

Population of the employed in period t : A population of size Ξ^t was born in the current period t and is employed. A population of size Ξ^{t-n} was born n periods ago, and a $\mathcal{X}^n$ portion of this population is still employed today. Summing across all previous generations, the total current employed population is:

$$\mathcal{E}_t = \sum_{n=0}^{\infty} \Xi^{t-n} \mathcal{X}^n = \Xi^t \cdot \sum_{n=0}^{\infty} \left(\frac{\mathcal{X}}{\Xi}\right)^n = \Xi^t \frac{1}{1 - \frac{\mathcal{X}}{\Xi}} = \frac{\Xi^{t+1}}{\Xi - \mathcal{X}}$$

Population of the unemployed in period t : For any period $t - n$, a population of size $\mathcal{U}\mathcal{E}_{t-n}$ entered unemployment at this time. Individuals in this population are still alive in proportion $\mathcal{X}^{n-1}$. Summing across all previous generations of newly unemployed, the total population of unemployed is:

$$\mathcal{U}_t = \sum_{n=1}^{\infty} \mathcal{X}^{t-n} \mathcal{U}\mathcal{E}_{t-n} = \sum_{n=1}^{\infty} \mathcal{U}\mathcal{X}^{t-n} \frac{\Xi^{t-n+1}}{\Xi - \mathcal{X}} = \frac{\mathcal{U}}{\Xi} \mathcal{E}_t \sum_{n=1}^{\infty} \left(\frac{\mathcal{X}}{\Xi}\right)^n = \frac{\mathcal{U}}{\Xi} \mathcal{E}_t \frac{1}{1 - \frac{\mathcal{X}}{\Xi}} = \frac{\mathcal{U}\mathcal{E}_t}{\Xi - \mathcal{X}} = \frac{\mathcal{U}\Xi^{t+1}}{(\Xi - \mathcal{U})(\Xi - \mathcal{X})}$$

Wage equation: GDP in period t is $\mathbf{P}_t = \mathbf{K}_t^\alpha (z_t \mathbf{L}_t)^{1-\alpha}$. The wage rate is the marginal product of labor:

$$\mathbf{W}_t = \frac{\partial \mathbf{P}_t}{\partial \mathbf{L}_t} = \mathbf{K}_t^\alpha (1 - \alpha) (z_t \mathbf{L}_t)^{1-\alpha} / \mathbf{L}_t = (1 - \alpha) \mathbf{P}_t$$

Euler equation for the employed consumer: The consumer's Euler equation is, as usual: $u'(\mathbf{c}_t) = \beta RE[u'(\mathbf{c}_{t+1})]$. For an employed consumer with a CRRA utility function, this can be written as:

$$u'(\mathbf{c}_t^e) = \beta R(\mathcal{X}u'(\mathbf{c}_{t+1}^e) + \mathcal{U}u'(\mathbf{c}_{t+1}^u)) = \beta R(\mathcal{X}(\mathbf{c}_{t+1}^e)^{-\rho} + \mathcal{U}(\mathbf{c}_{t+1}^u)^{-\rho})$$

Proof of eq:Explicit: First, note that for an employed consumer, total pay grows by: $\frac{W_{t+1}\ell_{t+1}}{W_t\ell_t} = \Gamma$, as wages grow at rate $\mathbf{G}$ while labor supply grows by $\mathbf{X}$ each period. Thus we can see that $\frac{c_{t+1}^e}{c_t^e} = \frac{c_{t+1}^e W_{t+1}\ell_{t+1}}{c_t^e W_t\ell_t} = \frac{c_{t+1}^e}{c_t^e} \Gamma$. Similarly, $\frac{(c_{t+1}^e)^{-\rho}}{(c_t^e)^{-\rho}} = (\frac{c_{t+1}^e}{c_t^e})^{-\rho} \Gamma^{-\rho}$. Thus we can divide both sides of the employed consumer's Euler equation by $(c_t^e)^{-\rho}$ to arrive immediately at eq:Explicit.

Proof of eq:cedel: Substituting in the definition of the growth patience factor, eq:Explicit now says: $1 = \mathbb{P}_\Gamma^\rho (\mathcal{V}(\frac{c_{t+1}^e}{c_t^e})^{-\rho} + \mathcal{U}(\frac{c_{t+1}^u}{c_t^e})^{-\rho})$. Divide both sides by $(\frac{c_{t+1}^e}{c_t^e})^{-\rho}$ to get:

$$(\frac{c_{t+1}^e}{c_t^e})^\rho = \mathbb{P}_\Gamma^\rho (\mathcal{V} + \mathcal{U}(\frac{c_{t+1}^u}{c_t^e})^{-\rho}) \Rightarrow (\frac{c_{t+1}^e}{c_t^e}) = \mathbb{P}_\Gamma (\mathcal{V} + \mathcal{U}(\frac{c_{t+1}^u}{c_t^e})^{-\rho})^{1/\rho} \Rightarrow (\frac{c_{t+1}^e}{c_t^e}) = \mathbb{P}_\Gamma (1 + \mathcal{U}((\frac{c_{t+1}^u}{c_t^e})^{-\rho})^{1/\rho})^{1/\rho}$$

Proof of eq:ce t+1: First, combine eq:normibc with the solution to the unemployed consumer's problem to find that: $c_{t+1}^u = \kappa^u \bar{\Gamma} (s_{t+1}^e - c_{t+1}^e + 1)$. Next, take eq:cedel, multiply both sides by c_t^e and raise both sides to the ρ power to get:

$$(c_{t+1}^e)^\rho = \mathbb{P}_\Gamma^\rho (c_t^e)^\rho (1 + \mathcal{U}((\frac{c_{t+1}^u}{c_t^e})^{-\rho})^{1/\rho})^{1/\rho}$$

Next, isolate the factors of $(c_{t+1}^e)^\rho$ to find that:

$$\mathcal{V} \mathbb{P}_\Gamma^\rho (c_t^e)^\rho = (c_{t+1}^e)^\rho (1 - \mathbb{P}_\Gamma (c_t^e)^\rho \mathcal{U}(c_{t+1}^u)^{-\rho})$$

Divide the equation by the factor on $(c_{t+1}^e)^\rho$ and raise to the power $1/\rho$:

$$c_{t+1}^e = \mathbb{P}_\Gamma \mathcal{V}^{1/\rho} c_t^e (1 - \mathcal{U} \mathbb{P}_\Gamma^\rho (\frac{c_t^e}{c_{t+1}^u})^{-\rho})^{-1/\rho}$$

Now we can substitute in the equation for c_{t+1}^u and rearrange a bit:

$$c_{t+1}^e = \mathbb{P}_\Gamma \mathcal{V}^{1/\rho} c_t^e (1 - \mathcal{U} \mathbb{P}_\Gamma^\rho (\frac{c_t^e}{\kappa^u \bar{\Gamma} (s_{t+1}^e - c_{t+1}^e + 1)})^{-\rho})^{-1/\rho} = \mathbb{P}_\Gamma \mathcal{V}^{1/\rho} c_t^e (1 - \mathcal{U} (\frac{\mathbb{P}_\Gamma \Gamma}{\mathbb{R} \kappa^u})^\rho (\frac{c_t^e}{s_{t+1}^e - c_{t+1}^e + 1})^{-\rho})^{-1/\rho}$$

Now we only have to show that $\frac{D_\Gamma \Gamma}{R \kappa^u} = \frac{1-\kappa^u}{\phi \kappa^u}$ and we are done. $D_\Gamma \Gamma = (\beta R)^{1/\rho}$ Further, $(\beta R)^{1/\rho}/R = (1 - \kappa^u)/\phi$, and so we arrive at the result.

Proof of eq:sTarg: To be completed.

Proof of eq:SuOpLev: Because $\frac{S_t^e}{P_t}$ is constant and P_t grows by ΞG each period, S^e must also grow by the factor ΞG . Individuals who became unemployed n periods ago have $(R(1 - \kappa^u))^n = (\phi(\beta R)^{1/\rho})^n$ portion of their original wealth. This starting wealth is $(\Xi G)^{-n} \frac{S_t^e}{P_t} \bar{U}$ (a $\bar{U}$ portion of total wealth belongs to newly unemployed individuals). Summing across all past cohorts to become unemployed, this is:

$$\frac{S_t^u}{P_t} = \bar{U} \frac{S^e}{P} \cdot \sum_{n=0}^{\infty} \left(\frac{\phi(\beta R)^{1/\rho}}{\Xi G} \right)^n = \bar{U} \frac{S^e}{P} \frac{1}{1 - \frac{\phi(\beta R)^{1/\rho}}{\Xi G}} = \frac{S^e}{P} \frac{\Xi G \bar{U}}{\Xi G - \phi(\beta R)^{1/\rho}}$$

Proof of eq:NOpLev: First, note that $S = S^e + S^u \Rightarrow \frac{S}{P} = \frac{S^e}{P} + \frac{S^u}{P}$. Using the result from eq:SuOpLev, we can also write that: $\frac{S}{P} = \frac{S^e}{P} \left(1 + \frac{\Xi G \bar{U}}{\Xi G - \phi(\beta R)^{1/\rho}} \right)$. S and K both grow by factor ΞG in each period, thus $K_{t+1} = \Xi G K_t$ and $S_{t+1} = \Xi G S_t$. Using this, eq:NFALev can be written as:

$$N_t = \Xi G (S_t R^{-1} - K_t) \Rightarrow \frac{N_t}{P_t} = \Xi G \left(\frac{S_t}{P_t} R^{-1} - \frac{K_t}{P_t} \right) = \Xi G \left(R^{-1} \left(1 + \frac{\Xi G \bar{U}}{\Xi G - \phi(\beta R)^{1/\rho}} \right) \frac{S^e}{P} - \frac{\alpha}{R - (1 - \delta)} \right)$$

Proof of eq:sTargWithSocIns: To be completed.