Drawbacks of Numerical Solutions

A Black Box

- Can Construct Solution to Model Without Really Understanding It
- Hard Even To Be Sure Your Numerical Solution Is Right
- Little Intuition for How Results Might Change With
  - Calibration
  - Structure
- Very Hard To Teach!

I Am A Big Fan Of Numerical Methods

- Have Done A Good Deal Of Work With Them Myself
- But As A Result, Have Felt All These Drawbacks Keenly
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The Gap This Paper Fills

Foundations For Microeconomic Household’s Problem With

- Uncertain Labor Income
- No Liquidity Constraints
- CRRA Utility
- (Problem with Liquidity Constraints Is A Limiting Case)
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Key Result

Restrictions On Parameter Values Such That

- Problem Defines A Contraction Mapping
  - $\Rightarrow \exists$ A Unique Consumption Function $c(m)$
- There Is A ‘Target’ Ratio Of Assets to Permanent Income
  - Requires A Key ‘Impatience’ Condition To Hold
  - Good News
    - Condition Is Weaker (Easier To Satisfy) Than Previous Papers Imposed
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  - Good news
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Limit as horizon \( T \) goes to infinity of

\[
\begin{align*}
\mathbf{a}_t &= \mathbf{m}_t - \mathbf{c}_t \\
\mathbf{b}_{t+1} &= \mathbf{a}_t \mathbf{R} \\
\mathbf{p}_{t+1} &= \mathbf{p}_t \left( \Gamma^{t+1} \psi_{t+1} \right) \\
&= \Gamma^{t+1} \\
\mathbf{m}_{t+1} &= \mathbf{b}_{t+1} + \mathbf{p}_{t+1} \xi_{t+1},
\end{align*}
\]

\[\xi_{t+n} = \begin{cases} 
0 & \text{with probability } \varnothing > 0 \\
\theta_{t+n}/(1 - \varnothing) & \text{with probability } (1 - \varnothing)
\end{cases}\]

\[
\mathbf{u}(\bullet) = \bullet^{1-\rho}/(1 - \rho); \quad \mathbf{E}_t[\psi_{t+n}] = \mathbf{E}_t[\xi_{t+n}] = 1 \quad \forall \ n > 0; \quad \beta < 1, \rho > 1
\]
Surely This Problem Has Been Solved?

No.

- Can’t Use Stokey et. al. theorems because CRRA utility
- Lit thru Matkowski and Nowak (2011) Can’t Handle Permanent Shocks
- Must Use Boyd’s ‘Weighted’ Contraction Mapping Theorem
- Surprisingly Subtle

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Benchmark: Perfect Foresight Model

Definitions:

- **Absolute Patience Factor**
  \[ \mathcal{P} = (R\beta)^{1/\rho} \]

- **Return Patience Factor**
  \[ \mathcal{P}_R = \frac{\mathcal{P}}{R} \]

- **Perfect Foresight Growth Patience Factor**
  \[ \mathcal{P}_\Gamma = \frac{\mathcal{P}}{\Gamma} \]

<table>
<thead>
<tr>
<th>Name</th>
<th>Condition</th>
<th>Implication</th>
</tr>
</thead>
<tbody>
<tr>
<td>(AIC) Absolute Impatience Condition</td>
<td>( \mathcal{P} &lt; 1 )</td>
<td>( c \downarrow ) over time</td>
</tr>
<tr>
<td>(RIC) Return Impatience Condition</td>
<td>( \mathcal{P}_R &lt; 1 )</td>
<td>( c/a \downarrow ) over time</td>
</tr>
<tr>
<td>(PFGIC) Growth Impatience Condition</td>
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</tbody>
</table>
When Does A Useful Limiting Solution Exist?

Finite Human Wealth (FHWC) condition:

\[ \Gamma < R \]  \hspace{1cm} (3)

Return Impatience Condition:

\[ \beta_R < R \]  \hspace{1cm} (4)
When Does A Useful Limiting Solution Exist?

Finite Human Wealth (FHWC) condition:

\[ \Gamma < R \quad (3) \]

Return Impatience Condition:

\[ \mathcal{R}_R < R \quad (4) \]
What If There Are Liquidity Constraints?

- FHWC is *not* necessary for solution to exist
- Other Key Condition For Useful Solution is
  ‘Perfect Foresight Finite Value of Autarky Condition (PFFVAC )’:

\[ \beta \Gamma^{1-\rho} < 1 \]  

(5)

- Without RIC, Constraints Are Irrelevant
  - Because Wealth Always Wants To Rise, So Constraint Never Binds
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Liquidity Constraints and Uncertainty

- Introduce permanent shocks to income
- Finite Value of Autarky Condition Becomes

\[ \beta \Gamma^{1-\rho} < 1 \]
\[ \beta < \Gamma^{\rho-1} \]
Finite Value of Autarky Condition: Same As In Liq Constr Problem!

\[
\beta \Gamma^{1-\rho} < 1 \\
\beta < \Gamma^{\rho-1}
\]

(7)

‘Weak Return Impatience Condition’ (WRIC)

\[
0 \leq \varphi^{1/\rho} D_R < 1
\]

(8)
Requirement For Existence Of A Target

Definitions: ‘Uncertainty-Adjusted’ Growth:

\[ \Gamma = \Gamma \psi \]  \hspace{2cm} (9)

Adjusted Growth Patience Factor:

\[ \Phi \prescript{\Gamma} = \Phi / \Gamma \]  \hspace{2cm} (10)

Growth Impatience Condition:

\[ \Phi \prescript{\Gamma} < 1 \]  \hspace{2cm} (11)

Why? Because it can be shown that

\[ \lim_{m_t \to \infty} E_t \left[ \frac{m_{t+1}}{m_t} \right] = \Phi \prescript{\Gamma} \]  \hspace{2cm} (12)
Five Propositions

1. $\lim_{m_t \to \infty} \mathbb{E}_t \left[ \frac{c_{t+1}}{c_t} \right] = \mathbb{P}$
2. $\lim_{m_t \to 0} \mathbb{E}_t \left[ \frac{c_{t+1}}{c_t} \right] = \infty$
3. $\exists$ a unique target value of $m$, called $\tilde{m}$
4. $\mathbb{E}_t \left[ \frac{c_{t+1}}{c_t} \big| m_t = \tilde{m} \right] = \Gamma - \epsilon$
5. $\left( \frac{d\mathbb{E}_t[c_{t+1}/c_t]}{dm_t} \right) < 0$
The Target Saving Figure

\[ \mathbb{E}_t[c_{t+1}/c_t] = (R\beta)^{1/\rho} \]

\[ \tilde{m} \]

\[ m_t \]

Growth Rate

\[ \Gamma \]
Bounds On the Consumption Function

\[ \overline{c}(m) = \overline{κ}m = (1 - \frac{1}{\rho_1/P_R})m \]

Upper Bound = Min \[ \{\overline{c}(m), \overline{c}(m)\} \]

\[ \overline{c}(m) = (m - 1 + h)\overline{κ} \]

\[ \underline{c}(m) = (1 - \frac{1}{\rho_1})κm \]
The Marginal Propensity to Consume

\[ \kappa(m) \equiv c'(m) \]

\[ \kappa \equiv (1 - \frac{p}{\rho}) \]

\[ (1 - \frac{\varphi^{1/\rho}p_R}{\rho}) \equiv \bar{\kappa} \]
The Consumption Function and Target Wealth

\[ c(m_t) \]

\[ \mathbb{E}_t[\Delta m_{t+1}] = 0 \]
Convergence To The Invariant Distribution

Szeidl (2012) Proves Existence of an Invariant Distribution of $m, c, a, \text{etc.}$
Balanced Growth Equilibrium

Achieved When Cross Section Distribution Reaches Invariance

\[ \frac{Y_{t+1}}{Y_t} = \frac{C_{t+1}}{C_t} = \Gamma \]  

(13)

Fisherian Separation Fails, Even Without Liquidity Constraints!

Insight:

- Precautionary Saving \( \approx \) Liquidity Constraints
- If \( \check{c}(m) \) is solution for constrained consumer,
  \[ \lim_{\varphi \downarrow 0} c(m; \varphi) = \check{c}(m) \]  

(14)
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(14)
The MPC Out Of Permanent Shocks

http://econ.jhu.edu/people/ccarroll/papers/MPCPerm.pdf

Lots of Recent Papers Trying to Measure the MPC

Paper Proves:

- MPCP < 1
- But not a lot less:
  - 0.75 to 0.95 (annual rate) for wide range of parameter values
Defined Conditions Under Which Widely Used Problem Has Solution
  - Finite Value of Autarky Condition Guarantees Contraction (with WRIC)
  - Growth Impatience Condition Prevents $m \to \infty$

Economy Of Buffer Stock Consumers Exhibits Balanced Growth
  - Even In Absence of General Equilibrium Adj of Interest Rate