcal{R}$. This follows from (34) below. It is equivalent to $\Pi = c^e_{t+1}/c^u_{t+1} > ((1 - \mathcal{R})/\mathcal{R})/\kappa^u$, i.e. the marginal propensity to consume in the unemployment state must be sufficiently small. The appendix shows that this condition is automatically satisfied if the RIC and GIC are imposed, thus no additional restrictions are needed to guarantee positive wealth in the state of unemployment.

[22] While this effect is offset by an increase in the downweighting factor associated with the transition to the unemployed state, the RIC already guarantees that the PDV of consumption, income, and value remain finite for the unemployed consumer, who is therefore irrelevant.

[23] Note that neither of these effects of $\mho$ is precautionary.

an approximation of the target level of market resources is

$$\check{m}^e \approx 1 + \left(\frac{1}{(\gamma - \mathsf{r}) + \vartheta(1 + (\gamma + \vartheta - \mathsf{r})/\mho)}\right). \tag{25}$$

The GIC and the RIC together guarantee that the denominator of (25) is positive.

This expression encapsulates several of the key intuitions of the model. The human wealth effect of labor income growth (conditional upon remaining employed) is captured by the first γ term in the denominator; for any calibration for which the denominator is positive, increasing γ reduces the target level of wealth. This reflects the fact that a consumer who anticipates being richer in the future will choose to save less in the present, and the result of lower saving is smaller wealth. The human wealth effect of interest rates is correspondingly captured by the $-\mathsf{r}$ term, which goes in the opposite direction to the effect of income growth, because an increase in the rate at which future labor income is discounted constitutes a reduction in human wealth. An increase in the rate at which future utility is discounted, ϑ, reduces the target wealth level. Finally, a reduction in unemployment risk raises $(\gamma + \vartheta - \mathsf{r})/\mho$ and therefore reduces the target wealth level.[24,25]

Note that the different effects *interact* with each other, in the sense that the strength of, say, the human wealth effect of interest rates will vary depending on the values of the other parameters.

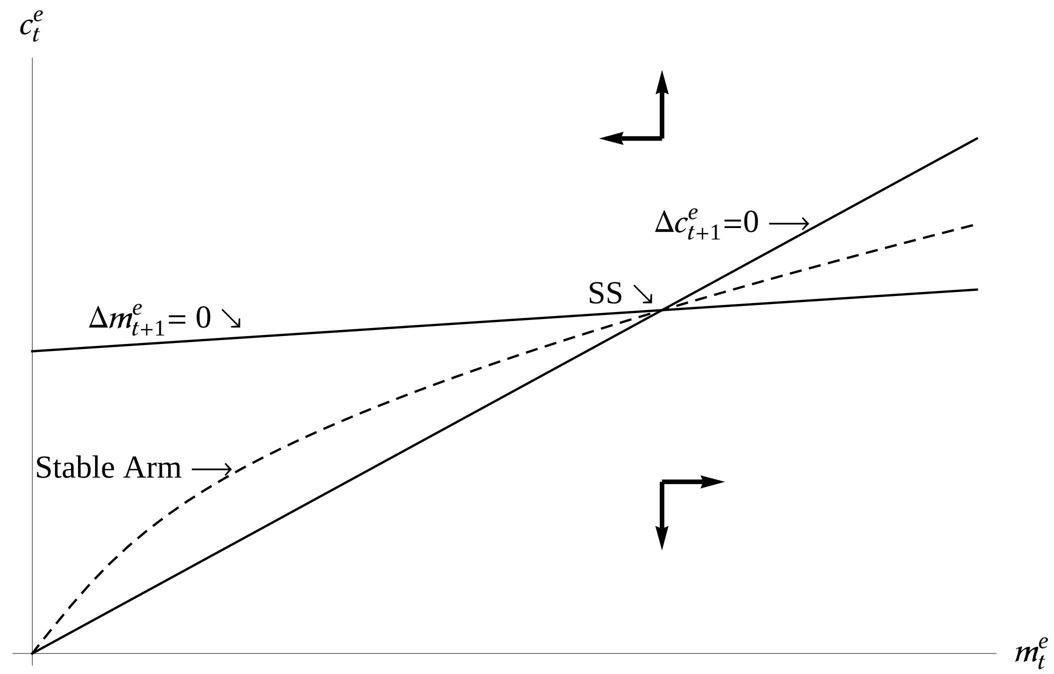
The assumption of log utility is implausible; empirical estimates from structural estimation exercises (e.g. Gourinchas and Parker (2002), Cagetti (2003), or the subsequent literature) typically find estimates considerably in excess of $\rho = 1$, and evidence from Barsky, Juster, Kimball, and Shapiro (1997) suggests that values of 5 or higher are not implausible. Another special case helps to illuminate how results change for $\rho > 1$. The appendix shows that, in the special case where $\vartheta = \mathsf{r}$, the target level of wealth is:

$$\check{m} \approx 1 + \left(\frac{1}{(\gamma - \mathsf{r}) + \vartheta(1 + (\gamma/\mho)(1 - (\gamma/\mho)\omega))}\right). \tag{26}$$

Compare the target level in (26) with (25). The key difference is that (26) contains an extra term involving ω, which measures the amount by which prudence exceeds the logarithmic benchmark. An increase in ω reduces the denominator of (26) and thereby raises the target level of wealth, just as would be expected from an increase in the intensity of the precautionary motive.

In the $\omega > 0$ case, the interaction effects between parameter values are particularly intense for the $(\gamma/\mho)^2$ term that multiplies ω; this implies, e.g., that a given increase in unemployment risk can have a greater effect on the target level of wealth for a consumer who is more prudent.

Figure 1 Phase Diagram



2.2.7 The Phase Diagram

Figure 1 presents the phase diagram of system (23)-(24) under our baseline parameter values. Since the employed consumer never borrows, market resources never fall below the value of current labor income, which is the value selected for the origin of the diagram.[26] An intuitive interpretation is that the $\Delta m_t^e = 0$ locus characterized by (24) shows how much consumption c_t^e would be required to leave resources m_t^e unchanged so that $m_t^e = m_t^e$.[27] Thus, any point below the $\Delta m_t^e = 0$ line would have consumption below the break-even amount, implying that wealth would rise. Conversely for points above $\Delta m_t^e = 0$. This is the logic behind the horizontal arrows of motion in the diagram: Above $\Delta m_t^e = 0$ the arrows point leftward, below $\Delta m_t^e = 0$ the arrows point rightward.

The intuitive interpretation of the $\Delta c_t^e = 0$ locus characterized by (23) is more subtle. Recall that expected consumption growth depends on the amount by which consumption would fall if the unemployment state were realized. At a given level of resources, the farther actual consumption (if employed) is below the break-even (sustainable) amount, the smaller the c_t^e/c_t^u ratio is, and therefore the smaller consumption growth is. Points below the $\Delta c_t^e = 0$ locus are

---

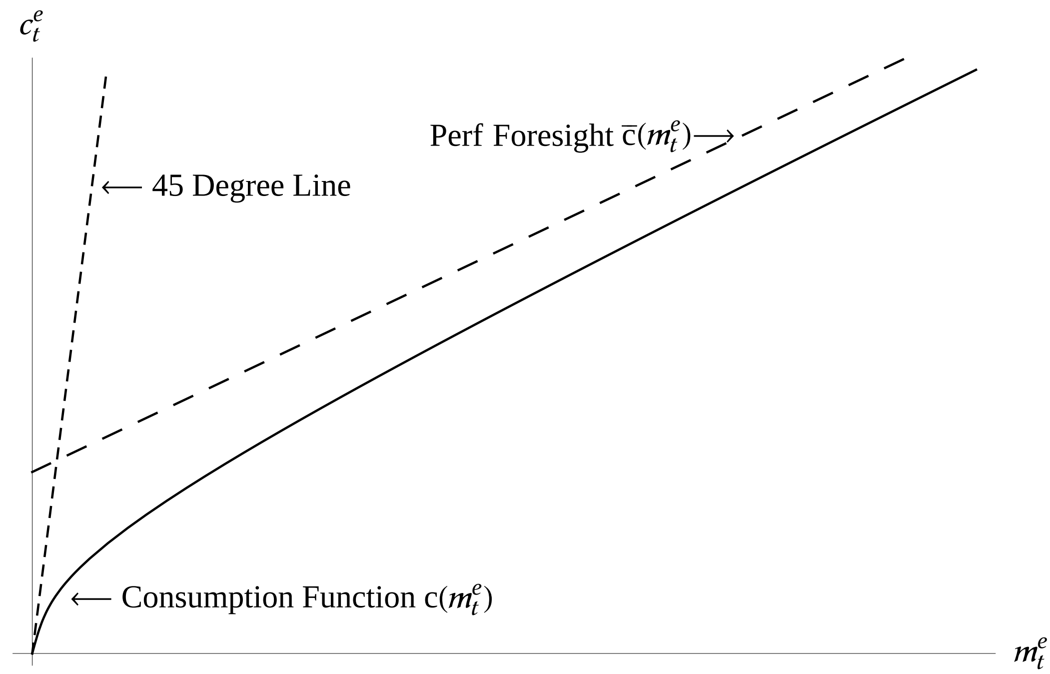
[24] $(\gamma + \vartheta - \mathrm{r}) > 0$ is guaranteed by (22) under log utility ($\rho = 1$).

[25] This discussion omits the fact that an increase in $\mho$ requires an adjustment to γ via (12) which induces a human wealth effect that goes in the opposite direction from the direct effect of uncertainty. For sufficiently large values of $\mho$, this effect can dominate the direct effect of uncertainty and the target wealth-to-income ratio declines. See the illustration below of the effects of an increase in uncertainty for further discussion. The same qualitative results may be found by a direct analysis of the partial derivatives of equation (34).

[26] Our parameterization is not indended to maximize realism, but instead to generate well-proportioned figures that illustrate the mechanisms of the model as clearly as possible. The parameter values are encapsulated in the file `ParametersBase.m` in the online archive.

[27] Some authors refer to $\Delta m_t^e = 0$ as the level of 'permanent income.' However, this definition differs from Friedman (1957)'s and, moreover, is a potential source of confusion with permanent labor income' $\mathsf{W}_t \ell_t$; we prefer to describe the locus as depicting the level of 'sustainable consumption.'

Figure 2 The Consumption Function for the Employed Consumer



associated with negative values of Δc_t^e. This is the logic behind the vertical arrows of motion in the diagram: Above $\Delta c_t^e = 0$ the arrows point upward, below $\Delta c_t^e = 0$ the arrows point downward.

2.2.8 The Consumption Function

Figure 2 shows the optimal consumption function $c(m)$ for an employed consumer (dropping the e superscript to reduce clutter). This is of course the stable arm of the phase diagram. Also plotted are the 45 degree line along which $c_t = m_t$ and

$$\bar{c}(m) \quad = \quad (m - 1 + h)\kappa^u, \tag{27}$$

where

$$h \quad = \quad \left(\frac{1}{1 - \mathsf{G}/\mathsf{R}} \right) \tag{28}$$

is the level of (normalized) human wealth. $\bar{c}(m)$ is the solution to the no-risk version of the model; it is depicted in order to introduce another property of the model: As wealth approaches infinity, the solution to the problem with risky labor income approaches the solution to the no-risk problem arbitrarily closely.[28,29]

The consumption function $c(m)$ is *concave*: The marginal propensity to consume $\boldsymbol{\kappa}(m) \equiv dc(m)/dm$ is higher at low levels of m because the intensity of the precautionary motive

[28] This limiting result requires that we impose the additional assumption $\Gamma < \mathsf{R}$, because the no-risk consumption function is not defined if $\Gamma \geq \mathsf{R}$.

[29] If the horizontal axis is stretched far enough, the two consumption functions appear to merge (visually), with the 45 degree line merging (visually) with the vertical axis. The current scaling is chosen both for clarity and to show realistic values of wealth.

increases as resources m decline.[30] The MPC is higher at lower levels of m because the *relaxation* in the intensity of the precautionary motive induced by a small increase in m (Kimball, 1990) is relatively larger for a consumer who starts with less than for a consumer who starts with more resources (Carroll and Kimball, 1996).

To see this important point, consider a counterfactual. Suppose the consumer were to spend all his resources in period t, i.e. $c_t = m_t$. In this situation, if the consumer were to become unemployed in the next period, he would then be left with resources $m^u_{t+1} = (m_t - c_t)\mathcal{R} = 0$, which would induce consumption $c^u_{t+1} = \kappa^u m^u_{t+1} = 0$, yielding negative infinite utility. A rational, optimizing consumer will always avoid such an eventuality, no matter how small its likelihood may be. Thus the consumer never spends all available resources.[31] This implication is illustrated in figure 2 by the fact that consumption function always remains below the 45 degree line.

2.2.9 *Expected Consumption Growth Is Downward Sloping in m^e*

Figure 3 illustrates some of the key points in a different way. It depicts the growth rate of consumption $\boldsymbol{c}^e_{t+1}/\boldsymbol{c}^e_t$ as a function of m^e_t.

Figure 3 illustrates the result that consumption growth is equal to what it would be in the absence of risk, plus a precautionary term; for algebraic verification, multiply both sides of (15) by Γ to obtain

$$\frac{\boldsymbol{c}^e_{t+1}}{\boldsymbol{c}^e_t}\Bigg)\Bigg(= (\mathsf{R}\beta)^{1/\rho}\left\{1 + \mho\left[\left(\frac{c^e_{t+1}}{c^u_{t+1}}\right)^{\rho} - 1\right]\right\}^{1/\rho}, \tag{29}$$

and observe that the contribution of the precautionary motive becomes arbitrarily large as $m_t \rightarrow 0$, because $c^u_{t+1} = m^u_{t+1}\kappa^u = (m_t - \mathrm{c}(m_t))\mathcal{R}\kappa^u$ approaches zero as $m_t \rightarrow 0$; that is, as resources m^e_t decline, expected consumption growth approaches infinity. The point where consumption growth is equal to income growth is at the target value of m^e.

2.2.10 *Summing Up the Intuition*

We are finally in position to get an intuitive understanding of how the model works and why a target wealth ratio exists. On the one hand, consumers are growth-impatient: It cannot be optimal for them to let wealth become arbitrarily large in relation to income. On the other hand, consumers have a precautionary motive that intensifies as the level of wealth falls. The two effects work in opposite directions. As resources fall, the precautionary motive becomes stronger, eventually offsetting the impatience motive. The point at which prudence becomes exactly large enough to match impatience defines the target wealth-to-income ratio.

It is instructive to work through a couple of comparative dynamics exercises. In doing so, we assume that all changes to the parameters are exogenous, unexpected, and permanent. Figure 4 depicts the effects of increasing the interest rate to $\grave{\mathsf{r}} > \mathsf{r}$. The no-risk consumption growth locus shifts up to the higher value $\grave{\text{þ}}_{\mathsf{r}} \approx \rho^{-1}(\grave{\mathsf{r}} - \vartheta)$, inducing a corresponding increase in the

[30] Carroll and Kimball (1996) prove that the consumption function must be concave for a general class of stochastic processes and utility functions – including almost all commonly-used model assumptions except for the knife-edge cases explicitly chosen to avoid concavity.

[31] This is an implication not just of the CRRA utility function used here but of the general class of continuously differentiable utility functions that satisfy the *Inada condition* $\mathrm{u}'(0) = \infty$.

Figure 3 Income and Consumption Growth

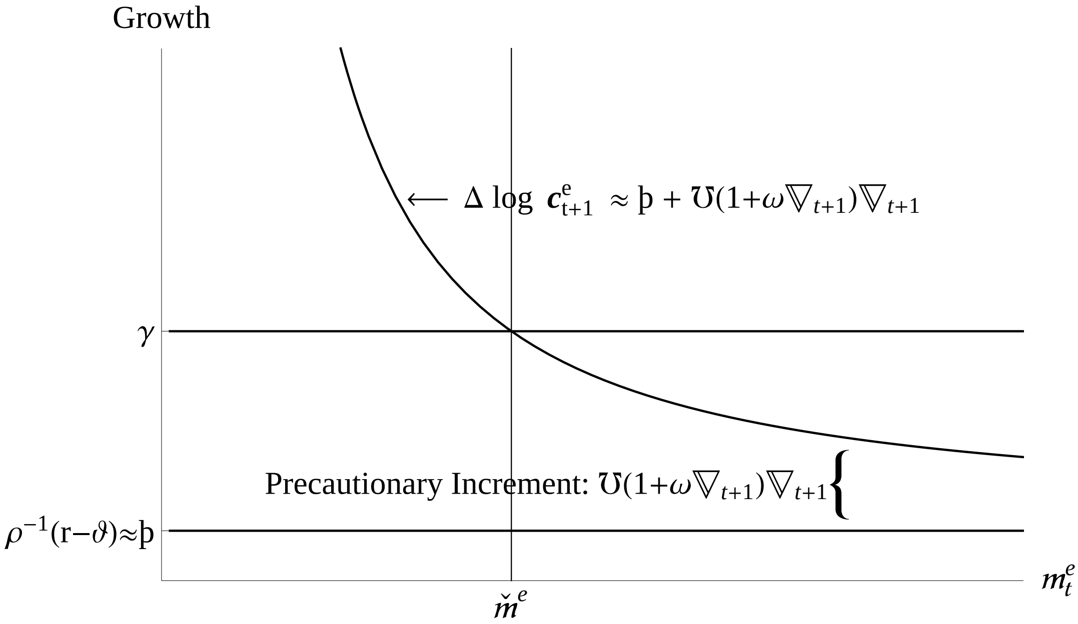


Figure 4 Effect of an Increase in r

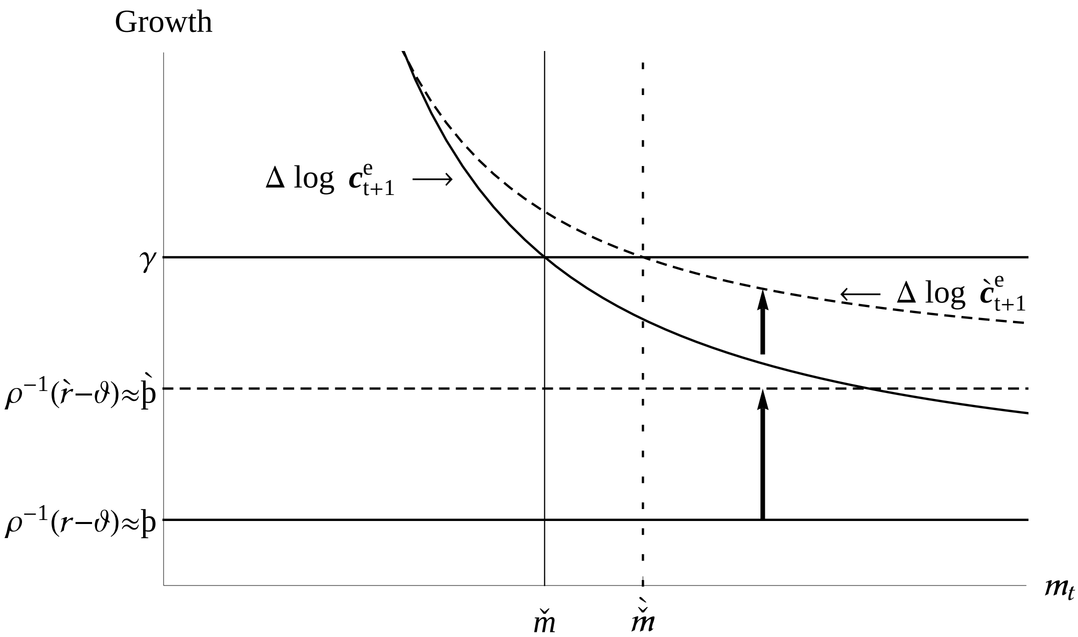
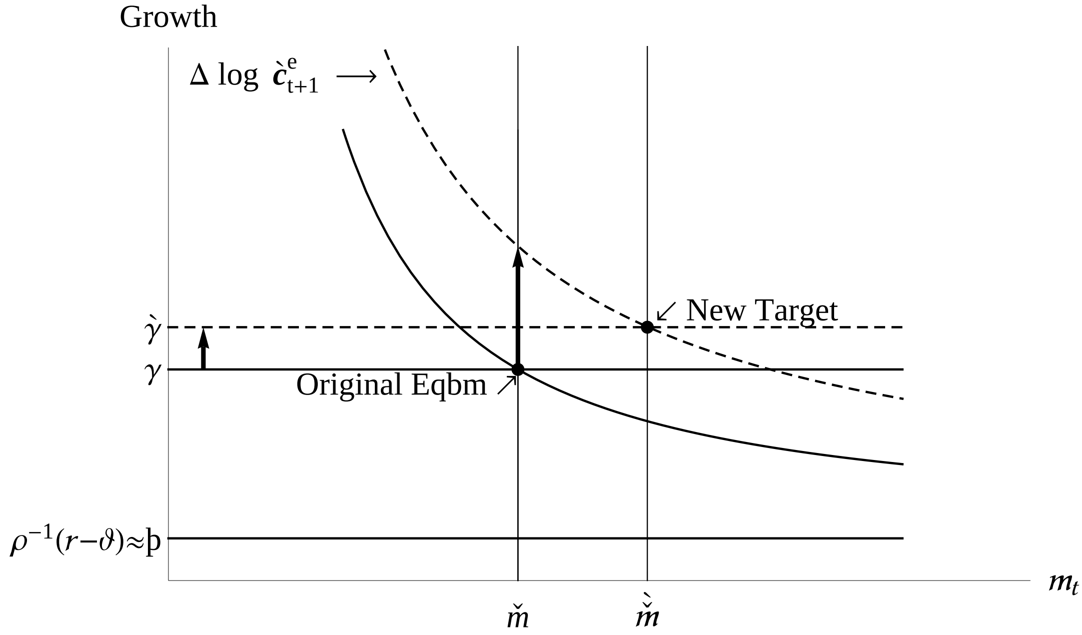


Figure 5 Effect of an Increase in Unemployment Risk $\mho$ to $\grave{\mho}$



expected consumption growth locus. Since the expected growth rate of labor income remains unchanged, the new target level of resources $\grave{\check{m}}^e$ is higher. Thus, an increase in the interest rate raises the target level of wealth, an intuitive result that carries over to more elaborate models of buffer-stock saving with more realistic assumptions about the income process (Carroll (2011)).

The next exercise is an increase in the risk of unemployment $\mho$. The principal effect we are interested in is the upward shift in the expected consumption growth locus to $\Delta\grave{\boldsymbol{c}}_{t+1}$. If the household starts at the original target level of resources $\check{m}$, the size of the upward shift at that point is captured by the arrow orginating at $\{\check{m}, \gamma\}$.

In the absence of other consequences of the rise in $\mho$, the effect on the target level of m would be unambiguously positive. However, recall our adjustment to the growth rate conditional upon employment, (12); this induces the shift in the income growth locus to $\grave{\gamma}$ which has an offsetting effect on the target m ratio. Under our benchmark parameter values, the target value of m is higher than before the increase in risk even after accounting for the effect of higher γ, but in principle it is possible for the γ effect to dominate the direct effect. Note, however, that even if the target value of m is lower, it is possible that the *saving rate* will be higher; this is possible because the faster rate of γ makes a given saving rate translate into a lower ratio of wealth to income. In any case, our view is that most useful calibrations of the model are those for which an increase in uncertainty results in either an increase in the saving rate or an increase in the target ratio of resources to permanent income. This is partly because our intent is to use the model to illustate the general features of precautionary behavior, including the qualitative effects of an increase in the magnitude of transitory shocks, which unambiguously increase both target m and saving rates.

2.2.11 Death to the Log-Linearized Consumption Euler Equation!

Our simple model may help explain why the attempt to estimate preference parameters like the degree of relative risk aversion or the time preference rate using consumption Euler equations has been so signally unsuccessful (Carroll (2001)). On the one hand, as illustrated in figures 3 and 4, the steady state growth rate of consumption, for impatient consumers, is equal to the steady-state growth rate of income,

$$\Delta \log \boldsymbol{c}^{e}_{t+1} \quad = \quad \gamma. \tag{30}$$

On the other hand, under logarithmic utility our approximation of the Euler equation for consumption growth, obtained from equation (29), seems to tell a different story,

$$\Delta \log \boldsymbol{c}^{e}_{t+1} \quad \approx \quad þ + ℧\nabla_{t+1}, \tag{31}$$

where the last line uses the Taylor approximations used to obtain (16). The approximate Euler equation (31) does not contain any term *explicitly* involving income growth. How can we reconcile (30) and (31) and resolve the apparent contradiction? The answer is that the size of the precautionary term $℧\nabla_{t+1}$ is *endogenous* (and depends on γ). To see this, solve (30)- (31): In steady-state,

$$℧\check{\nabla} \quad \approx \quad \gamma - þ. \tag{32}$$

The expression in (32) helps to understand the relationship between the model parameters and the steady-state level of wealth. From figure 3 it is apparent that $\nabla_{t+1}(m^{e}_{t})$ is a downward-sloping function of m^{e}_{t}. At low levels of current wealth, much of the spending of an employed consumer is financed by current income. In the event of job loss, such a consumer must suffer a large drop in consumption, implying a large value of ∇_{t+1}.

To illustrate further the workings of the model, consider an increase in the growth rate of income. On the one hand, the right-hand side of (32) rises. But, lower wealth raises consumption risk, so that the new target level of $\check{m}$ must be lower, and this raises the left-hand side of (32). In equilibrium, both sides of the expression rise by the same amount.

The fact that consumption growth equals income growth in the steady-state poses major problems for empirical attempts to estimate the Euler equation. To see why, suppose we had a collection of countries indexed by i, identical in all respects except that they have different interest rates r^{i}. In the spirit of Hall (1988), one might be tempted to estimate an equation of the form

$$\Delta \log \boldsymbol{c}^{i} \quad = \quad \eta_0 + \eta_1 \mathsf{r}^{i} + \epsilon^{i}, \tag{33}$$

and to interpret the coefficient on r^{i} as an empirical estimate of the value of ρ^{-1}. This empirical strategy will fail. To see why, consider the following stylized scenario. Suppose that all the countries are inhabited by impatient workers with optimal buffer-stock target rules, but each country has a different after-tax interest rate (measured by r^{i}. Suppose that the workers are not far from their wealth-to-income target, so that $\Delta \log \boldsymbol{c}^{i} = \gamma^{i}$. Suppose further that all countries have *the same* steady-state income growth rate and *the same* unemployment rate.[32]

[32] The key point holds if countries have different growth rates; this stylized example is merely an illustration.

A regression of the form of (33) would return the estimates

$$\eta_0 = \gamma$$
$$\eta_1 = 0.$$

The regression specification suffers from an *omitted variable* bias caused by the influence of the (endogenous) $\mho\nabla^i$ term. In our scenario, the omitted term is correlated with the included variable r^i (and if our scenario is exact, the correlation is perfect). Thus, estimates obtained from the log-linearized Euler equation specification in (33) will be biased estimates of ρ^{-1}. For a thorough discussion of this econometric problem, see Carroll (2001). For a demonstration that the problem is of pratical importance in (macroeconomic) empirical studies, see Parker and Preston (2005).

2.2.12 Dynamics Following An Increase in Patience

We now consider a final experiment: Figure 6 depicts the effect on consumption of a decrease in the rate of time preference (the change is exogenous, unexpected, permanent), starting from a steady-state position. A decrease in the discount rate (an increase in patience) causes an immediate drop in the level of consumption; successive points in time are reflected in the series of dots in the diagram. The new consumption path (or consumption function) starts from a lower consumption *level* and has a higher consumption *growth* than before the decrease in ϑ.[33]

Consumption eventually approaches the new, higher equilibrium target level. This higher level of consumption is financed, in the long run, by the higher interest income provided by the higher target level of wealth.

Note again, however, that equilibrium steady-state consumption growth is still equal to the growth rate of income (this follows from the fact that there is a steady-state *level* for the *ratio* of consumption to income). The higher target level of the wealth-to-income ratio is precisely enough to reduce the precautionary term by an amount that exactly offsets the effect of the rise in $-\rho^{-1}\vartheta$.

Figures 8 and 9 depict the time paths of consumption, market wealth, and the marginal propensity to consume following the decrease in ϑ. The dots are spread out evenly over time to give a sense of the rate at which the model adjusts toward the steady state.

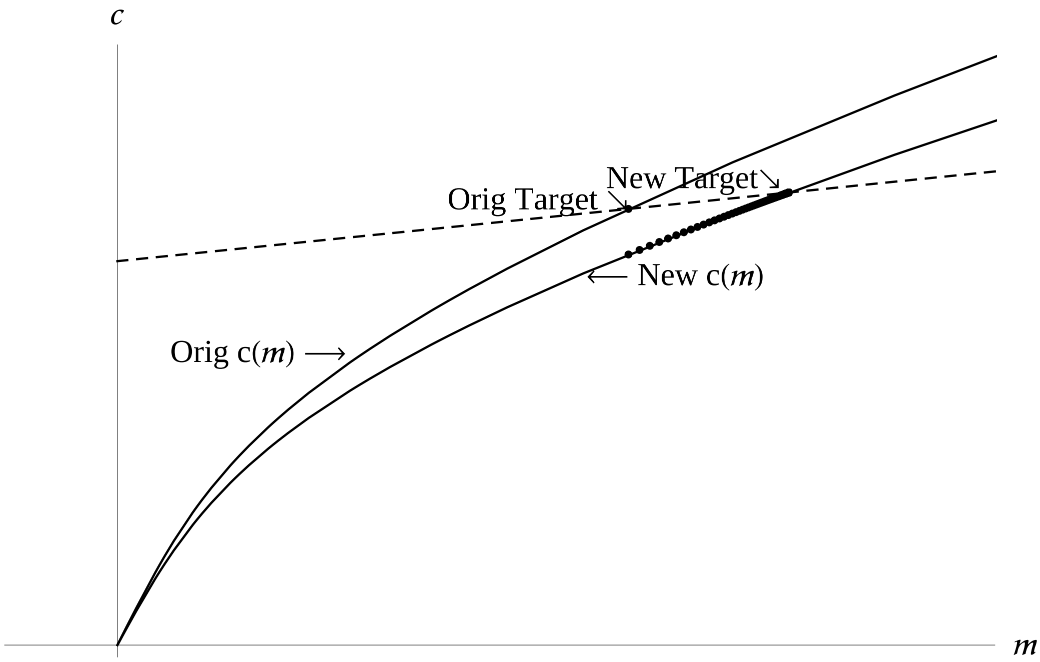
3 Conclusions

Despite its simplicity, the core logic of the model analyzed above emerges in almost every detail (after much more work) under more realistic assumptions about risk that allow for transitory shocks, permanent shocks, and unemployment in a form that is calibrated to match a large literature exploring the details of the household income process (Carroll (2011)).

We hope that the simplicity of our framework will encourage its use as a building block for analyzing questions that have so far been resistant to a transparent treatment of the role of nonreturn risk. For example, Carroll and Jeanne (2009) construct a fully articulated model

[33]The effect of changes in productivity growth is essentially the same as the effect of an increase the interest rate depicted in figure 4.

Figure 6 Effect of Lower ϑ On Consumption Function



of international capital mobility for a small open economy using the model analyzed here as the core element. We can envision a variety of other direct purposes the model could serve, including applications to topical questions such as the effects of risk in a search model of unemployment.

Figure 7 Path of c^e Before and After ϑ Decline

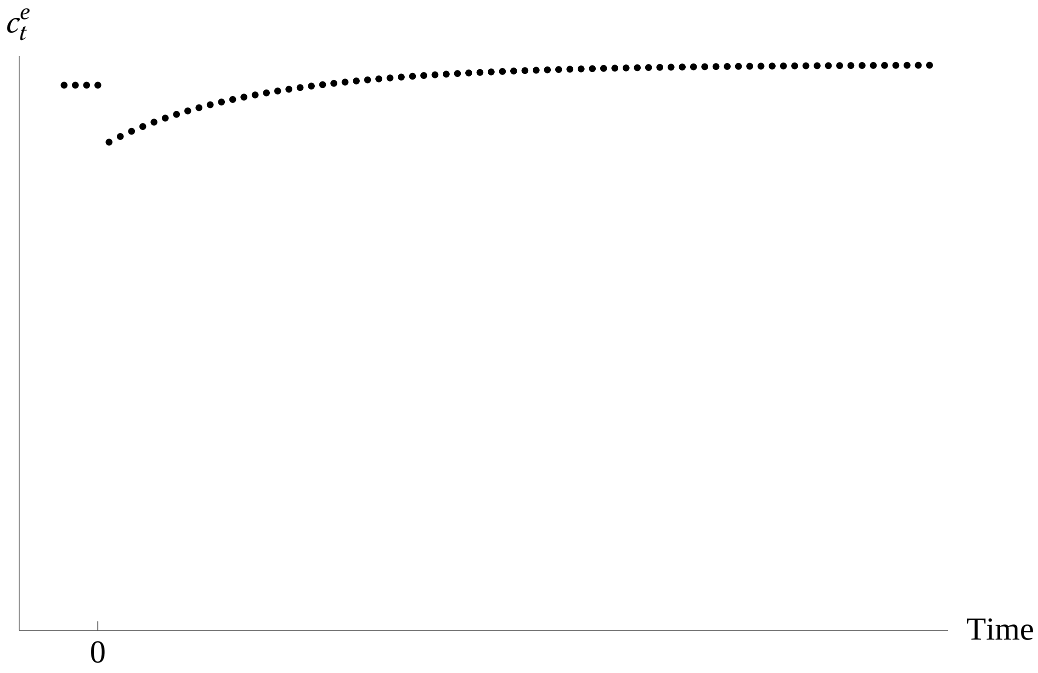


Figure 8 Path of m^e Before and After ϑ Decline

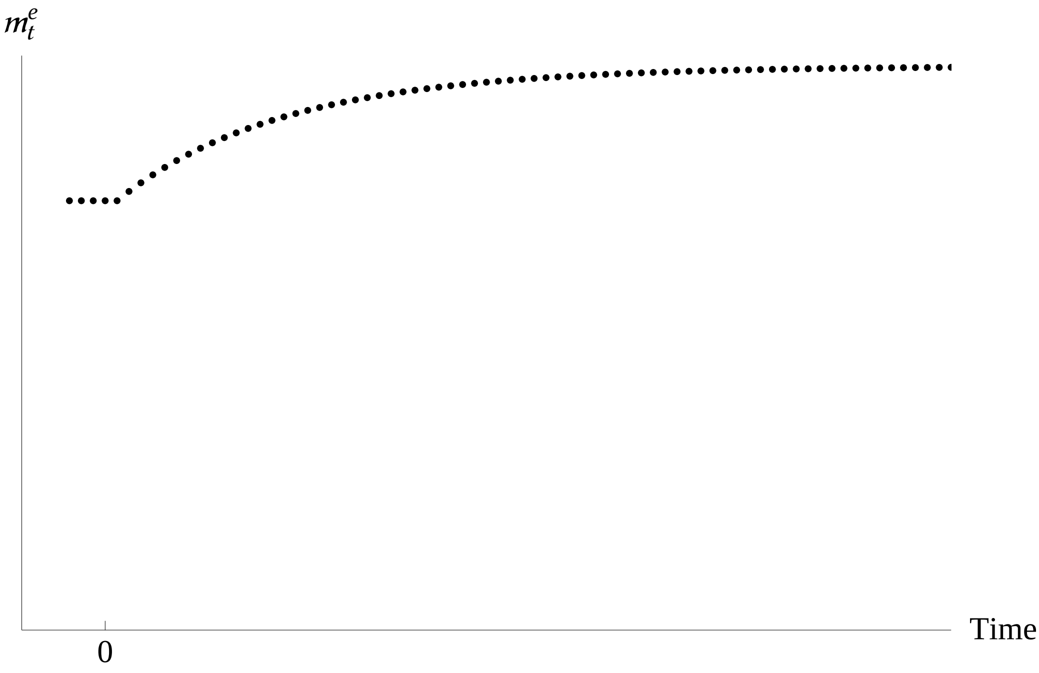


Figure 9 Marginal Propensity to Consume κ_t Before and After ϑ Decline

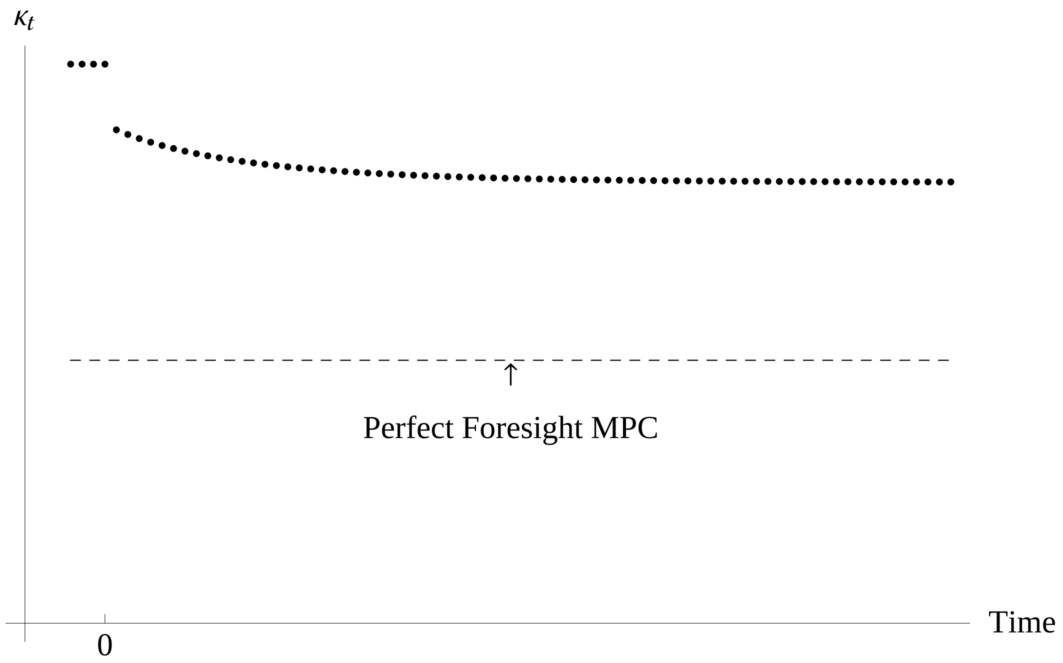


Table 1 Summary of Notation

a	-	end-of-period t assets (after consumption decision)
b	-	middle-of-period t balances (before consumption decision)
c	-	consumption
ℓ	-	personal labor productivity
m	-	market resources (capital, capital income, and labor income)
R, r	-	interest factor, rate
W	-	aggregate wage
G	-	growth factor for aggregate wage rate W
$\Gamma \equiv \mathsf{G}/\cancel{\mho}$	-	conditional (on employment) growth factor for individual labor income
γ	-	$\log \Gamma$, conditional growth *rate* for labor income
β	-	time preference factor ($= 1/(1+\vartheta)$)
ξ	-	dummy variable indicating the employment state, $\xi \in \{0, 1\}$
κ	-	marginal propensity to consume
ρ	-	coefficient of relative risk aversion
ϑ	-	time preference rate ($\approx -\log \beta$)
$\mho$	-	probability of falling into permanent unemployment
$\cancel{\mho} = 1 - \mho$	-	probability of staying in employment from one period to the next
$\text{Þ}, \text{þ}$	-	absolute patience factor, rate
$\text{Þ}_{\Gamma}, \text{þ}_{\gamma}$	-	growth patience factor, rate
$\text{Þ}_{\mathsf{R}}, \text{þ}_{\mathsf{r}}$	-	return patience factor, rate
ω	-	excess prudence factor ($= (\rho - 1)/2$)
∇	-	proportional consumption drop upon entering unemployment
$\mathcal{R}$	-	short for R/Γ
Π	-	short for $\left(\frac{\text{Þ}_{\Gamma}^{-\rho} - \cancel{\mho}}{\mho}\right)^{1/\rho}$

Appendix

A Taylor Approximation for Consumption Growth

Applying a Taylor approximation to (15), simplifying, and rearranging yields

$$\begin{aligned}
\left\{1+\mho\left[\left(\frac{c^e_{t+1}}{c^u_{t+1}}\right)^{\rho}-1\right]\right\}^{1/\rho} &= \left\{1+\mho\left[\left(\frac{c^u_{t+1}+c^e_{t+1}-c^u_{t+1}}{c^u_{t+1}}\right)^{\rho}-1\right]\right\}^{1/\rho} \\
&= \{1+\mho\,[(1+\nabla_{t+1})^{\rho}-1]\}^{1/\rho} \\
&\approx \left\{1+\mho\left[1+\rho\nabla_{t+1}+\rho(\nabla_{t+1})^2\omega-1\right]\right\}^{1/\rho} \\
&= \left\{1+\rho\mho(\nabla_{t+1}+(\nabla_{t+1})^2\omega)\right\}^{1/\rho} \\
&\approx 1+\mho\,(1+\nabla_{t+1}\omega)\,\nabla_{t+1}.
\end{aligned}$$

B The Exact Formula for $\check{m}$

The steady-state value of m^e will be where both (23) and (24) hold. To simplify the algebra, define $\zeta \equiv \mathcal{R}\kappa^u\Pi$ so that $\mathsf{R}\kappa^u\Pi = \zeta\Gamma$. Then:

$$\begin{aligned}
\left(\frac{\zeta}{1+\zeta}\right)\check{m} &= (1-\mathcal{R}^{-1})\check{m}+\mathcal{R}^{-1} \\
\mathcal{R}\left(\frac{\zeta}{1+\zeta}\right)\check{m} &= (\mathcal{R}-1)\check{m}+1 \\
\left(\mathcal{R}\left\{\frac{\zeta}{1+\zeta}-1\right\}+1\right)\check{m} &= 1 \\
\left(\mathcal{R}\left\{\frac{\zeta-(1+\zeta)}{1+\zeta}\right\}+\frac{1+\zeta}{1+\zeta}\right)\check{m} &= 1 \\
\left(\frac{1+\zeta-\mathcal{R}}{1+\zeta}\right)\check{m} &= 1 \\
\check{m} &= \left(\frac{1+\zeta}{1+\zeta-\mathcal{R}}\right) \\
\check{m} &= \left(\frac{1+\zeta+\mathcal{R}-\mathcal{R}}{1+\zeta-\mathcal{R}}\right) \\
&= 1+\left(\frac{\mathcal{R}}{1+\zeta-\mathcal{R}}\right) \\
&= 1+\left(\frac{\mathsf{R}}{\Gamma+\zeta\Gamma-\mathsf{R}}\right). \qquad (34)
\end{aligned}$$

A first point about this formula is suggested by the fact that

$$\zeta\Gamma \;=\; \mathsf{R}\kappa^u \left(1 + \left(\frac{\text{Þ}_\Gamma^{-\rho} - 1}{\mho}\right)\right)^{1/\rho} \tag{35}$$

which is likely to increase as $\mho$ approaches zero.[34] Note that the limit as $\mho \to 0$ is infinity, which implies that $\lim_{\mho\to 0} \check{m} = 1$. This is precisely what would be expected from this model in which consumers are impatient but self-constrained to have $m^e > 1$: As the risk gets infinitesimally small, the amount by which target m^e exceeds its minimum possible value shrinks to zero.

We now show that the RIC and GIC ensure that the denominator of the fraction in (34) is positive:

$$\begin{aligned}
\Gamma + \zeta\Gamma - \mathsf{R} &= \Gamma + \mathsf{R}\kappa^u\Pi - \mathsf{R} \\
&= \Gamma + \mathsf{R}\left(1 - \frac{(\mathsf{R}\beta)^{1/\rho}}{\mathsf{R}}\right)\left(\left(\frac{(\frac{(\mathsf{R}\beta)^{1/\rho}}{\Gamma})^{-\rho} - 1}{\mho} + 1\right)\right)^{1/\rho} - \mathsf{R} \\
&> \Gamma + \mathsf{R}\left(1 - \frac{(\mathsf{R}\beta)^{1/\rho}}{\mathsf{R}}\right)\left(\left(\frac{(\frac{(\mathsf{R}\beta)^{1/\rho}}{\Gamma})^{-\rho} - 1}{1} + 1\right)\right)^{1/\rho} - \mathsf{R} \\
&= \Gamma + \mathsf{R}\left(1 - \frac{(\mathsf{R}\beta)^{1/\rho}}{\mathsf{R}}\right)\left(\frac{\Gamma}{(\mathsf{R}\beta)^{1/\rho}}\right) - \mathsf{R} \\
&= \Gamma + \mathsf{R}\frac{\Gamma}{(\mathsf{R}\beta)^{1/\rho}} - \Gamma - \mathsf{R} \\
&= \mathsf{R}\left(\frac{\Gamma}{(\mathsf{R}\beta)^{1/\rho}} - 1\right) \\
&> 0.
\end{aligned}$$

However, note that $\mho$ also affects Γ; thus, the first inequality above does not necessarily imply that the denominator is decreasing as $\mho$ moves from 0 to 1.

C An Approximation for $\check{m}$

Now defining

$$\aleph \;=\; \left(\frac{\text{Þ}_\Gamma^{-\rho} - 1}{\mho}\right),$$

we can obtain further insight into (34) using a judicious mix of first- and second-order Taylor expansions (along with $\kappa^u = -\text{þ}_{\mathsf{r}}$):

$$\begin{aligned}
\zeta\Gamma &= \mathsf{R}\kappa^u\,(1 + \aleph)^{1/\rho} \\
&\approx -\mathsf{R}\text{þ}_{\mathsf{r}}\left(1 + \rho^{-1}\aleph + (\rho^{-1})(\rho^{-1} - 1)(\aleph^2/2)\right)
\end{aligned}$$

[34] 'Likely' but not certain because of the fact that $\mho$ affects Þ_Γ as well as appearing in the denominator of (34); however, for plausible calibrations the effect of the denominator predominates.

$$= -\mathsf{R}\text{þ}_{\mathsf{r}} \left(1 + \rho^{-1}\aleph\left\{\left(1 + \left(\frac{1-\rho}{\rho}\right)(\aleph/2)\right)\right\}\right) \tag{36}$$

But

$$\begin{aligned} \aleph &= \left(\frac{(1+\text{þ}_\gamma)^{-\rho} - 1}{\mho}\right) \\ &\approx \left(\frac{1 - \rho\text{þ}_\gamma - 1}{\mho}\right) \\ &\approx -\left(\frac{\rho\text{þ}_\gamma}{\mho}\right) \end{aligned} \tag{37}$$

which can be substituted into (36) to obtain

$$\begin{aligned} \zeta\Gamma &\approx -\mathsf{R}\text{þ}_{\mathsf{r}}\left(1 - (\text{þ}_\gamma/\mho)(1 + (1-\rho)(-\text{þ}_\gamma/\mho)/2)\right) \\ &\approx \underbrace{-\mathsf{R}\text{þ}_{\mathsf{r}}}_{>0}\left\{1 \underbrace{-(\text{þ}_\gamma/\mho)}_{>0}\left(1 + \underbrace{(1-\rho)}_{<0}\underbrace{(-\text{þ}_\gamma/\mho)}_{>0}/2\right)\right\} \end{aligned} \tag{38}$$

Letting ω capture the excess of prudence over the logarithmic case,

$$\omega \equiv \left(\frac{\rho - 1}{2}\right), \tag{39}$$

(34) can be approximated by

$$\begin{aligned} \check{m} &\approx 1 + \left(\frac{1}{\Gamma/\mathsf{R} - \text{þ}_{\mathsf{r}}\left(1 - (\text{þ}_\gamma/\mho)(1 - (-\text{þ}_\gamma/\mho)\omega)\right) - 1}\right) \\ &\approx 1 + \left(\frac{1}{(\gamma - \mathsf{r}) + (-\text{þ}_{\mathsf{r}})\left(1 + (-\text{þ}_\gamma/\mho)(1 - (-\text{þ}_\gamma/\mho)\omega)\right)}\right) \end{aligned} \tag{40}$$

where negative signs have been preserved in front of the þ_{r} and þ_γ terms as a reminder that the GIC and the RIC imply these terms are themselves negative (so that $-\text{þ}_{\mathsf{r}}$ and $-\text{þ}_\gamma$ are positive). *Ceteris paribus*, an increase in relative risk aversion ρ will increase ω and thereby decrease the denominator of (40). This suggests that greater risk aversion will result in a larger target level of wealth.[35]

The formula also provides insight about how the human wealth effect works in equilibrium. All else equal, the human wealth effect is captured by the $(\gamma - \mathsf{r})$ term in the denominator of (40), and it is obvious that a larger value of γ will result in a smaller target value for m. But it is also clear that the size of the human wealth effect will depend on the magnitude of the patience and prudence contributions to the denominator, and that those terms can easily dominate the human wealth effect.

For (40) to make sense, we need the denominator of the fraction to be a positive number;

[35] "Suggests" because this derivation used some dubious approximations; the suggestion can be verified, however, for plausible numerical calibrations.

defining

$$\hat{\text{þ}}_\gamma \equiv \text{þ}_\gamma(1 - (-\text{þ}_\gamma/\mho)\omega), \tag{41}$$

this means that we need:

$$\begin{aligned}
(\gamma - \mathsf{r}) &> \text{þ}_\mathsf{r} - \text{þ}_\mathsf{r}\hat{\text{þ}}_\gamma/\mho \\
&= \left(\rho^{-1}(\mathsf{r} - \vartheta) - \mathsf{r}\right) - \text{þ}_\mathsf{r}\hat{\text{þ}}_\gamma/\mho \\
\gamma &> \rho^{-1}(\mathsf{r} - \vartheta) - \text{þ}_\mathsf{r}\hat{\text{þ}}_\gamma/\mho \\
0 &> \underbrace{\rho^{-1}(\mathsf{r} - \vartheta) - \gamma}_{\text{þ}_\gamma} - \text{þ}_\mathsf{r}(\hat{\text{þ}}_\gamma/\mho) \\
0 &> \text{þ}_\gamma - \text{þ}_\mathsf{r}(\hat{\text{þ}}_\gamma/\mho).
\end{aligned} \tag{42}$$

But since the RIC guarantees $\text{þ}_\mathsf{r} < 0$ and the GIC guarantees $\text{þ}_\gamma < 0$ (which, in turn, guarantees $\hat{\text{þ}}_\gamma < 0$), this condition must hold.[36]

The same set of derivations imply that we can replace the denominator in (40) with the negative of the RHS of (42), yielding a more compact expression for the target level of resources,

$$\begin{aligned}
\check{m} &\approx 1 + \left(\frac{1}{\text{þ}_\mathsf{r}(\hat{\text{þ}}_\gamma/\mho) - \text{þ}_\gamma}\right) \\
&= 1 + \left(\frac{1/(-\text{þ}_\gamma)}{1 + (-\text{þ}_\mathsf{r}/\mho)(1 + (-\text{þ}_\gamma/\mho)\omega)}\right).
\end{aligned} \tag{43}$$

This formula makes plain the fact that an increase in either form of impatience, by increasing the denominator of the fraction in (43), will reduce the target level of assets.

We are now in position to discuss (40), understanding that the impatience conditions guarantee that its numerator is a positive number.

Two specializations of the formula are particularly useful. The first is the case where $\rho = 1$ (logarithmic utility). In this case,

$$\begin{aligned}
\text{þ}_\mathsf{r} &= -\vartheta \\
\text{þ}_\gamma &= \mathsf{r} - \vartheta - \gamma \\
\omega &= 0
\end{aligned}$$

and the approximation becomes

$$\check{m} \approx 1 + \left(\frac{1}{(\gamma - \mathsf{r}) + \vartheta(1 + (\gamma + \vartheta - \mathsf{r})/\mho)}\right) \tag{44}$$

which neatly captures the effect of an increase in human wealth (via either increased γ or reduced r), the effect of increased impatience ϑ, or the effect of a reduction in unemployment risk $\mho$ in reducing target wealth.

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[36] In more detail: For the second-order Taylor approximation in (36), we implicitly assume that the absolute value of the second-order term is much smaller than that of the first-order one, i.e. $|\rho^{-1}\aleph| \geq |(\rho^{-1})(\rho^{-1} - 1)(\aleph^2/2)|$. Substituting (37), the above could be simplified to $1 \geq (-\text{þ}_\gamma/\mho)\omega$, therefore we have $\hat{\text{þ}}_\gamma < 0$. This simple justification is based on the proof above that RIC and GIC guarantee the denominator of the fraction in (34) is positive.

The other useful case to consider is where $\mathsf{r} = \vartheta$ but $\rho > 1$. In this case,

$$\begin{aligned} \text{þ}_{\mathsf{r}} &= -\vartheta \\ \text{þ}_{\gamma} &= -\gamma \\ \hat{\text{þ}}_{\gamma} &= -\gamma(1 - (\gamma/\mho)\omega) \end{aligned}$$

so that

$$\check{m} \approx 1 + \left(\frac{1}{(\gamma - \mathsf{r}) + \vartheta(1 + (\gamma/\mho)(1 - (\gamma/\mho)\omega))} \right) \tag{45}$$

where the additional term involving ω in this equation captures the fact that an increase in the prudence term ω shrinks the denominator and thereby boosts the target level of wealth.[37]

[37] It would be inappropriate to use the equation to consider the effect of an increase in r because the equation was derived under the assumption $\vartheta = \mathsf{r}$ so r is not free to vary.

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