A Tractable Model of Buffer Stock Saving

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Abstract

We present a tractable model of the effects of nonfinancial risk on intertemporal choice. Our purpose is to provide a simple framework that can be adopted in fields like representative-agent macroeconomics, corporate finance, or political economy, where most modelers have chosen not to incorporate serious nonfinancial risk because available methods were too complex to yield transparent insights. Our model produces an intuitive analytical formula for target assets, and we show how to analyze transition dynamics using a familiar Ramsey-style phase diagram. Despite its starkness, our model captures most of the key implications of nonfinancial risk for intertemporal choice.

Keywords risk, uncertainty, precautionary saving, buffer stock saving

JEL codes C61, D11, E24

PDF: http://econ.jhu.edu/people/ccarroll/papers/ctDiscrete.pdf
Web: http://econ.jhu.edu/people/ccarroll/papers/ctDiscrete/
Archive: http://econ.jhu.edu/people/ccarroll/papers/ctDiscrete.zip
(Contains Mathematica and Matlab code solving the model)

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1 Introduction

The Merton (1969)-Samuelson (1969) model of portfolio choice is the foundation for the vast literature analyzing financial risk, not because it provides insights that are unavailable in any other framework, but because those insights are packaged in a form that is tractable, transparent, and easy to use. These qualities make the Merton-Samuelson model the natural starting point (though often not the finishing point) for any problem where rate-of-return risk is the only kind of risk worth worrying about.

Unfortunately, nonfinancial risks (such as unemployment risk for a consumer) have proven much more difficult to analyze. Of course, there is a large and sophisticated literature that carefully examines the theoretical effects of realistically calibrated nonfinancial risks. But much of the economic literature, and much graduate-level instruction, dodges the question of how nonfinancial risk influences choices, by assuming perfect insurance markets or perfect foresight or risk neutrality or quadratic utility or Constant Absolute Risk Aversion, or by calibrating models to match aggregate risks which are orders of magnitude smaller than idiosyncratic risks. These assumptions rob the question of its essence, either by assuming that markets transform nonfinancial risk into financial risk or by making implausible assumptions that yield the conclusion that decisions are largely or entirely unaffected by such risk.[1]

Often, nonreturn risk is avoided not because economists judge it to be unimportant, but because they have a perception that a fully realistic treatment would entail too much additional complexity. The specialized literature on precautionary saving and heterogeneous-agents macroeconomic models[2] has reinforced that perception by showing just how much effort can be required to properly analyze behavior in the presence of empirically plausible specifications of risk.

This paper offers a compromise. We present a tractable model that captures the key qualitative features of models that incorporate a serious treatment of nonreturn risk. Our model is a natural extension of the benchmark perfect foresight framework, and we show how to analyze the model using a phase diagram that will look familiar to every economist because of its close kinship to the Ramsey model of economic growth universally taught in graduate school.

Our model's tractability springs from our distillation of all nonreturn risk into a stark and simple possibility: The decisionmaker might experience an uninsurable one-time permanent reduction in the flow of nonfinancial income. When that decisionmaker is an employed household, this can be interpreted as an exogenous and permanent transition into unemployment (or disability, or retirement). A similar risk is faced by a country whose exports are dominated by a commodity whose price might collapse (e.g., oil exporters, if cold fusion had worked). The model could even be interpreted as applying

[1] CARA utility with only labor income risk is included as a 'dodge' because Carroll and Kimball (1996) show that it is a knife-edge case that is unrepresentative of the broader effects of uncertainty (notably, it fails to exhibit the consumption concavity that holds for virtually every other combination of assumptions); indeed, the addition of rate-of-return risk renders the optimal consumption function concave even under CARA utility. (The other traditional objection to CARA utility is that the optimal consumption plan under CARA utility generally involves setting consumption to a negative value in some states of the world, which is difficult to make sense of.)

[2] Well known heterogeneous-agents macro models include Carroll (1992), Aiyagari (1994), and Krusell and Smith (1998); roots go back to Schechtman and Escudero (1977) and Bewley (1977), with other important contributions by Clarida (1987), Zeldes (1989), and Chamberlain and Wilson (2000).

to the behavior of a firm controlled by a risk-neutral manager, so long as the collapse of a line of business could have the effect of reducing the firm's collateral value and therefore increasing its cost of external finance *a la* Bernanke, Gertler, and Gilchrist (1996).[3]

The optimal response to this risk is to aim to accumulate a buffer stock of precautionary assets, as a form of "self-insurance." The existing literature has employed cumbersome numerical solution and simulation methods to explore the determinants of the target stock of wealth under alternative assumptions. In contrast, we are able to derive an analytical formula for the target level of wealth, and show transparently how the precautionary motive interacts with the other saving motives that have been well understood since Irving Fisher (1930)'s work: The income, substitution, and human wealth effects.

The literature's principal other approach (besides numerical solutions) to analyzing precautionary behavior has been the examination of a generalized approximation to the consumption Euler equation that incorporates nonlinear (higher-order) terms. The influences determining the magnitude of the higher-order terms, especially for a consumer away from the target level of assets, have mostly been treated as an impenetrable mystery. We derive a simple expression that shows how the familiar perfect-foresight consumption Euler equation is modified in an intuitive way by our one-shot risk; whether or not the consumer's assets are at the target, the effect of the risk on consumption growth is related to the probability of the bad event, its magnitude, the degree of risk aversion, and the consumer's wealth position. At the target, we are able to obtain an exact analytical expression for the combined value of the higher-order terms.

Our chief ambition is to persuade nonspecialist modelers that incorporating a serious treatment of nonreturn risk is not as hard as they think. (Specialists are already aware of how difficult the problem can be; but they may be surprised at how simple it can be, when stripped down to its essence). The treatment of risk may need to be stylized (as ours is) to preserve tractability, but incorporating a stylized treatment of risk is much better than ignoring it altogether.[4]

2 The Decision Problem

For concreteness, we analyze the problem of an individual consumer facing a labor income risk. Other interpretations (like the ones mentioned in the introduction) are left for future work or other authors.

We couch the problem in discrete time, but in most cases we provide the logarithmic approximations that will correspond to the exact solution to the corresponding problem in continuous time.[5]

[3]In this case, the convex increase in borrowing rates when cash drops plays the same role as the convexity of the marginal utility function for a consumer; see also Berk, Stanton, and Zechner (2009) for an argument that senior firm managers are not risk neutral even if shareholders are, because poor performance under their tenure will reduce their own future employment opportunities. A firm controlled by such managers may behave very much like a risk-averse household.

[4]In order to assist authors in modifying our model for other purposes, we have constructed a public archive that contains Matlab and Mathematica programs that produce all the results and figures reported in this paper, along with some other examples of uses to which the model could be put. The archive is available on the first author's website.

[5]See Toche (2005) for an explicit but brief treatment of a closely related model in continuous time.

The aggregate wage rate, W_t, grows by a constant factor G from the current time period to the next, reflecting exogenous productivity growth:

$$\mathsf{W}_{t+1} = \mathsf{G}\mathsf{W}_t. \tag{1}$$

The interest rate is exogenous and constant (the economy is small and open); the interest factor is denoted R. Define $\boldsymbol{m}$ as market resources (financial wealth plus current income), $\boldsymbol{a}$ as end-of-period assets after all actions have been accomplished (specifically, after the consumption decision), and $\boldsymbol{b}$ as bank balances before receipt of labor income. Individuals are subject to a dynamic budget constraint (DBC) that can be decomposed into the following elements:

$$\boldsymbol{a}_t = \boldsymbol{m}_t - \boldsymbol{c}_t \tag{2}$$
$$\boldsymbol{b}_{t+1} = \mathsf{R}\boldsymbol{a}_t \tag{3}$$
$$\boldsymbol{m}_{t+1} = \boldsymbol{b}_{t+1} + \ell_{t+1}\mathsf{W}_{t+1}\xi_{t+1} \tag{4}$$

where ℓ measures the consumer's labor productivity (hours of work for an employed consumer are assumed to be exogenous and fixed) and ξ is a dummy variable indicating the consumer's employment state: Everyone in this economy is either employed ($\xi = 1$, a state indicated by the letter 'e') or unemployed ($\xi = 0$, a state indicated by 'u'). Thus, labor income is zero for unemployed consumers.[6]

2.1 The Unemployed Consumer's Problem

There is no way out of unemployment; once an individual becomes unemployed, that individual remains unemployed forever, $\xi_t = 0 \Longrightarrow \xi_{t+1} = 0 \ \forall \ t$. Consumers have a CRRA utility function $\mathrm{u}(\bullet) = \bullet^{1-\rho}/(1-\rho)$, with $\rho > 1$, and discount future utility geometrically by β per period. The solution to the unemployed consumer's optimization problem is simply:[7]

$$\boldsymbol{c}_t^u = \kappa^u \boldsymbol{b}_t, \tag{5}$$

where κ^u is the marginal propensity to consume, which can be derived from $\kappa^u = 1 - \text{Þ}_{\mathsf{R}}$ where[8]

$$\text{Þ}_{\mathsf{R}} \equiv \mathsf{R}^{-1}(\mathsf{R}\beta)^{1/\rho} \tag{6}$$

is the 'return patience factor' (see Carroll (2004) for a detailed discussion).[9] We will show below that the simplicity of the unemployed consumer's behavior (in particular, the closed-form consumption function (5)) is what makes the problem of the employed consumer tractable (given our assumption that the employed consumer faces only a single kind of risk).

The κ^u for the problem without risk is strictly below the MPC for the problem with risk (Carroll and Kimball, 1996). We impose what Carroll (2004) calls the 'return impatience

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[6] This is without loss of generality. We could allow for unemployment insurance by modifying the value of ξ associated with unemployment. On this, see also footnote 4 in Toche (2005).

[7] This is a standard result from the literature; a derivation can be found, for example, in the lecture notes on the first author's web page.

[8] Table 1 compactly summarizes our notation as an aid to the reader's memory.

[9] **Þ** is the Old English letter 'thorn'; its modern equivalent is the digraph 'th.'

condition' (RIC),

$$\text{Þ}_{\mathsf{R}} < 1, \tag{7}$$

which embodies sufficient impatience to guarantee that $\kappa^u > 0$. The interpretation is that the consumer must not be so patient that a boost to total wealth would fail to boost consumption (for the unemployed, wealth consists in balances $\boldsymbol{b}$ only).[10] An alternative (equally correct) interpretation is that the condition guarantees that the present discounted value (PDV) of consumption for the unemployed consumer remains finite. Þ_{R} is the 'return patience factor' because it defines desired perfect-foresight consumption growth *relative* to the rate of return R. We define the 'return patience rate' as the lower-case version:

$$\text{þ}_{\mathsf{r}} \equiv \log \text{Þ}_{\mathsf{R}} \approx \text{Þ}_{\mathsf{R}} - 1 = -\kappa^u.$$

For short, we will sometimes say that a consumer is 'return impatient' (or, 'the RIC holds') if $\text{Þ}_{\mathsf{R}} < 1$ or if $\text{þ}_{\mathsf{r}} < 0$ or if $\kappa^u > 0$, all three conditions being equivalent.[11] A consumer who is return impatient is someone who will be spending enough to make the ratio of consumption to total wealth decline over time.

The return patience factor can be compared to the 'absolute patience factor'

$$\text{Þ} \;=\; (\mathsf{R}\beta)^{1/\rho} \tag{8}$$

which is the growth factor for consumption in the perfect foresight model. We say that a consumer is 'absolutely impatient' if

$$\text{Þ} \;<\; 1, \tag{9}$$

in which case the consumer will choose to spend more than the amount that would permit constant consumption; such a consumer's absolute level of wealth declines over time, and therefore consumption itself declines, since consumption is proportional to total wealth. Analogously to (8), we define the absolute patience rate as

$$\text{þ} \;\equiv\; \log \text{Þ} \approx \rho^{-1}(\mathsf{r} - \vartheta). \tag{10}$$

2.2 The Employed Consumer's Problem

The consumer's preferences are the same in the employment and unemployment states; only exposure to risk differs.

2.2.1 A Human-Wealth-Preserving Spread in Unemployment Risk

A consumer who is *employed* in the current period has $\xi_t = 1$; if this person is still employed next period ($\xi_{t+1} = 1$), market resources will be:

$$\boldsymbol{m}^e_{t+1} = (\boldsymbol{m}^e_t - \boldsymbol{c}^e_t)\mathsf{R} + \mathsf{W}_{t+1}\ell_{t+1}. \tag{11}$$

[10] 'Pathologically patient' consumers who do not satisfy this condition can be thought of as people who would hoard any incremental resources in order to enable even more extra spending in the distant future.

[11] Throughout, we casually treat logs of factors like Þ_{R} as equivalent to the level minus 1; that is, we treat expressions like $\log \text{Þ}_{\mathsf{R}}$ and $\text{Þ}_{\mathsf{R}} - 1$ as interchangeable, which is an appropriate approximation so long as the factor is 'close' to 1. In practice, the approximation is very good.

However, there is no guarantee that the consumer will remain employed: Employed consumers face a constant risk $\mho$ of becoming unemployed. It is convenient to define $\not\mho \equiv 1-\mho$, the complementary probability that a consumer does *not* become unemployed. We assume that ℓ grows by a factor $\not\mho^{-1}$ every period,

$$\ell_{t+1} = \ell_t/\not\mho, \tag{12}$$

because under this assumption, for a consumer who remains employed, labor income will grow by factor $\Gamma = \mathsf{G}/\not\mho$, so that the *expected* labor income growth factor for employed consumers is the same G as in the perfect foresight case:

$$\begin{aligned}
\mathbb{E}_t[\mathsf{W}_{t+1}\ell_{t+1}\xi_{t+1}] &= \left(\frac{\ell_t\mathsf{G}\mathsf{W}_t}{\not\mho}\right)(\mho \times 0 + \not\mho \times 1) \\
\frac{\mathbb{E}_t[\mathsf{W}_{t+1}\ell_{t+1}\xi_{t+1}]}{\mathsf{W}_t\ell_t} &= \mathsf{G}
\end{aligned}$$

implying that an increase in $\mho$ is a pure increase in risk with no effect on the PDV of expected labor income – a mean-preserving spread in the intertemporal sense. Thus, any change in behavior that results from a change in $\mho$ will be cleanly interpretable as reflecting an effect of uncertainty rather than the effect of a change in human wealth.

2.2.2 First Order Optimality Condition

The usual steps lead to the standard consumption Euler equation. Using $i \in \{e, u\}$ to stand for the two possible states,

$$\begin{aligned}
\mathrm{u}'(\boldsymbol{c}^e_t) &= \mathsf{R}\beta\,\mathbb{E}_t\left[\mathrm{u}'(\boldsymbol{c}^i_{t+1})\right] \\
1 &= \mathsf{R}\beta\,\mathbb{E}_t\left[\left(\frac{\boldsymbol{c}^i_{t+1}}{\boldsymbol{c}^e_t}\right)^{-\rho}\right].
\end{aligned} \tag{13}$$

Henceforth nonbold variables will be used to represent the bold equivalent divided by the level of permanent labor income for an employed consumer, e.g. $c^e_t = \boldsymbol{c}^e_t/(\mathsf{W}_t\ell_t)$; thus we can rewrite the consumption Euler equation as:

$$\begin{aligned}
1 &= \mathsf{R}\beta\,\mathbb{E}_t\left[\left(\frac{c^i_{t+1}\mathsf{W}_{t+1}\ell_{t+1}}{c^e_t\mathsf{W}_t\ell_t}\right)^{-\rho}\right] \\
&= \mathsf{R}\beta\,\mathbb{E}_t\left[\left(\frac{c^i_{t+1}}{c^e_t}\Gamma\right)^{-\rho}\right] \\
&= \Gamma^{-\rho}\mathsf{R}\beta\left\{(1-\mho)\left(\frac{c^e_{t+1}}{c^e_t}\right)^{-\rho} + \mho\left(\frac{c^u_{t+1}}{c^e_t}\right)^{-\rho}\right\},
\end{aligned} \tag{14}$$

where the term in braces is a probability-weighted average of the growth rates of marginal utility in the case where the consumer remains employed (the first term) and the case where the consumer becomes unemployed (the second term).

2.2.3 *Analysis and Intuition Of Consumption Growth Path*

It will be useful now to define a 'growth patience factor' $\text{Þ}_{\Gamma} = (\mathsf{R}\beta)^{1/\rho}/\Gamma$ which is the factor by which the consumption-income ratio c^e would grow in the absence of labor income risk. With this notation, (14) can be written as:

$$
\begin{aligned}
1 &= \text{Þ}_{\Gamma}^{\rho} \left(\frac{c^e_{t+1}}{c^e_t}\right)^{-\rho} \left\{1 - \mho + \mho \left[\left(\frac{c^u_{t+1}}{c^e_t}\right)\left(\frac{c^e_t}{c^e_{t+1}}\right)\right]^{-\rho}\right\} \\
\left(\frac{c^e_{t+1}}{c^e_t}\right) &= \text{Þ}_{\Gamma} \left\{1 + \mho \left[\left(\frac{c^e_{t+1}}{c^u_{t+1}}\right)^{\rho} - 1\right]\right\}^{1/\rho}. \qquad (15)
\end{aligned}
$$

To understand (15), it is useful to consider an approximation. Define $\nabla_{t+1} \equiv \left(\frac{c^e_{t+1} - c^u_{t+1}}{c^u_{t+1}}\right)$, the proportion by which consumption next period would drop in the event of a transition into unemployment; we refer to this loosely as the size of the 'consumption risk.' Define ω, the 'excess prudence' factor, as $\omega = (\rho - 1)/2$.[12] Applying a Taylor approximation to (15) (see appendix A) yields:

$$
\left(\frac{c^e_{t+1}}{c^e_t}\right) \approx (1 + \mho(1 + \omega\nabla_{t+1})\nabla_{t+1})\, \text{Þ}_{\Gamma} \qquad (16)
$$

which simplifies further in the logarithmic utility case (since $\omega = 0$) to

$$
\left(\frac{c^e_{t+1}}{c^e_t}\right) \approx (1 + \mho\nabla_{t+1})\, \text{Þ}_{\Gamma}. \qquad (17)
$$

Consumption growth depends on the employment outcome because insurance markets are missing (by assumption);[13] consumption if employed next period c^e_{t+1} is greater than consumption if unemployed c^u_{t+1}, so that ∇_{t+1} is positive. Recall that c^e_{t+1}/c^e_t approaches Þ_{Γ} as the risk vanishes. Thus equation (16) shows that risk boosts consumption growth for the employed consumer by an amount proportional to the probability of becoming unemployed $\mho$ multiplied by a factor that is increasing in the amount of 'consumption risk' ∇. In the logarithmic case, equation (17) shows that the precautionary boost to consumption growth is directly proportional to the size of the consumption risk.

For any given m^e_t, an increase in risk does not change the PDV of future labor income, so that the human wealth term in the intertemporal budget constraint is not affected by an increase in $\mho$. But the larger $\mho$ is, the faster consumption growth must be, as equation (16) shows. For consumption growth to be faster while keeping the PDV constant, the *level* of *current* c^e must be lower. Thus, the introduction of a risk of unemployment $\mho$ induces a (precautionary) increase in saving.

In the (persuasive) case where $\rho > 1$, (16) implies that a consumer with a higher degree of prudence (larger ρ and therefore larger ω) will anticipate greater consumption growth. This reflects the greater precautionary saving motive induced by a higher degree of prudence.

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[12] It is 'excess' in the sense of exceeding the benchmark case of logarithmic utility which corresponds to $\rho = 1$. Logarithmic utility is often viewed as a lower bound on the possible degree of risk aversion.

[13] The no-slavery provisions of the U.S. Constitution prohibit even indentured servitude, providing a moral hazard explanation for why this insurance market should be missing. Adverse selection arguments provide an even better explanation.

To perform a phase-diagram analysis of this model, we must find the $\Delta c^e_{t+1} = 0$ and $\Delta m^e_{t+1} = 0$ loci. Consider a consumer who is unemployed in period $t + 1$. Dividing both sides of (4) by $\mathsf{W}_{t+1}\ell_{t+1}$ yields $m^u_{t+1} = b^u_{t+1} = (m^e_t - c^e_t)\mathcal{R}$, where the shorthand $\mathcal{R} \equiv \mathsf{R}/\Gamma$ has been used.

Substituting $c^e_{t+1} = c^e_t$ and $c^u_{t+1} = m^u_{t+1}\kappa^u$ into (15) yields:

$$
\begin{aligned}
1 &= \left\{1 + \mho\left[\left(\frac{c^e_{t+1}}{\kappa^u m^u_{t+1}}\right)^{\rho} - 1\right]\right\}\text{Þ}^{\rho}_{\Gamma} \\
\frac{c^e_{t+1}}{(m^e_{t+1} - c^e_{t+1})\mathcal{R}\kappa^u} &= \left(\frac{\text{Þ}^{-\rho}_{\Gamma} - \cancel{\mho}}{\mho}\right)^{1/\rho} \equiv \Pi \\
c^e_{t+1} &= (m^e_{t+1} - c^e_{t+1})\mathcal{R}\kappa^u\Pi \qquad (18)
\end{aligned}
$$

where the expression $m^u_{t+1} = (m^e_t - c^e_t)\mathcal{R}$ has been used in the second line.

We know that $m^e_{t+1} - c^e_{t+1} > 0$ because a consumer facing the risk of perpetual unemployment will never borrow. Since the RIC imposes $\kappa^u > 0$, (18) implies that steady-state consumption is positive only if Π is positive. From the definition of Π above, we need the condition

$$
\text{Þ}_{\Gamma} < (1 - \mho)^{\rho}. \qquad (19)
$$

Recall that $\text{Þ}_{\Gamma} = (\mathsf{R}\beta)^{1/\rho}/\Gamma$. In the limit as $\mho$ approaches zero, (19) therefore reduces to a requirement that the growth patience factor Þ_{Γ} be less than one,

$$
\text{Þ}_{\Gamma} < 1. \qquad (20)
$$

Following Carroll (2004), we call the condition (20) the 'perfect foresight growth impatience' condition (PF-GIC), by analogy with the 'return impatience' condition (7) imposed earlier (and recognizing that if $\mho = 0$ the consumer knows with perfect certainty what will happen in the future; the PF-GIC ensures that a consumer facing no risk would be sufficiently impatient to choose a wealth-to-permanent-income ratio that would be falling over time.[14]

Using $\gamma \equiv \log \Gamma$, we similarly define the corresponding 'growth patience rate'

$$
\text{þ}_{\gamma} \equiv \log \text{Þ}_{\Gamma} \qquad (21)
$$

so that the PF-GIC can also be written

$$
\text{þ}_{\gamma} \approx \rho^{-1}(\mathsf{r} - \vartheta) - \gamma < 0. \qquad (22)
$$

2.2.4 Why Increased Unemployment Risk Increases Growth Impatience

Under the maintained assumption that the RIC holds, the (generalized) GIC in (19) slackens (becomes easier to satisfy) as unemployment risk rises because, with relative risk aversion $\rho > 1$, an increase in $\mho$ reduces the right-hand side of (19). This occurs for two reasons. First, an increase in $\mho$ is like a reduction in the future downweighting factor (that is, a decrease in patience), conditional on the consumer remaining employed.[15] Second,

[14] The PF-GIC is a slightly stronger condition than is strictly necessary; the necessary condition is (19). However, the PF-GIC guarantees that the solution is well behaved as the risk vanishes, which lends itself to a more intuitive interpretation.

[15] While this effect is offset by an increase in the downweighting factor associated with the transition to the unemployed state, the RIC already guarantees that the PDV of consumption, income, and value remain finite for the unemployed consumer, who is therefore irrelevant.

an increase in $\mho$ slackens the GIC because our mean-preserving-spread assumption requires that labor productivity growth be adjusted so that the value of human wealth is independent of $\mho$ – see (12). The higher $\mho$ is, the faster growth is *conditional on remaining employed.* As income growth (conditional on employment) increases, the continuously-employed (lucky) consumer is effectively more 'impatient' in the sense of desiring consumption growth below employment-conditional income growth.[16]

The fact that the GIC is easier to satisfy as $\mho$ increases means that if the PF-GIC (20) is satisfied, then (19) must be satisfied.

2.2.5 The Target Level of m^e

We first characterize the steady state. Consider first $\Delta c^e_{t+1} = 0$. Imposing the RIC and the GIC, we substitute $c^e_{t+1} = c^e_{t+1}$ into equation (18):

$$\begin{aligned} c^e_{t+1} &= m^e_{t+1}\mathcal{R}\kappa^u\Pi - c^e_{t+1}\mathcal{R}\kappa^u\Pi \\ c^e_{t+1} &= \left(\frac{\mathcal{R}\kappa^u\Pi}{1+\mathcal{R}\kappa^u\Pi}\right) m^e_{t+1}. \end{aligned} \tag{23}$$

Consider next $\Delta m^e_{t+1} = 0$. From the normalized version of the DBC in (11),

$$\begin{aligned} m^e_{t+1} &= (m^e_t - c^e_t)\mathcal{R} + 1 \\ c^e_t &= (1-\mathcal{R}^{-1})m^e_{t+1} + \mathcal{R}^{-1}. \end{aligned} \tag{24}$$

The steady-state levels of m^e and c^e are the values for which both (24) and (23) hold. This system of two equations in two unknowns can be solved explicitly (see the appendix). For illustration, consider the special case of logarithmic utility ($\rho = 1$). The appendix shows that an approximation of the target level of market resources is

$$\check{m}^e \approx 1 + \left(\frac{1}{(\gamma - \mathsf{r}) + \vartheta(1 + (\gamma + \vartheta - \mathsf{r})/\mho)}\right). \tag{25}$$

The GIC and the RIC together guarantee that the denominator of (25) is positive.

This expression encapsulates several of the key intuitions of the model. The human wealth effect of income growth is captured by the first γ term in the denominator; for any calibration for which the denominator is positive, increasing γ reduces the target level of wealth. This reflects the fact that a consumer who anticipates being richer in the future will choose to save less in the present, and the result of lower saving is smaller wealth. The human wealth effect of interest rates is correspondingly captured by the $-\mathsf{r}$ term, which goes in the opposite direction to the effect of income growth, because an increase in the rate at which future labor income is discounted constitutes a reduction in human wealth. (Less human wealth results in lower consumption and therefore higher target wealth). An increase in the rate at which future happiness is discounted, ϑ, reduces the target wealth level. Finally, a reduction in unemployment risk raises $(\gamma + \vartheta - \mathsf{r})/\mho$ and therefore reduces the target wealth level.[17,18]

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[16] Note that neither of these effects of $\mho$ is precautionary.

[17] $(\gamma + \vartheta - \mathsf{r}) > 0$ is guaranteed by (22) under log utility ($\rho = 1$).

[18] We neglect here the fact that an increase in $\mho$ requires an adjustment to γ via (12) which induces a human wealth effect that goes in the opposite direction from the direct effect of uncertainty. For sufficiently large values of $\mho$, this effect can dominate the direct effect of

Note that the different effects *interact* with each other, in the sense that the strength of, say, the human wealth effect of interest rates will vary depending on the values of the other parameters.

The assumption of log utility is implausible; empirical estimates from structural estimation exercises (e.g. Gourinchas and Parker (2002), Cagetti (2003), or the subsequent literature) regularly find estimates considerably in excess of $\rho = 1$, and evidence from Barsky, Juster, Kimball, and Shapiro (1997) suggests that values of 5 or higher are not implausible. Another special case helps to illuminate how results change for $\rho > 1$. The appendix shows that, in the special case where $\vartheta = \mathsf{r}$, the target level of wealth is:

$$\check{m} \approx 1 + \left(\frac{1}{(\gamma - \mathsf{r}) + \vartheta(1 + (\gamma/\mho)(1 - (\gamma/\mho)\omega))} \right). \tag{26}$$

Compare the target level in (26) with (25) (where $\vartheta - \mathsf{r} = 0$). The key difference is that (26) contains an extra term involving ω, which measures the amount by which prudence exceeds the logarithmic benchmark. An increase in ω reduces the denominator of (26) and thereby raises the target level of wealth, just as would be expected from an increase in the intensity of the precautionary motive.

In the $\omega > 0$ case, the interaction effects between parameter values are particularly intense for the $(\gamma/\mho)^2$ term that multiplies ω; this implies, e.g., that a given increase in unemployment risk (say, from 5 percent to 10 percent) can have a *much* more powerful effect on the target level of wealth for a consumer who is more prudent.

2.2.6 The Phase Diagram

Figure 1 presents the phase diagram of system (23)-(24) under our baseline parameter values.[19] An intuitive interpretation is that the $\Delta m^e_{t+1} = 0$ locus characterized by (24) shows how much consumption c^e_t would be required to leave resources m^e_t unchanged so that $m^e_{t+1} = m^e_t$.[20] Thus, any point below the $\Delta m^e_{t+1} = 0$ line would have consumption below the break-even amount, implying that wealth would rise. Conversely for points above $\Delta m^e_{t+1} = 0$. This is the logic behind the horizontal arrows of motion in the diagram: Above $\Delta m^e_{t+1} = 0$ the arrows point leftward, below $\Delta m^e_{t+1} = 0$ the arrows point rightward.

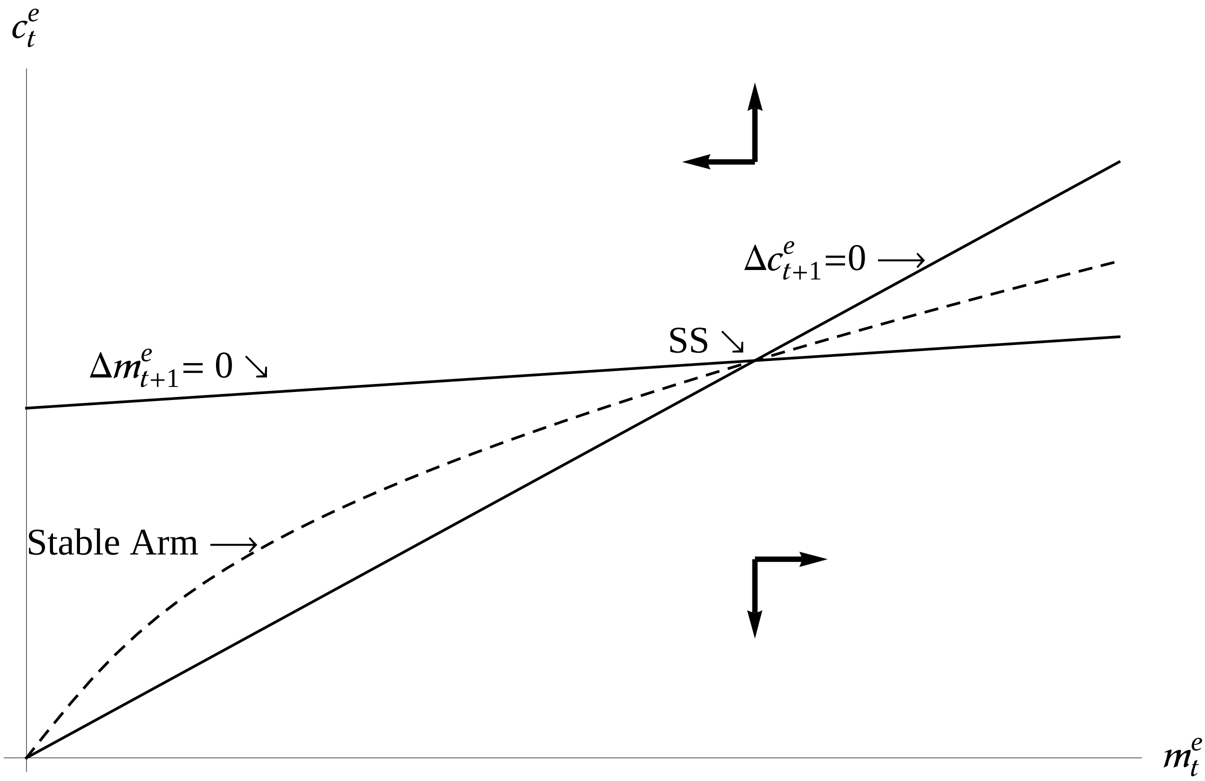
The intuitive interpretation of the $\Delta c^e_{t+1} = 0$ locus characterized by (23) is more subtle. Recall that expected consumption growth depends on the amount by which consumption would fall if the unemployment state were realized. At a given level of resources, the farther actual consumption (if employed) is below the break-even (sustainable) amount, the smaller the c^e_{t+1}/c^u_{t+1} ratio is, and therefore the smaller consumption growth is. Points below the $\Delta c^e_{t+1} = 0$ locus are associated with negative values of Δc^e_{t+1}. This is the logic

uncertainty and the target wealth-to-income ratio declines. See the illustration below of the effects of an increase in uncertainty for further discussion.

[19] Our parameterization is not indended to maximize realism, but instead to generate well-proportioned figures that illustrate the mechanisms of the model as clearly as possible. The parameter values are encapsulated in the file `ParametersBase.m` in the online archive.

[20] Some authors refer to $\Delta m^e_{t+1} = 0$ as the level of 'permanent income.' However, this definition differs from Friedman (1957)'s and, moreover, is a potential source of confusion with permanent labor income' $\mathsf{W}_t \ell_t$; we prefer to describe the locus as depicting the level of 'sustainable consumption.'

Figure 1 Phase Diagram



behind the vertical arrows of motion in the diagram: Above $\Delta c_{t+1}^e = 0$ the arrows point upward, below $\Delta c_{t+1}^e = 0$ the arrows point downward.

2.2.7 The Consumption Function

Figure 2 shows the optimal consumption function $\mathrm{c}(m)$ for an employed consumer (dropping the e superscript to reduce clutter). This is of course the stable arm of the phase diagram. Also plotted are the 45 degree line along which $c_t = m_t$; and

$$\bar{\mathrm{c}}(m) \;=\; (m - 1 + h)\kappa^u, \tag{27}$$

where

$$h \;=\; \left(\frac{1}{1 - \mathsf{G}/\mathsf{R}}\right) \tag{28}$$

is the level of (normalized) human wealth. $\bar{\mathrm{c}}(m)$ is the solution to the no-risk version of the model; it is depicted in order to introduce another property of the model: As wealth approaches infinity, the solution to the problem with risky labor income approaches the solution to the no-risk problem arbitrarily closely.[21,22] See the appendix for details.

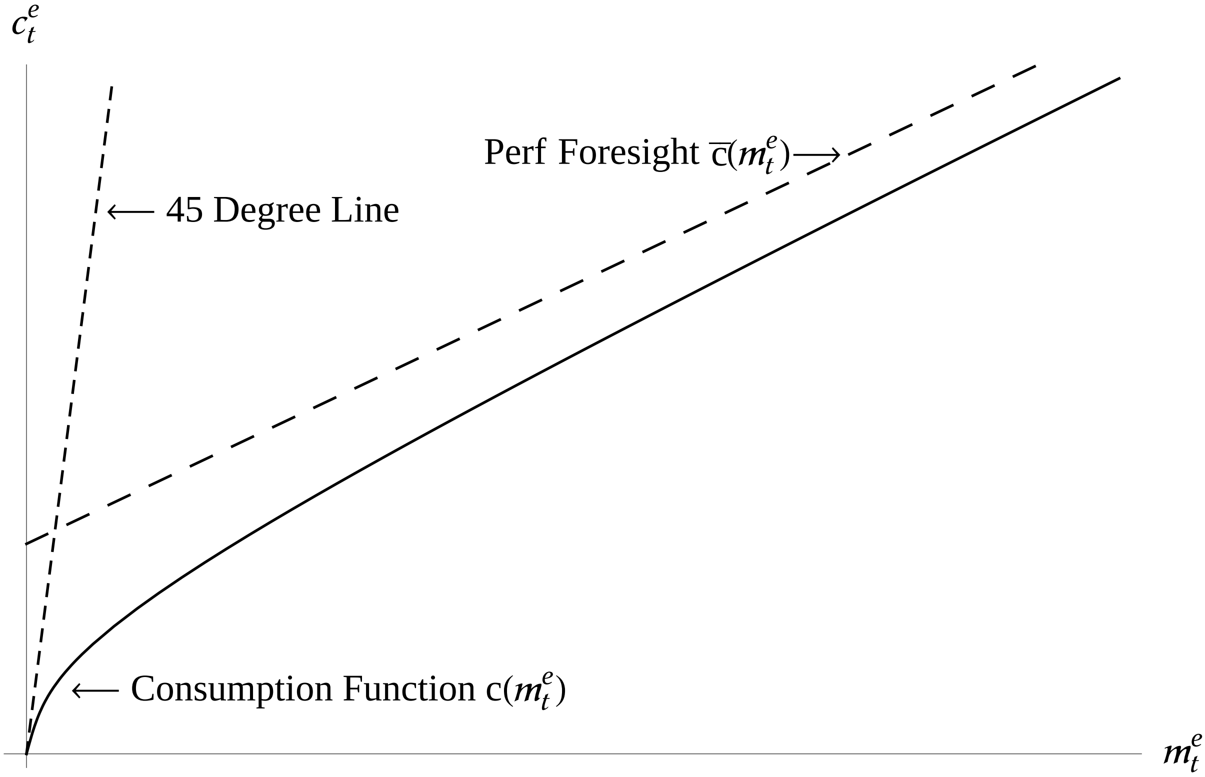
The consumption function $\mathrm{c}(m)$ is *concave*: The marginal propensity to consume $\boldsymbol{\kappa}(m) \equiv d\mathrm{c}(m)/dm$ is higher at low levels of m because the intensity of the precautionary motive increases as resources m decline.[23] The MPC is higher at lower levels of m because the *relaxation* in the intensity of the precautionary motive induced by a small increase

[21] This limiting result requires that we impose the additional assumption $\Gamma < \mathsf{R}$, because the no-risk consumption function is not defined if $\Gamma \geq \mathsf{R}$.

[22] If the horizontal axis is stretched far enough, the two consumption functions appear to merge (visually), with the 45 degree line merging (visually) with the vertical axis. The current scaling is chosen both for clarity and to show realistic values of wealth.

[23] Carroll and Kimball (1996) prove that the consumption function must be concave for a general class of stochastic processes and utility functions – including almost all commonly-used model assumptions except for the knife-edge cases explicitly chosen to avoid concavity.

Figure 2 The Consumption Function for the Employed Consumer



in m (Kimball, 1990) is relatively larger for a consumer who starts with less than for a consumer who starts with more resources (Carroll and Kimball, 1996).

This important point is clearest as m approaches zero. Consider a counterfactual. Suppose the consumer were to spend all his resources in period t, i.e. $c_t = m_t$. In this situation, if the consumer were to become unemployed in the next period, he would then be left with resources $m_{t+1}^u = (m_t - c_t)\mathcal{R} = 0$, which would induce consumption $c_{t+1}^u = \kappa^u m_{t+1}^u = 0$, yielding negative infinite utility. A rational, optimizing consumer will always avoid such an eventuality, no matter how small its likelihood may be. Thus the consumer never spends all available resources.[24]

This implication is illustrated in figure 2 by the fact that consumption function always remains below the 45 degree line.

2.2.8 Expected Consumption Growth Is Downward Sloping in m^e

Figure 3 illustrates some of the key points in a different way. It depicts the growth rate of consumption $\boldsymbol{c}_{t+1}^e/\boldsymbol{c}_t^e$ as a function of m_t^e. Since $\mho \geq 0$, the no-risk GIC for this model implies:

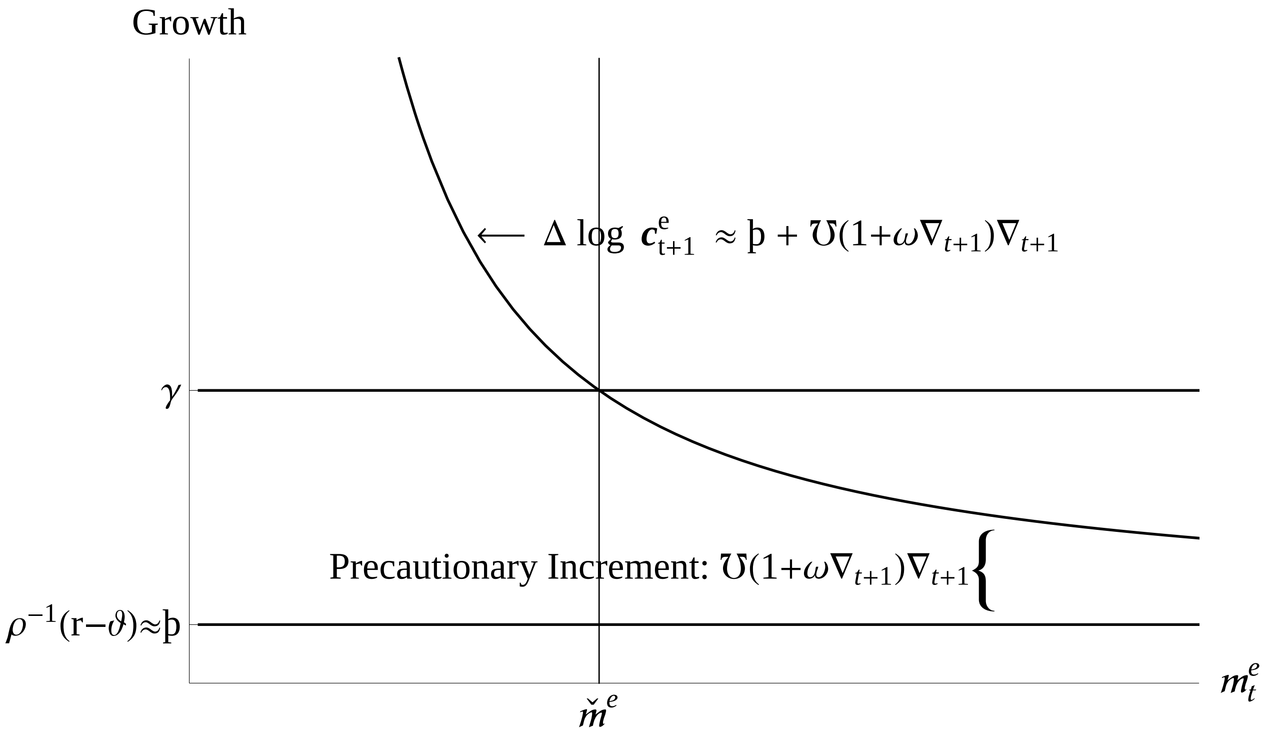
$$\gamma \;>\; \rho^{-1}(\mathsf{r} - \vartheta) \approx \text{Þ}_{\mathsf{r}}. \tag{29}$$

This condition can be visually verified for our benchmark calibration.

Figure 3 illustrates the result that consumption growth is equal to what it would be in the absence of risk, plus a precautionary term; for algebraic verification, multiply both

[24]This is an implication not just of the CRRA utility function used here but of the general class of continuously differentiable utility functions that satisfy the *Inada condition* $\mathrm{u}'(0) = \infty$.

Figure 3 Income and Consumption Growth



sides of (15) by Γ to obtain

$$\left(\frac{\boldsymbol{c}^{e}_{t+1}}{\boldsymbol{c}^{e}_{t}}\right) = (\mathsf{R}\beta)^{1/\rho}\left\{1+\mho\left[\left(\frac{c^{e}_{t+1}}{c^{u}_{t+1}}\right)^{\rho}-1\right]\right\}^{1/\rho}, \tag{30}$$

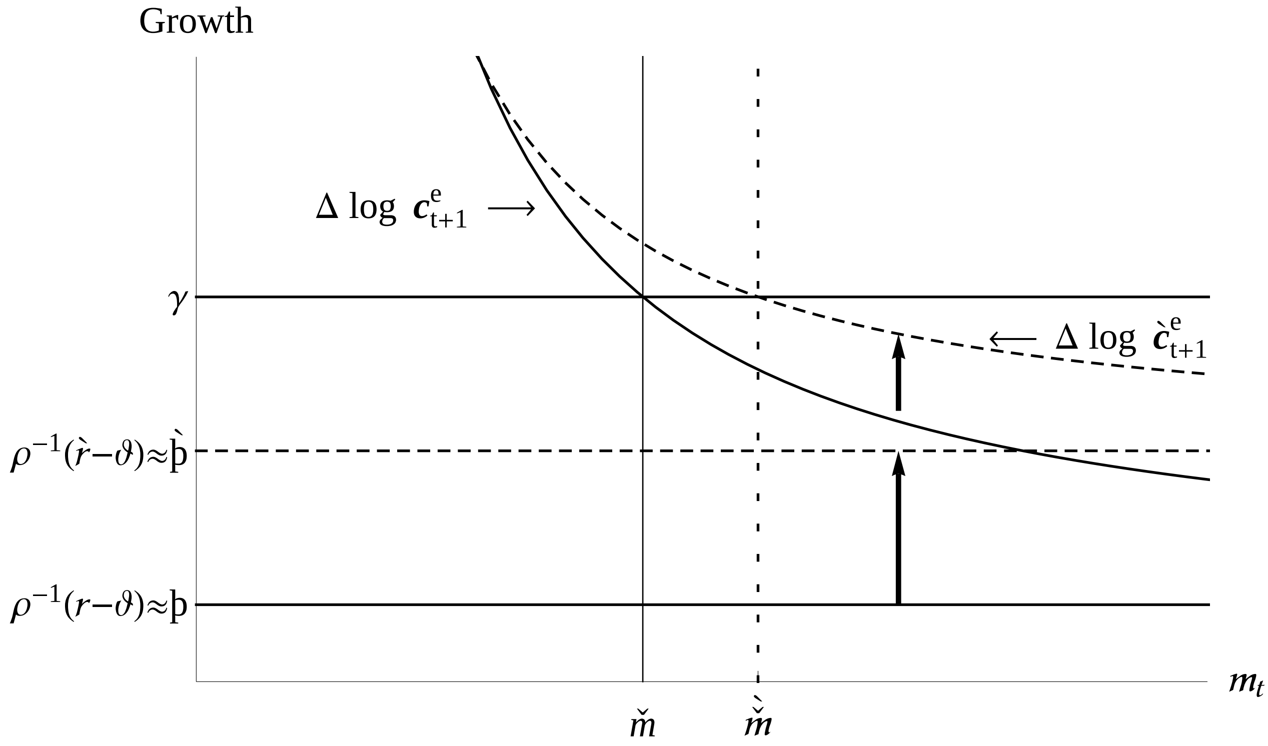
and observe that the contribution of the precautionary motive becomes arbitrarily large as $m_t \rightarrow 0$, because $c^{u}_{t+1} = m^{u}_{t+1}\kappa^{u} = (m_t - \mathrm{c}(m_t))\mathcal{R}\kappa^{u}$ approaches zero as $m_t \rightarrow 0$; that is, as resources m^{e}_{t} decline, expected consumption growth approaches infinity. The point where consumption growth is equal to income growth is at the target value of m^{e}.

2.2.9 Summing Up the Intuition

We are finally in position to get an intuitive understanding of how the model works and why a target wealth ratio exists. On the one hand, consumers are growth-impatient: It cannot be optimal for them to let wealth become arbitrarily large in relation to income. On the other hand, consumers have a precautionary motive that intensifies as the level of wealth falls. The two effects work in opposite directions. As resources fall, the precautionary motive becomes stronger, eventually offsetting the impatience motive. The point at which prudence becomes exactly large enough to match impatience defines the target wealth-to-income ratio.

It is instructive to work through a couple of comparative dynamics exercises. In doing so, we assume that all changes to the parameters are exogenous, unexpected, and permanent. Figure 4 depicts the effects of increasing the interest rate to $\grave{\mathsf{r}} > \mathsf{r}$. The no-risk consumption growth locus shifts up to the higher value $\grave{\text{þ}}_{\mathsf{r}} \approx \rho^{-1}(\grave{\mathsf{r}} - \vartheta)$, inducing a corresponding increase in the expected consumption growth locus. Since the expected growth rate of labor income remains unchanged, the new target level of resources $\grave{\check{m}}^{e}$

Figure 4 Effect of An Increase In r

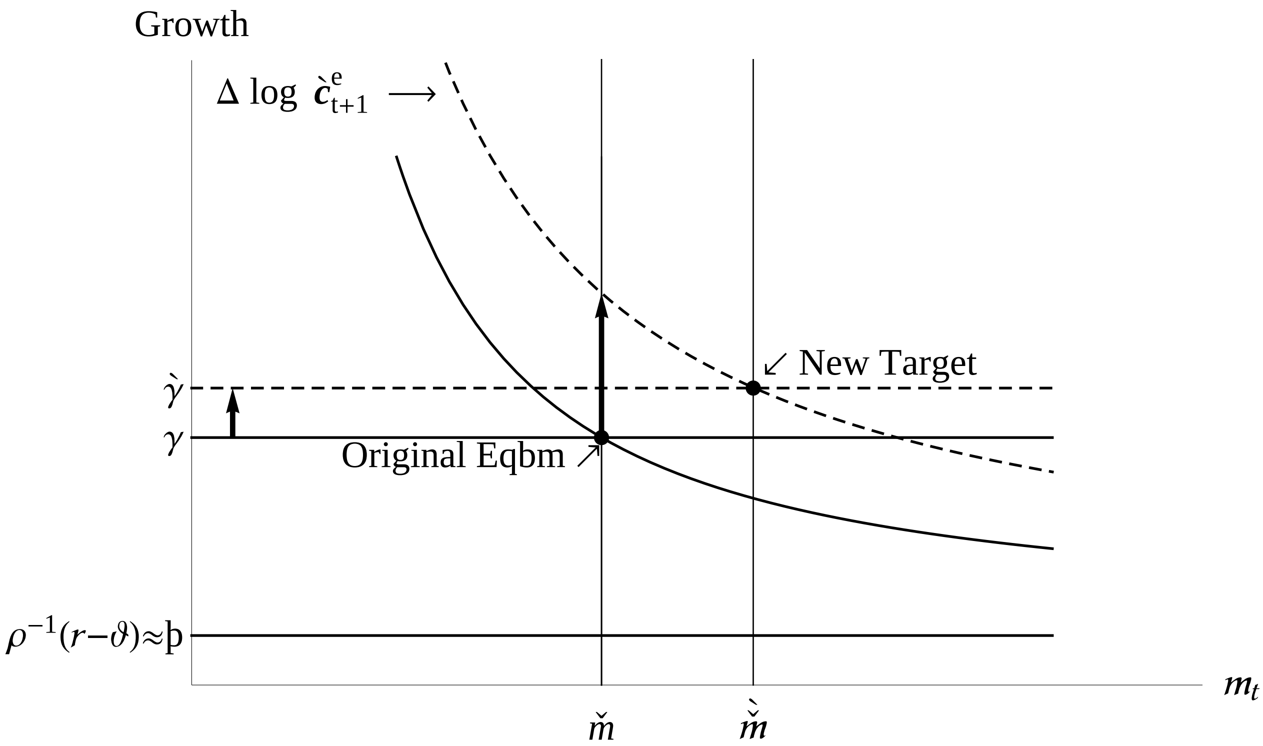


is higher. Thus, an increase in the interest rate raises the target level of wealth, an intuitive result that carries over to more elaborate models of buffer-stock saving with more realistic assumptions about the income process (Carroll (2004)).

The next exercise is an increase in the risk of unemployment $\mho$. The principal effect we are interested in is the upward shift in the expected consumption growth locus to $\Delta\grave{\mathbf{c}}_{t+1}$. If the household starts at the original target level of resources $\check{m}$, the size of the upward shift at that point is captured by the arrow orginating at $\{\check{m}, \gamma\}$.

In the absence of other consequences of the rise in $\mho$, the effect on the target level of m would be unambiguously positive. However, recall our adjustment to the growth rate conditional upon employment, (12); this induces the shift in the income growth locus to $\grave{\gamma}$ which has an offsetting effect on the target m ratio. Under our benchmark parameter values, the target value of m is higher than before the increase in risk even after accounting for the effect of higher γ, but in principle it is possible for the γ effect to dominate the direct effect. Note, however, that even if the target value of m is lower, it is possible that the *saving rate* will be higher; this is possible because the faster rate of γ makes a given saving rate translate into a lower ratio of wealth to income. In any case, our view is that most useful calibrations of the model are those for which an increase in uncertainty results in either an increase in the saving rate or an increase in the target ratio of resources to permanent income. This is partly because our intent is to use the model to illustate the general features of precautionary behavior, including the qualitative effects of an increase in the magnitude of transitory shocks, which unambiguously increase both target m and saving rates.

Figure 5 Effect of an Increase in Unemployment Risk $\mho$ to $\grave{\mho}$



2.2.10 Death to the Log-Linearized Consumption Euler Equation!

Our simple model may help explain why the attempt to estimate preference parameters like the degree of relative risk aversion or the time preference rate using consumption Euler equations has been so signally unsuccessful (Carroll (2001)). On the one hand, as illustrated in figures 3 and 4, the steady state growth rate of consumption, for impatient consumers, is equal to the steady-state growth rate of income,

$$\Delta \log \, \boldsymbol{c}^{e}_{t+1} \;=\; \gamma. \tag{31}$$

On the other hand, under logarithmic utility our approximation of the Euler equation for consumption growth, obtained from equation (30), seems to tell a different story,

$$\Delta \log \, \boldsymbol{c}^{e}_{t+1} \;\approx\; \text{þ} + \mho\nabla_{t+1}, \tag{32}$$

where the last line uses the Taylor approximations used to obtain (16). The approximate Euler equation (32) does not contain any term *explicitly* involving income growth. How can we reconcile (31) and (32) and resolve the apparent contradiction? The answer is that the size of the precautionary term $\mho\nabla_{t+1}$ is *endogenous* (and depends on γ). To see this, solve (31)- (32): In steady-state,

$$\mho\check{\nabla} \;\approx\; \gamma - \text{þ}. \tag{33}$$

The expression in (33) helps to understand the relationship between the model parameters and the steady-state level of wealth. From figure 3 it is apparent that $\nabla_{t+1}(m^{e}_{t})$ is a downward-sloping function of m^{e}_{t}. At low levels of current wealth, much of the spending of an employed consumer is financed by current income. In the event of job loss, such a consumer must suffer a large drop in consumption, implying a large value of ∇_{t+1}.

To illustrate further the workings of the model, consider an increase in the growth rate of income. On the one hand, the right-hand side of (33) rises. But, lower wealth raises consumption risk, so that the new target level of $\check{m}$ must be lower, and this raises the left-hand side of (33). In equilibrium, both sides of the expression rise by the same amount.

The fact that consumption growth equals income growth in the steady-state poses major problems for empirical attempts to estimate the Euler equation. To see why, suppose we had a collection of countries indexed by i, identical in all respects except that they have different interest rates r^i. In the spirit of Hall (1988), one might be tempted to estimate an equation of the form

$$\Delta \log \boldsymbol{c}^i = \eta_0 + \eta_1 \mathsf{r}^i + \epsilon^i, \tag{34}$$

and to interpret the coefficient on r^i as an empirical estimate of the value of ρ^{-1}. This empirical strategy will fail. To see why, consider the following stylized scenario. Suppose that all the countries are inhabited by impatient workers with optimal buffer-stock target rules, but each country has a different after-tax interest rate (measured by r^i. Suppose that the workers are not far from their wealth-to-income target, so that $\Delta \log \boldsymbol{c}^i = \gamma^i$. Suppose further that all countries have *the same* steady-state income growth rate and *the same* unemployment rate.[25]

A regression of the form of (34) would return the estimates

$$\begin{aligned} \eta_0 &= \gamma \\ \eta_1 &= 0. \end{aligned}$$

The regression specification suffers from an *omitted variable* bias caused by the influence of the (endogenous) $\mho\nabla^i$ term. In our scenario, the omitted term is correlated with the included variable r^i (and if our scenario is exact, the correlation is perfect). Thus, estimates obtained from the log-linearized Euler equation specification in (34) will be biased estimates of ρ^{-1}. For a thorough discussion of this econometric problem, see Carroll (2001). For a demonstration that the problem is of pratical importance in (macroeconomic) empirical studies, see Parker and Preston (2005).

2.2.11 Dynamics Following An Increase in Patience

We now consider a final experiment: Figure 6 depicts the effect on consumption of a decrease in the rate of time preference (the change is exogenous, unexpected, permanent), starting from a steady-state position. A decrease in the discount rate (an increase in patience) causes an immediate drop in the level of consumption; successive points in time are reflected in the series of dots in the diagram. The new consumption path (or consumption function) starts from a lower consumption *level* and has a higher consumption *growth* than before the decrease in ϑ.[26]

[25] The key point holds if countries have different growth rates; this stylized example is merely an illustration.

[26] The effect of changes in productivity growth is essentially the same as the effect of an increase the interest rate depicted in figure 4.

Consumption eventually approaches the new, higher equilibrium target level. This higher level of consumption is financed, in the long run, by the higher interest income provided by the higher target level of wealth.

Note again, however, that equilibrium steady-state consumption growth is still equal to the growth rate of income (this follows from the fact that there is a steady-state *level* for the *ratio* of consumption to income). The higher target level of the wealth-to-income ratio is precisely enough to reduce the precautionary term by an amount that exactly offsets the effect of the rise in $-\rho^{-1}\vartheta$.

Figures 8 and 9 depict the time paths of consumption, market wealth, and the marginal propensity to consume following the decrease in ϑ. The dots are spread out evenly over time to give a sense of the rate at which the model adjusts toward the steady state.

3 Conclusions

Despite its simplicity, the core logic of the model as analyzed above is reflected in almost every detail (after much more work) under more realistic assumptions about risk that allow for transitory shocks, permanent shocks, and unemployment in a form that is calibrated to match a large literature exploring the details of the household income process (Carroll (2004)).

We hope that the simplicity of our framework will encourage its use as a building block for analyzing questions that have so far been resistant to a transparent treatment of the role of nonreturn risk. For example, Carroll and Jeanne (2009) construct a fully articulated model of international capital mobility for a small open economy using the model analyzed here as the core element. We can envision a variety of other direct purposes the model could serve, including applications to topical questions such as the effects of risk in a search model of unemployment.

Figure 6 Effect of Lower ϑ On Consumption Function

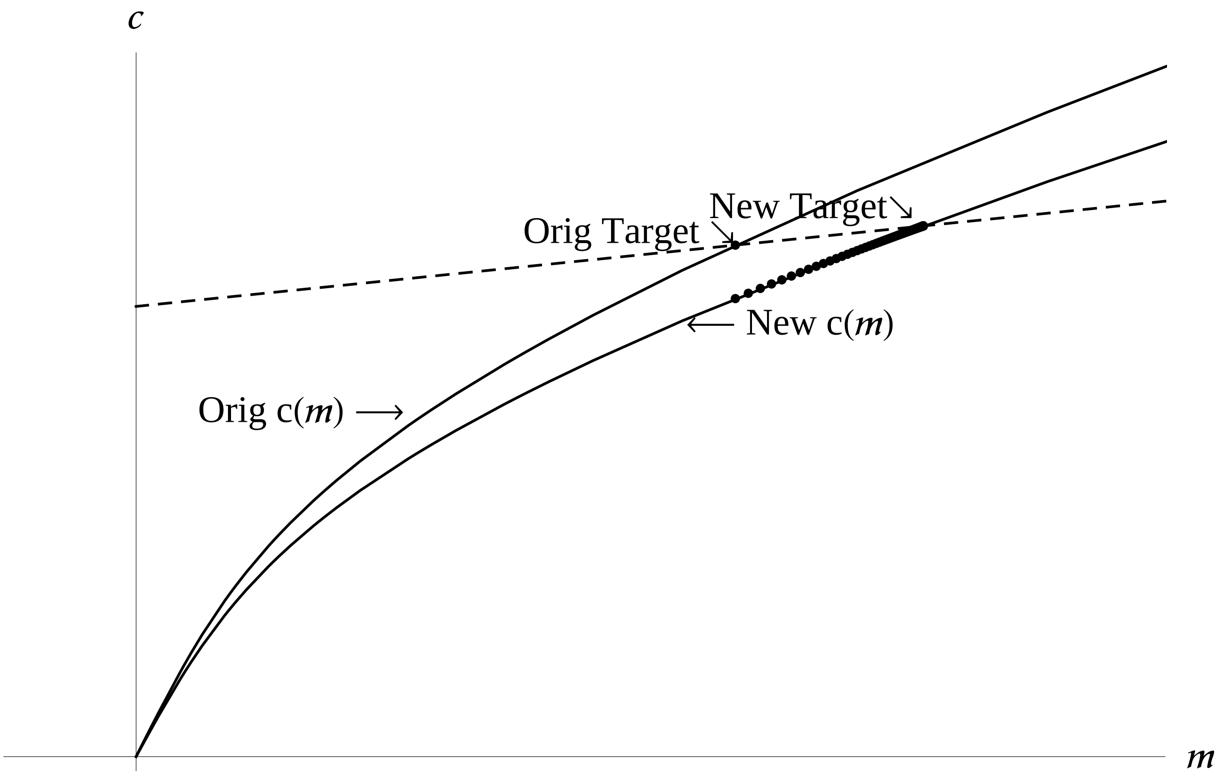


Figure 7 Path of c^e Before and After ϑ Decline

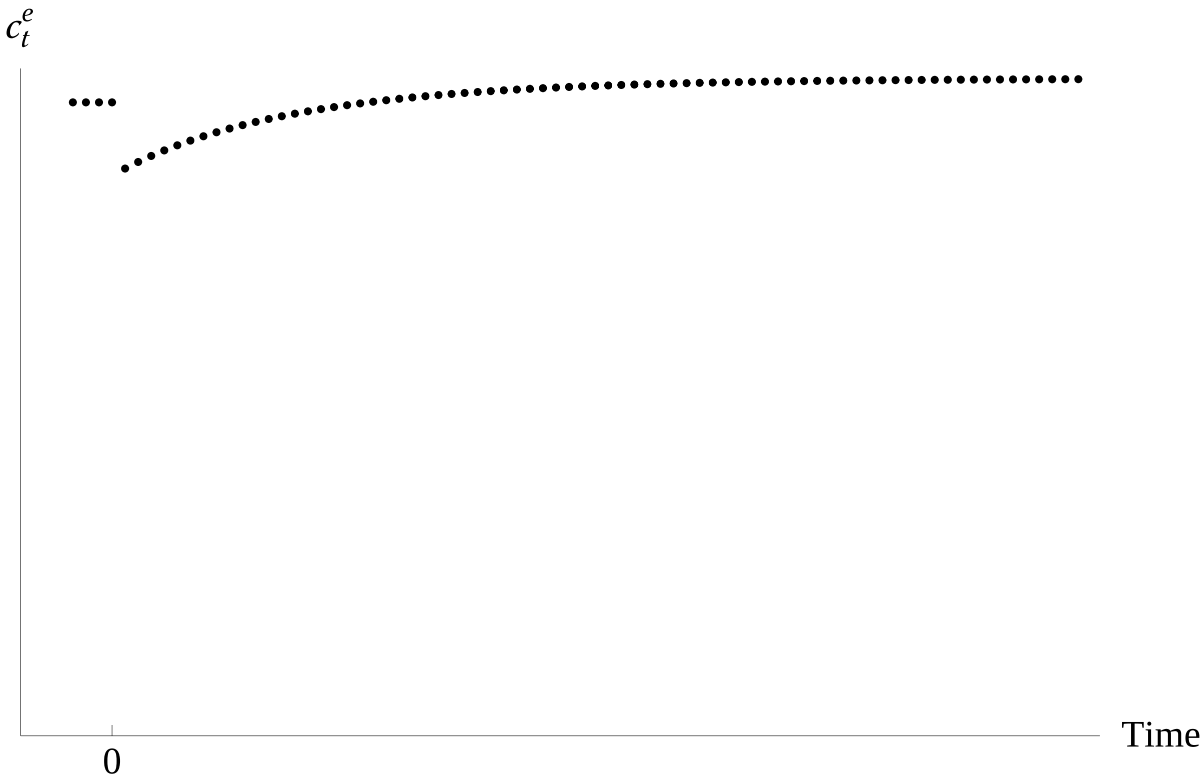


Figure 8 Path of m^e Before and After ϑ Decline

Figure 9 Marginal Propensity to Consume κ_t Before and After ϑ Decline

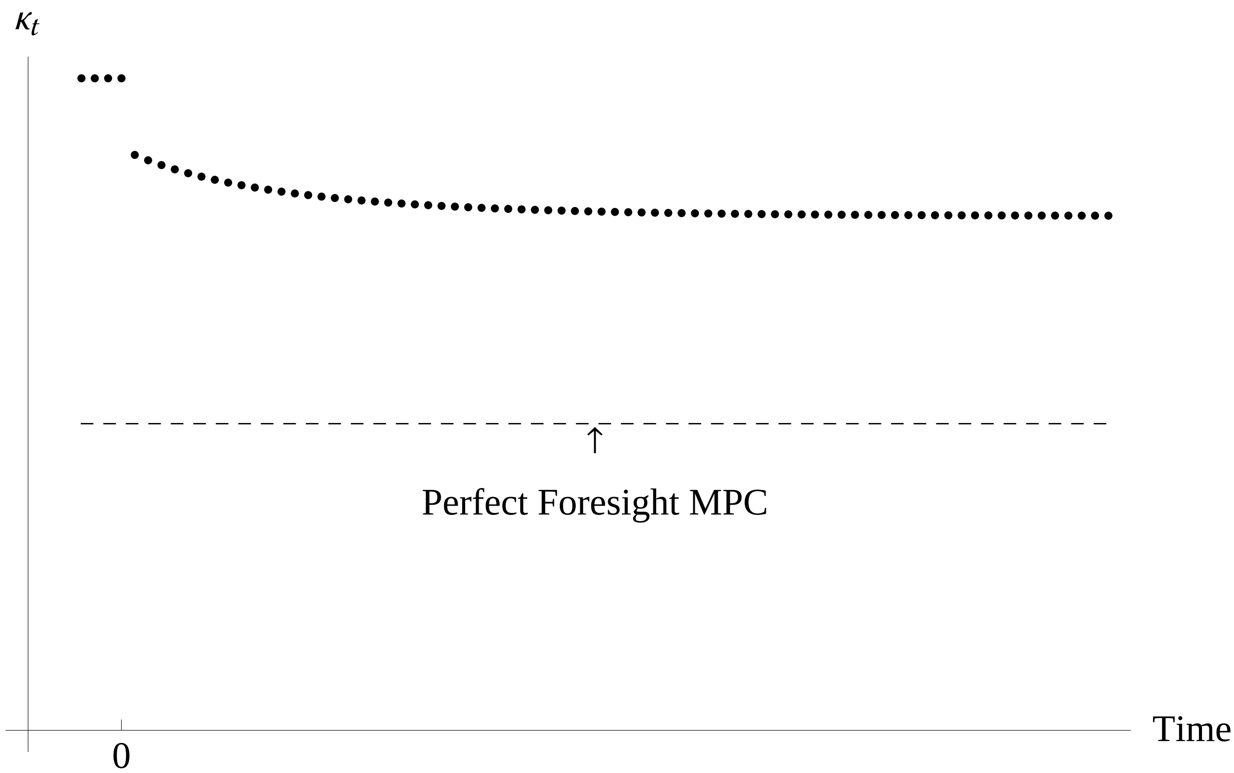


Table 1 Summary of Notation

a	-	end-of-period t assets (after consumption decision)
b	-	middle-of-period t balances (before consumption decision)
c	-	consumption
ℓ	-	personal labor productivity
m	-	market resources (capital, capital income, and labor income)
R, r	-	interest factor, rate
W	-	aggregate wage
G	-	growth factor for aggregate wage rate W
$\Gamma \equiv \mathsf{G}/\cancel{\mho}$	-	conditional (on employment) growth factor for individual labor income
γ	-	$\log \Gamma$, conditional growth *rate* for labor income
β	-	time preference factor $(= 1/(1+\vartheta))$
ξ	-	dummy variable indicating the employment state, $\xi \in \{0, 1\}$
κ	-	marginal propensity to consume
ρ	-	coefficient of relative risk aversion
ϑ	-	time preference rate $(\approx -\log \beta)$
$\mho$	-	probability of falling into permanent unemployment
$\cancel{\mho} = 1 - \mho$	-	probability of staying in employment from one period to the next
$\mathbf{Þ}$, þ	-	absolute patience factor, rate
$\mathbf{Þ}_{\Gamma}$, þ_{γ}	-	growth patience factor, rate
$\mathbf{Þ}_{\mathsf{R}}$, þ_{r}	-	return patience factor, rate
ω	-	excess prudence factor $(= (\rho - 1)/2)$
∇	-	proportional consumption drop upon entering unemployment
$\mathcal{R}$	-	short for R/Γ
Π	-	short for $\left(\frac{\mathbf{Þ}_{\Gamma}^{-\rho} - \cancel{\mho}}{\mho}\right)^{1/\rho}$

Appendix

A Taylor Approximation for Consumption Growth

Applying a Taylor approximation to (15), simplifying, and rearranging yields

$$\begin{aligned}
\left\{1+\mho\left[\left(\frac{c^e_{t+1}}{c^u_{t+1}}\right)^\rho-1\right]\right\}^{1/\rho} &= \left\{1+\mho\left[\left(\frac{c^u_{t+1}+c^e_{t+1}-c^u_{t+1}}{c^u_{t+1}}\right)^\rho-1\right]\right\}^{1/\rho} \\
&= \{1+\mho\left[(1+\nabla_{t+1})^\rho-1\right]\}^{1/\rho} \\
&\approx \left\{1+\mho\left[1+\rho\nabla_{t+1}+\rho(\nabla_{t+1})^2\omega-1\right]\right\}^{1/\rho} \\
&= \left\{1+\rho\mho(\nabla_{t+1}+(\nabla_{t+1})^2\omega)\right\}^{1/\rho} \\
&\approx 1+\mho\left(1+\nabla_{t+1}\omega\right)\nabla_{t+1}.
\end{aligned}$$

B The Exact Formula for $\check{m}$

The steady-state value of m^e will be where both (23) and (24) hold. To simplify the algebra, define $\zeta \equiv \mathcal{R}\kappa^u\Pi$ so that $\mathsf{R}\kappa^u\Pi = \zeta\Gamma$. Then:

$$\begin{aligned}
\left(\frac{\zeta}{1+\zeta}\right)\check{m} &= (1-\mathcal{R}^{-1})\check{m}+\mathcal{R}^{-1} \\
\left(\mathcal{R}\frac{\zeta}{1+\zeta}\right)\check{m} &= (\mathcal{R}-1)\check{m}+1 \\
\left(\mathcal{R}\left\{\frac{\zeta}{1+\zeta}-1\right\}+1\right)\check{m} &= 1 \\
\left(\mathcal{R}\left\{\frac{\zeta-(1+\zeta)}{1+\zeta}\right\}+\frac{1+\zeta}{1+\zeta}\right)\check{m} &= 1 \\
\left(\frac{1+\zeta-\mathcal{R}}{1+\zeta}\right)\check{m} &= 1 \\
\check{m} &= \left(\frac{1+\zeta}{1+\zeta-\mathcal{R}}\right) \\
\check{m} &= \left(\frac{1+\zeta+\mathcal{R}-\mathcal{R}}{1+\zeta-\mathcal{R}}\right) \\
&= 1+\left(\frac{\mathcal{R}}{1+\zeta-\mathcal{R}}\right) \\
&= 1+\left(\frac{\mathsf{R}}{\Gamma+\zeta\Gamma-\mathsf{R}}\right). \qquad (35)
\end{aligned}$$

A first point about this formula is suggested by the fact that

$$\zeta\Gamma = \mathsf{R}\kappa^u\left(1+\left(\frac{\text{Þ}_\Gamma^{-\rho}-1}{\mho}\right)\right)^{1/\rho} \qquad (36)$$

which is likely to increase as $\mho$ approaches zero.[27] Note that the limit as $\mho \to 0$ is infinity, which implies that $\lim_{\mho \to 0} \check{m} = 1$. This is precisely what would be expected from this model in which consumers are impatient but self-constrained to have $m^e > 1$: As the risk gets infinitesimally small, the amount by which target m^e exceeds its minimum possible value shrinks to zero.

We now show that the RIC and GIC ensure that the denominator of the fraction in (35) is positive:

$$
\begin{aligned}
\Gamma + \zeta\Gamma - \mathsf{R} &= \Gamma + \mathsf{R}\kappa^u \Pi - \mathsf{R} \\
&= \Gamma + \mathsf{R}\left(1 - \frac{(\mathsf{R}\beta)^{1/\rho}}{\mathsf{R}}\right)\left(\frac{(\frac{(\mathsf{R}\beta)^{1/\rho}}{\Gamma})^{-\rho} - 1}{\mho} + 1\right)^{1/\rho} - \mathsf{R} \\
&> \Gamma + \mathsf{R}\left(1 - \frac{(\mathsf{R}\beta)^{1/\rho}}{\mathsf{R}}\right)\left(\frac{(\frac{(\mathsf{R}\beta)^{1/\rho}}{\Gamma})^{-\rho} - 1}{1} + 1\right)^{1/\rho} - \mathsf{R} \\
&= \Gamma + \mathsf{R}\left(1 - \frac{(\mathsf{R}\beta)^{1/\rho}}{\mathsf{R}}\right)\frac{\Gamma}{(\mathsf{R}\beta)^{1/\rho}} - \mathsf{R} \\
&= \Gamma + \mathsf{R}\frac{\Gamma}{(\mathsf{R}\beta)^{1/\rho}} - \Gamma - \mathsf{R} \\
&= \mathsf{R}\left(\frac{\Gamma}{(\mathsf{R}\beta)^{1/\rho}} - 1\right) \\
&> 0.
\end{aligned}
$$

However, note that $\mho$ also affects Γ; thus, the first inequality above does not necessarily imply that the denominator is decreasing as $\mho$ moves from 0 to 1.

C An Approximation for $\check{m}$

Now defining

$$\aleph = \left(\frac{\mathbf{Þ}_\Gamma^{-\rho} - 1}{\mho}\right),$$

we can obtain further insight into (35) using a judicious mix of first- and second-order Taylor expansions (along with $\kappa^u = -\text{þ}_\mathsf{r}$):

$$
\begin{aligned}
\zeta\Gamma &= \mathsf{R}\kappa^u (1 + \aleph)^{1/\rho} \\
&\approx -\mathsf{R}\text{þ}_\mathsf{r}\left(1 + \rho^{-1}\aleph + (\rho^{-1})(\rho^{-1} - 1)(\aleph^2/2)\right) \\
&= -\mathsf{R}\text{þ}_\mathsf{r}\left(1 + \rho^{-1}\aleph\left\{1 + \left(\frac{1-\rho}{\rho}\right)(\aleph/2)\right\}\right)
\end{aligned}
\tag{37}
$$

But

$$\aleph = \left(\frac{(1 + \text{þ}_\gamma)^{-\rho} - 1}{\mho}\right) \tag{38}$$

[27]'Likely' but not certain because of the fact that $\mho$ affects $\mathbf{Þ}_\Gamma$ as well as appearing in the denominator of (35); however, for plausible calibrations the effect of the denominator predominates.

$$\approx \left(\frac{1-\rho \text{þ}_\gamma - 1}{\mho}\right)$$
$$\approx -\left(\frac{\rho \text{þ}_\gamma}{\mho}\right)$$

which can be substituted into (37) to obtain

$$\begin{aligned} \zeta\Gamma &\approx -\mathsf{R}\text{þ}_\mathsf{r}\left(1 - (\text{þ}_\gamma/\mho)(1 + (1-\rho)(-\text{þ}_\gamma/\mho)/2)\right) \\ &\approx \underbrace{-\mathsf{R}\text{þ}_\mathsf{r}}_{>0}\left\{1 \underbrace{-(\text{þ}_\gamma/\mho)}_{>0}\left(1 + \underbrace{(1-\rho)}_{<0}\underbrace{(-\text{þ}_\gamma/\mho)}_{>0}/2\right)\right\}. \end{aligned} \tag{39}$$

Letting ω capture the excess of prudence over the logarithmic case,

$$\omega \equiv \left(\frac{\rho - 1}{2}\right), \tag{40}$$

(35) can be approximated by

$$\begin{aligned} \check{m} &\approx 1 + \left(\frac{1}{\Gamma/\mathsf{R} - \text{þ}_\mathsf{r}\left(1 - (\text{þ}_\gamma/\mho)(1 - (-\text{þ}_\gamma/\mho)\omega)\right) - 1}\right) \\ &\approx 1 + \left(\frac{1}{(\gamma - \mathsf{r}) + (-\text{þ}_\mathsf{r})\left(1 + (-\text{þ}_\gamma/\mho)(1 - (-\text{þ}_\gamma/\mho)\omega)\right)}\right) \end{aligned} \tag{41}$$

where negative signs have been preserved in front of the þ_r and þ_γ terms as a reminder that the GIC and the RIC imply these terms are themselves negative (so that $-\text{þ}_\mathsf{r}$ and $-\text{þ}_\gamma$ are positive). *Ceteris paribus*, an increase in relative risk aversion ρ will increase ω and thereby decrease the denominator of (41). This suggests that greater risk aversion will result in a larger target level of wealth.[28]

The formula also provides insight about how the human wealth effect works in equilibrium. All else equal, the human wealth effect is captured by the $(\gamma - \mathsf{r})$ term in the denominator of (41), and it is obvious that a larger value of γ will result in a smaller target value for m. But it is also clear that the size of the human wealth effect will depend on the magnitude of the patience and prudence contributions to the denominator, and that those terms can easily dominate the human wealth effect.

For (41) to make sense, we need the denominator of the fraction to be a positive number; defining

$$\hat{\text{þ}}_\gamma \equiv \text{þ}_\gamma(1 - (-\text{þ}_\gamma/\mho)\omega), \tag{42}$$

this means that we need:

$$\begin{aligned} (\gamma - \mathsf{r}) &> \text{þ}_\mathsf{r} - \text{þ}_\mathsf{r}\hat{\text{þ}}_\gamma/\mho \\ &= \left(\rho^{-1}(\mathsf{r} - \vartheta) - \mathsf{r}\right) - \text{þ}_\mathsf{r}\hat{\text{þ}}_\gamma/\mho \\ \gamma &> \rho^{-1}(\mathsf{r} - \vartheta) - \text{þ}_\mathsf{r}\hat{\text{þ}}_\gamma/\mho \end{aligned}$$

[28] "Suggests" because this derivation used some dubious approximations; the suggestion can be verified, however, for plausible numerical calibrations.

$$
\begin{aligned}
0 &> \underbrace{\rho^{-1}(\mathsf{r}-\vartheta)-\gamma}_{\text{þ}_\gamma} - \text{þ}_\mathsf{r}(\hat{\text{þ}}_\gamma/\mho) \\
0 &> \text{þ}_\gamma - \text{þ}_\mathsf{r}(\hat{\text{þ}}_\gamma/\mho).
\end{aligned} \tag{43}
$$

But since the RIC guarantees $\text{þ}_\mathsf{r} < 0$ and the GIC guarantees $\text{þ}_\gamma < 0$ (which, in turn, guarantees $\hat{\text{þ}}_\gamma < 0$), this condition must hold.[29]

The same set of derivations imply that we can replace the denominator in (41) with the negative of the RHS of (43), yielding a more compact expression for the target level of resources,

$$
\begin{aligned}
\check{m} &\approx 1 + \left(\frac{1}{\text{þ}_\mathsf{r}(\hat{\text{þ}}_\gamma/\mho) - \text{þ}_\gamma} \right) \\
&= 1 + \left(\frac{1/(-\text{þ}_\gamma)}{1 + (-\text{þ}_\mathsf{r}/\mho)(1 + (-\text{þ}_\gamma/\mho)\omega)} \right).
\end{aligned} \tag{44}
$$

This formula makes plain the fact that an increase in either form of impatience, by increasing the denominator of the fraction in (44), will reduce the target level of assets.

We are now in position to discuss (41), understanding that the impatience conditions guarantee that its numerator is a positive number.

Two specializations of the formula are particularly useful. The first is the case where $\rho = 1$ (logarithmic utility). In this case,

$$
\begin{aligned}
\text{þ}_\mathsf{r} &= -\vartheta \\
\text{þ}_\gamma &= \mathsf{r} - \vartheta - \gamma \\
\omega &= 0
\end{aligned}
$$

and the approximation becomes

$$
\check{m} \approx 1 + \left(\frac{1}{(\gamma - \mathsf{r}) + \vartheta(1 + (\gamma + \vartheta - \mathsf{r})/\mho)} \right) \tag{45}
$$

which neatly captures the effect of an increase in human wealth (via either increased γ or reduced r), the effect of increased impatience ϑ, or the effect of a reduction in unemployment risk $\mho$ in reducing target wealth.

The other useful case to consider is where $\mathsf{r} = \vartheta$ but $\rho > 1$. In this case,

$$
\begin{aligned}
\text{þ}_\mathsf{r} &= -\vartheta \\
\text{þ}_\gamma &= -\gamma \\
\hat{\text{þ}}_\gamma &= -\gamma(1 - (\gamma/\mho)\omega)
\end{aligned}
$$

so that

$$
\check{m} \approx 1 + \left(\frac{1}{(\gamma - \mathsf{r}) + \vartheta(1 + (\gamma/\mho)(1 - (\gamma/\mho)\omega))} \right) \tag{46}
$$

where the additional term involving ω in this equation captures the fact that an increase

[29] In more detail: For the second-order Taylor approximation in (37), we implicitly assume that the absolute value of the second-order term is much smaller than that of the first-order one, i.e. $|\rho^{-1}\aleph| \geq |(\rho^{-1})(\rho^{-1}-1)(\aleph^2/2)|$. Substituting (38), the above could be simplified to $1 \geq (-\text{þ}_\gamma/\mho)\omega$, therefore we have $\hat{\text{þ}}_\gamma < 0$. This simple justification is based on the proof above that RIC and GIC guarantee the denominator of the fraction in (35) is positive.

in the prudence term ω shrinks the denominator and thereby boosts the target level of wealth.[30]

[30] It would be inappropriate to use the equation to consider the effect of an increase in r because the equation was derived under the assumption $\vartheta = \mathsf{r}$ so r is not free to vary.

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