# On the Concavity of the Consumption Function

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Abstract

At least since Keynes (1935), many economists have had the intuition that the marginal propensity to consume out of wealth declines as wealth increases. Nonetheless, standard perfect-certainty and certainty equivalent versions of intertemporal optimizing models of consumption imply a marginal propensity to consume that is unrelated to the level of household wealth. We show that adding income uncertainty to the standard optimization problem induces a concave consumption function in which, as Keynes suggested, the marginal propensity to consume out of wealth or transitory income declines with the level of wealth.

**Keywords**: consumption function, policy rule, uncertainty, concavity, stochastic income, certainty equivalent, precautionary saving, marginal propensity to consume.

**JEL Classification Codes:** C6, D91, E21

# 1 Introduction

One of the more surprising discoveries in macroeconomic theory over the past ten years has been how much the solution to the optimal intertemporal consumption problem can change when income uncertainty is introduced into the problem. In an important early paper, Zeldes (1989), using numerical methods, found that introducing labor income uncertainty made the consumption function concave, with the marginal propensity to consume everywhere higher than in the certainty case. Kimball (1990a, 1990b) gives the analytical explanation for the increase in the slope of the consumption function, but until now there has been no analytical explanation for the concavity of the consumption function that income uncertainty seemed to induce.

The idea that the consumption function is concave is an old one, dating at least to the origin of Keynesian macroeconomics: *The General Theory of Employment, Interest, and Money* emphasized the central importance of the consumption function, and explicitly argued that the consumption function is concave.[1] Nonetheless, standard (and widely used) perfect-certainty and certainty-equivalent versions of intertemporal optimizing models imply a marginal propensity to consume that is unrelated to the level of household wealth. We show that, except under very special conditions, adding income uncertainty to the standard optimization problem induces a concave consumption function in which, as Keynes suggested, the marginal propensity to consume out of wealth or transitory income declines with the level of wealth.

There appears to be little recent direct empirical evidence on whether the consumption function is concave. We are aware of only two papers, one by Lusardi (1992) and one by Souleles (1995). Both of these papers find that the marginal propensity to consume is substantially higher for consumers with low wealth or low income than for consumers with high wealth or income.[2] There is a substantial older literature, dat-

[1]Keynes (1935) (p. 31) writes: "... the marginal propensity to consume [is] weaker in a wealthy community ..."; and, later (p. 349), "... with the growth in wealth [comes] the diminishing marginal propensity to consume ..." See also the discussion of a diminishing marginal propensity to consume out of current income on page 120.

[2]Specifically, Souleles (1995) finds, for the low-wealth and low-income households, a higher marginal propensity to consume nondurables out of income tax refunds, and Lusardi (1992) finds similar results

ing to early tests of the Permanent Income Hypothesis, which found some evidence of concavity.[3]

Concavity of the consumption function is interesting for several reasons. The most important is probably that strict concavity implies that consumption growth depends on the level of wealth, which is serially correlated, so the implication of Hall (1978) that consumption should follow a random walk at the household level no longer holds. In practice, most Euler equation tests have implicitly assumed that the consumption function is linear, or approximately linear. [4] At the aggregate level, concavity means that the entire wealth distribution is an omitted variable when estimating aggregate consumption Euler equations, and so the random walk implication again fails.[5]

At a more conceptual level, Carroll (1995a, 1995b) shows that the concavity of the consumption function induced by uncertainty is a key element in understanding how "buffer-stock" (target saving) behavior can emerge from the standard unconstrained Life Cycle/Permanent Income Hypothesis consumption model when labor income uncertainty is introduced.

Another result that follows from the concavity of the consumption function is a strengthening of Kimball's (1990a, 1990b) proof that uncertainty increases the marginal propensity to consume. Kimball (1990a, 1990b) shows that uncertainty increases the marginal propensity to consume at a given level of consumption, but not necessarily at a given level of wealth. Concavity of the consumption function implies that uncertainty increases the marginal propensity to consume at a given level of *wealth* even more than

when she estimates the MPC out of predictable changes in income.

[3]Bodkin (1959) found a high MPC for recipients of an unexpected windfall that the U.S. government conferred on certain World War II veterans. Kreinin (1961), by contrast, found that Israeli recpients of restitution payments from Germany in the late 1950s had much lower MPCs. The discrepancy in these two results was attributed by some observers to the fact that the Israeli windfalls were much larger, relative to recipient income, than the WWII windfalls, an explanation which implies a concave consumption function. Landsberger (1966) sorted the data by the ratio of the windfall to income, and found that the MPC fell as the size of the payment increased, implying a concave consumption function.

[4]Of the 25 household-level studies summarized in the recent survey by Browning and Lusardi (1995), only two allow for a nonlinear consumption function by allowing the variance term in the log-linearized Euler equation to vary across households.

[5]For more discussion of this point, see Carroll (1992, 1995a).

at a given level of consumption.[6]

One final implication of concavity of the consumption function is a corollary of the link Kimball (1990a) shows between the marginal propensity to consume and the effective risk aversion of the value function. A higher marginal propensity to consume raises the effective risk aversion of the value function by raising the consumption covariance resulting from a given financial risk. A concave consumption function implies that the effective level of risk aversion will fall more quickly with wealth than it would if the marginal propensity to consume were constant. As a consequence, labor income risk will make financial risk-taking more strongly related to wealth than it otherwise would be, even when the financial risks are statistically independent of the labor income risk.[7]

These implications of the concavity result illustrate why it is important for us to have a rigorous analytical understanding of why the consumption rule is concave. This paper provides an analytical demonstration of the concavity of the consumption function under uncertainty for a key class of utility functions. We show that the consumption rule is concave whenever the intertemporally separable period utility function is drawn from the class of functions which exhibit Hyperbolic Absolute Risk Aversion (HARA) and have a positive precautionary saving motive, i.e. those functions which satisfy the condition $u'''u'/u''^2 = k > 0$.[8] Most commonly used utility functions are of the HARA class: Quadratic utility is the borderline case, corresponding to $k = 0$, Constant Absolute Risk Aversion (CARA) corresponds to $k = 1$, and Constant Relative Risk Aversion (CRRA) utility functions satisfy $k > 1$.[9]

We further show that if $k > 0$ the consumption function will be *strictly* concave

[6]This follows because positive precautionary saving implies that the same level of consumption is at a higher level of wealth for the uncertain consumption function. The same level of wealth is lower on the uncertain consumption function and therefore must have an even higher marginal propensity to consume.

[7]This implication of a greater dependence of financial risk-taking on wealth goes beyond the results of Kimball (1993) which show only that the labor income risk will reduce the *level* of financial risk taking.

[8]This yields $-u''(x)/u'(x) = \frac{1}{A+(k-1)x}$ whence the name HARA. See the appendix for further analysis of this class.

[9]This class can also accomodate CRRA utility functions with a shifted origin, which are sometimes used to model a necessity level of consumption.

except under very special circumstances. The exceptions to strict concavity include two well-known cases: CARA utility if all of the risk is to labor income (no rate of return risk), and CRRA utility if all of the risk is rate-of-return risk (no labor income risk). These special cases have been widely used because of their analytical convenience (they yield a linear consumption function), but the analytical results in this paper bolster the argument (made forcefully by Kimball (1990a), Carroll (1995a), Deaton (1992) and others) that it is in most cases unwise to rely on these analytically convenient formulations because the behavior they imply is qualitatively quite different from behavior in the general case.

In the course of the proof, we also generalize the result of Neave (1971) that decreasing absolute risk aversion is inherited by the value function and of Sibley (1975) that a positive third derivative is inherited by the value function.

To understand our proof, it is helpful to see the value function as produced from the underlying utility function by two operations: first, addition of utility or value functions across states of nature at a given point in time; and, second, intertemporal aggregation, by maximizing the sum of utility in one period and the value from all subsequent periods. Since maximization of this sum requires equating (some ratio of) marginal utilities between present and future, and this marginal utility gives the marginal utility of the intertemporal aggregate, intertemporal aggregation involves a kind of horizontal addition of marginal utility, to match the vertical addition of marginal utility when an expectation is taken across states of nature at one point in time. It is easy to see that convexity of marginal utility ($u''' > 0$) is preserved under both vertical and horizontal addition of marginal utility. We show that the property $\frac{u'''u'}{u''}^2 \geq k$, or, equivalently, $-u'''/u'' \geq k(-u''/u')$ –that is, prudence (as defined by Kimball (1990b)) is greater than $k$ times risk aversion (as defined by Pratt, (1964))[10]–is preserved under both vertical (Lemma 1) and horizontal (Lemma 2) addition of marginal utility. Neither of these Lemmas depends on the assumption of HARA utility, and so both are quite general. The HARA assumption is used only in Lemma 3, where we show that if a period's utility function is HARA with prudence *equal* to $k$ times risk aversion and prudence is

[10]This is equivalent to convexity of $\frac{u'^{1-k}}{1-k}$ (or convexity of $\log u'$ when $k = 1$).

greater than or equal to $k$ times risk aversion for the future value function, then the current-period consumption rule is concave.

# 2 Proofs

We assume that the consumer is maximizing the time-additive present discounted value of utility from consumption $u(c)$. Denoting the (possibly stochastic) gross interest rate and time preference factors as $\tilde{R}_t \in (0, \infty)$ and $\tilde{\beta}_t \in (0, \infty)$, respectively, and labelling consumption $c_t$, stochastic labor income $\tilde{y}_t$, and gross wealth (inclusive of labor income) $w_t$, the consumer's problem can be written as:[11]

$$
\begin{aligned}
V_t(w_t) &= \max_{c_t} \; u(c_t) + E_t \sum_{s=t+1}^{T} \left( \prod_{j=t+1}^{s} \tilde{\beta}_j \right) u(c_s) && (1) \\
s.t. \quad & w_{t+1} = \tilde{R}_{t+1}(w_t - c_t) + \tilde{y}_{t+1} && (2) \\
& c_T \leq w_T. && (3)
\end{aligned}
$$

As usual, the recursive nature of the problem allows us to rewrite this equation as:

$$V_t(w_t) = \max_{c_t} \; u(c_t) + E_t \tilde{\beta}_{t+1} V_{t+1}(\tilde{R}_{t+1}(w_t - c_t) + \tilde{y}_{t+1}), \tag{4}$$

or, for convenience defining $\phi_t(s_t) = E_t \tilde{\beta}_{t+1} V_{t+1}(\tilde{R}_{t+1} s_t + \tilde{y}_{t+1})$, where $s_t = w_t - c_t$ is the portion of period t resources saved, we have:

$$V_t(w_t) = \max_{c_t} \; u(c_t) + \phi_t(w_t - c_t). \tag{5}$$

For notational simplicity we express the value function $V_t(w_t)$ and the expected discounted value function $\phi_t(s_t)$ as functions simply of wealth and savings, but implicitly these functions reflect the entire information set as of time t; if, for example, the

[11] We allow for a stochastic discount factor because some problems which contain a stochastic scale variable (such as permanent income) can be analyzed more easily by dividing the problem through by the scale variable; this division induces a term that effectively plays the role of a stochastic discount factor.

income process is not i.i.d., then information on lagged income or income shocks could be important in determining current optimal consumption. In the remainder of the paper the dependence of functions on the entire information set as of time t will be unobtrusively indicated, as here, by the presence of the t subscript. For example, we will call the policy rule in period t which indicates the optimal value of consumption $c_t^*(w_t)$. In contrast, because we assume that the utility function is the same from period to period, the utility function has no t subscript.

We are now in a position to state the main theorem of this paper. Defining a "permissible income process" as any income process which permits the agent to ensure that consumption remains within the domain over which $u(c)$ is defined, the theorem is:

**Theorem 1** *For utility functions in the HARA class, for any permissible income process, the optimal consumption rule is concave, i.e.* $c_t^{*''}(w_t) \leq 0$.

In order to prove this theorem we will first prove three lemmas.

The first Lemma establishes that the property that prudence is greater than $k$ times risk aversion is preserved when aggregating across states of nature (i.e. when taking an expectation). We called this "vertical aggregation" in the introduction.

**Lemma 1** *If* $\frac{V_{t+1}^{'''} V_{t+1}^{'}}{\left[V_{t+1}^{''}\right]^2} \geq k$ *then* $\frac{\phi_t^{'''} \phi_t^{'}}{\left[\phi_t^{''}\right]^2} \geq k$.

The expression $\frac{\phi_t^{'''} \phi_t^{'}}{\left[\phi_t^{''}\right]^2}$ will be $\geq k$ if and only if the expression $\phi_t^{'''} \phi_t^{'} - k \left[\phi_t^{''}\right]^2$ is nonnegative. This expression is merely the determinant of the following matrix:

$$\Phi_t = \left[ \begin{array}{cc} \phi_t^{'} & \sqrt{k} \ \left[\phi_t^{''}\right] \\ \sqrt{k} \ \left[\phi_t^{''}\right] & \phi_t^{'''} \end{array} \right]. \tag{6}$$

Therefore $\frac{\phi_t^{'''} \phi_t^{'}}{\left[\phi_t^{''}\right]^2} \geq k$ if and only if the matrix $\Phi_t$ is positive semidefinite. But we can rewrite $\Phi_t$ as:

$$\Phi_t = E_t \ \tilde{\beta}_{t+1} \left[ \begin{array}{cc} \tilde{R}_{t+1} \ V_{t+1}^{'} & \sqrt{k} \ \tilde{R}_{t+1}^2 \ \left[V_{t+1}^{''}\right] \\ \sqrt{k} \ \tilde{R}_{t+1}^2 \ \left[V_{t+1}^{''}\right] & \tilde{R}_{t+1}^3 \ V_{t+1}^{'''} \end{array} \right]. \tag{7}$$

Notice that, for all possible realizations of $\tilde{R}$, $\tilde{\beta}$, and $\tilde{y}$, the matrix whose expectation we are taking is positive semidefinite because $\tilde{R}^4_{t+1}V'_{t+1}V'''_{t+1} - k\tilde{R}^4_{t+1}\left[V''_{t+1}\right]^2 \in [0,\infty)$ from our initial assumptions that $V'_{t+1}V'''_{t+1}/\left[V''_{t+1}\right]^2 \geq k$, that $\tilde{R}_{t+1} > 0$, and that the period utility function is increasing and concave. (The value function inherits monotonicity and concavity from the utility function).[12] The expectation operator acts as a weighted sum across possible states, and because the sum of positive semidefinite matrices is positive semidefinite, $\Phi_t$ must be positive semidefinite, and the Lemma is proven.

The second Lemma establishes that the property of prudence exceeding $k$ times risk aversion is preserved by intertemporal aggregation (what we dubbed "horizontal addition" in the introduction). This generalizes Neave's (1971) proof that the value function inherits decreasing absolute risk aversion from the utility function.

**Lemma 2** *If* $\frac{\phi'''_t\phi'_t}{\left[\phi''_t\right]^2} \geq k$ *and* $\frac{u'''u'}{u''^2} \geq k$ *then* $\frac{V'''_tV'_t}{\left[V''_t\right]^2} \geq k$.[13]

Denoting the marginal utility of consumption at the optimal consumption level as $z_t = u'(c^*_t(w_t))$, we define the functions $f_t(z_t)$, $g_t(z_t)$, and $h_t(z_t)$ as the inverses of $u'$, $\phi'$, and $V'$:

$$\begin{aligned} f_t(z_t) &= u'^{-1}(z_t) = c_t, \\ g_t(z_t) &= \phi_t'^{-1}(z_t) = s_t, \\ h_t(z_t) &= V_t'^{-1}(z_t) = w_t. \end{aligned}$$

The proof of the lemma proceeds as follows. Dropping the time subscripts from f, g, and h for convenience, we have:

$$f' = \frac{1}{u''}, \tag{8}$$

$$f'' = -\frac{u'''}{u''^2}f' = -\frac{u'''}{u''^3}, \tag{9}$$

and so

$$-\frac{zf''}{f'} = \frac{u'''\,u'}{u''^2} \geq k. \tag{10}$$

[12]See Fama (1970).

[13]We prove the Lemma for $\frac{u'''u'}{u''^2} \geq k$ for generality; the main result uses the Lemma only for HARA utility functions for which $\frac{u'''u'}{u''^2} = k$.

Similarly,

$$-\frac{zg''}{g'} = \frac{\phi_t''' \ \phi_t'}{\left[\phi_t''\right]^2} \geq k \tag{11}$$

and

$$-\frac{zh''}{h'} = \frac{V_t''' \ V_t'}{\left[V_t''\right]^2}, \tag{12}$$

but because $h = f + g$, $h' = f' + g'$ and $h'' = f'' + g''$, so

$$\begin{aligned} -\frac{zh''}{h'} &= -z\frac{f'' + g''}{f' + g'} \qquad (13) \\ &= \underbrace{\frac{f'}{f' + g'}}_{>0} \underbrace{\left(\frac{-zf''}{f'}\right)}_{\geq k} + \underbrace{\frac{g'}{f' + g'}}_{>0} \underbrace{\left(\frac{-zg''}{g'}\right)}_{\geq k}. \qquad (14) \end{aligned}$$

Thus, $V_t'''V_t'/\left[V_t''\right]^2$ is a weighted average of two quantities both of which are greater than or equal to k, and therefore

$$\frac{V_t'''V_t'}{\left[V_t''\right]^2} \geq k, \tag{15}$$

and the Lemma is proven.

The third Lemma picks up where Lemma 2 leaves off. It states that if the value function has prudence $\geq k$ times risk aversion and the utility function is HARA with parameter $k$ (as we assume), then the consumption function is concave.

**Lemma 3** *If* $\frac{V_t'''V_t'}{\left[V_t''\right]^2} \geq k$ *and* $\frac{u'''u'}{u''^2} = k$, *then the optimal consumption policy rule is concave,* $c_t^{*''}(w_t) \leq 0$.

We begin by defining a function which yields the amount of saving corresponding to any optimally chosen level of consumption: $\theta_t^*(c_t) = w_t^*(c_t) - c_t$ where $w_t^*(c_t)$ is the inverse of the optimal consumption rule $c_t^*(w_t)$. Note that an alternative definition of $\theta_t^*(c_t)$ is $\theta_t^*(c_t) = g(f^{-1}(c_t))$. Because $w_t^*(c_t) = c_t + \theta_t^*(c_t)$ it is clear that if $\theta_t^*(c_t)$ is convex, $w_t^*(c_t)$ is convex. But if $w_t^*(c_t)$ is convex, its inverse, $c_t^*(w_t)$, must be concave, because both are increasing functions. Thus, if we can prove that $\theta_t^*(c_t)$ is convex we will have proven that the consumption rule must be concave.

The proof that $\theta_t^*(c_t)$ is convex closely follows proofs in Pratt (1964) and Kimball (1990b). It proceeds by directly calculating the derivatives of $\theta_t^*(c_t) = g(f^{-1}(c_t))$. Taking the derivative of both sides with respect to $c_t$ (and dropping the $c_t$ argument on $f^{-1}(c_t)$ for clarity), we have:

$$\theta_t^*(c_t) = g(f^{-1}), \tag{16}$$

$$\theta_t^{*'}(c_t) = \frac{g'(f^{-1})}{f'(f^{-1})}, \tag{17}$$

$$\theta_t^{*''}(c_t) = \frac{\left[g''(f^{-1})f'(f^{-1}) - g'(f^{-1})f''(f^{-1})\right]/f'(f^{-1})}{\left[f'(f^{-1})\right]^2} \tag{18}$$

$$= \frac{g'(f^{-1})}{\left[f'(f^{-1})\right]^2}\left[\frac{g''(f^{-1})}{g'(f^{-1})} - \frac{f''(f^{-1})}{f'(f^{-1})}\right] \tag{19}$$

which, from Lemma 2,

$$= \frac{g'(z_t)}{\left[f'(z_t)\right]^2}\underbrace{\underbrace{\left(\frac{1}{z_t}\right)}_{>0}\left[\underbrace{\frac{-z_t f''(z_t)}{f'(z_t)}}_{=k} - \underbrace{\frac{-z_t g''(z_t)}{g'(z_t)}}_{\geq k}\right]}_{\leq 0}. \tag{20}$$

So $sign\left(\theta_t^{*''}(c_t)\right) = -sign\left(g'(z_t)\right))$. But

$$g'(z_t) = \frac{1}{\phi_t''(s_t)} \tag{21}$$

and

$$\phi_t''(s_t) = E_t\tilde{\beta}_{t+1}\left(\tilde{R}_{t+1}^2\right)V_{t+1}''(w_{t+1}) \leq 0. \tag{22}$$

Therefore $sign(g'(z_t)) \leq 0$, and $sign\left(\theta_t^{*''}(c_t)\right) \geq 0$. Thus $\theta_t^*(c_t)$ is convex, implying that $c_t^*(w_t)$ is concave, and the Lemma is proven.

We are now in a position to prove the main theorem of the paper.

*Proof of Theorem 1.* First, note that in the last period of life, $V_T(w_t) = u(w_t)$, so $V_T''' V_T' / \left[V_T''\right]^2 = u''' u' / u''^2 = k$. Therefore, by Lemma 1 $\phi_{T-1}''' \phi_{T-1}' / \left[\phi_{T-1}''\right]^2 \geq k$, and by Lemma 2 we know that $V_{T-1}''' V_{T-1}' / \left[V_{T-1}''\right]^2 \geq k$. Continued iteration using Lemma 1 and Lemma 2 demonstrates that for any $t < T$, $V_t''' V_t' / \left[V_t''\right]^2 \geq k$. Then Lemma 3 shows that the consumption rule is concave in all time periods t, and the theorem is proven.

The final question we address is under what conditions the consumption rule is *strictly* concave. We will consider separately the cases where $k$ is a positive number other than 1, and where $k = 1$.

**Corollary 1** *For utility functions in the HARA class, for any permissible income process, the optimal consumption rule is strictly concave, i.e.* $c_t^{*''}(w_t) < 0$, *whenever* $k > 0$ *and* $k \neq 1$ *and future labor income is to any degree uncertain, and is not perfectly correlated with the future interest rate.*

Proving this corollary requires us to revisit Lemma 1. We will call the new version Lemma 4.

**Lemma 4** *If* $\frac{V_{t+1}''' V_{t+1}'}{\left[V_{t+1}''\right]^2} = k \neq 1$ *and labor income in period t+1,* $\tilde{y}_{t+1}$*, is uncertain and imperfectly correlated with* $\tilde{R}_{t+1}$*, then* $\frac{\phi_t''' \phi_t'}{\left[\phi_t''\right]^2} > k$.

Recall our matrix $\Phi$

$$\Phi_t = E_t \; \tilde{\beta}_{t+1} \left[ \begin{array}{cc} \tilde{R}_{t+1} \; V_{t+1}' & \sqrt{k} \; \tilde{R}_{t+1}^2 \; \left[V_{t+1}''\right] \\ \sqrt{k} \; \tilde{R}_{t+1}^2 \; \left[V_{t+1}''\right] & \tilde{R}_{t+1}^3 \; V_{t+1}''' \end{array} \right]. \tag{23}$$

In proving Lemma 1 we showed that for any possible realizations of $\tilde{R}$, $\tilde{\beta}$, and $\tilde{y}$, the $\Phi$ matrix is positive semidefinite. In order for the expectation (the weighted sum across states) to be positive *definite*, we need only that the matrices being added not be scalar multiples of each other. The conditions under which the matrices are scalar multiples are quite restrictive. In the appendix, we derive explicit expressions for $V(x), V'(x)$, and $V''(x)$ when $V'''V'/[V'']^2 = k$. Taking one more derivative gives:

$$V_{t+1}'''(x_{t+1}) = \left[\frac{A}{1-k}\right]^2 k[Ax_{t+1}+B]^{k/(1-k)-1}. \tag{24}$$

In order for the matrices to be scalar multiples whatever the realizations of $\tilde{R}$ and $\tilde{y}$, we need to have, for example, $\tilde{R}_{t+1}V_{t+1}'/\tilde{R}_{t+1}^3 V_{t+1}'''$ be constant. But

$$\left[\frac{V_{t+1}'}{\tilde{R}_{t+1}^2 V_{t+1}'''}\right] = \left[\left(\frac{A}{1-k}\right)^{-2} \frac{(Ax_{t+1}+B)^2}{k\tilde{R}_{t+1}^2}\right]. \tag{25}$$

Substituting for $x_{t+1} = \tilde{R}_{t+1}s_t + \tilde{y}_{t+1}$,

$$\left[\frac{V_{t+1}'}{\tilde{R}_{t+1}^2 V_{t+1}'''}\right] = \left[\left(\frac{A}{1-k}\right)^{-2} \frac{(As_t + (A\tilde{y}_{t+1}+B)/\tilde{R}_{t+1})^2}{k}\right]. \tag{26}$$

For this expression to be constant it is necessary to have $(A\tilde{y}+B)/\tilde{R} = \gamma$ where $\gamma$ is a constant. This can be achieved in only a few ways. First, if there is no labor income uncertainty then $y$ could be constant at $-B/A$, making the numerator of $(A\tilde{y}+B)/\tilde{R}$ zero. This reflects the well-known result that, under HARA utility, if there is no labor income uncertainty it is possible to solve for a linear analytical consumption function (Merton (1969)). If there is both labor income uncertainty and rate of return uncertainty this expression will be constant only if the two forms of uncertainty are perfectly correlated and are related by $\tilde{y} = (\gamma\tilde{R} - B)/A$.[14]

The next lemma is used to establish that $\frac{V_t'''V_t'}{\left[V_t''\right]^2} > k$ whenever there is any income uncertainty in any future period, even if the uncertainty is not in the successive period (as in Lemma 4).

**Lemma 5** *If* $\frac{V_{t+1}'''V_{t+1}'}{\left[V_{t+1}''\right]^2} \geq k$ *and* $> k$ *with positive probability, and* $\frac{u'''u'}{u''^2} \geq k$*, then* $\frac{V_t'''V_t'}{\left[V_t''\right]^2} > k$.

Consider again the definition of the matrix $\Phi_t$ in equation (23). For each possible realization of $\tilde{R}$, $\tilde{\beta}$, and $\tilde{y}$, the matrix whose expectation we are taking is positive definite

[14]This equation also handles the case of a stochastic necessity level of consumption $B$.

when $\frac{V'''_{t+1}V'_{t+1}}{\left[V''_{t+1}\right]^2} > k$ with $V'_{t+1}$ and $V'''_{t+1}$ both positive. As the sum of these positive definite matrices (and perhaps some positive semi-definite matrices), $\Phi_t$ itself must be positive definite, implying that $\phi'''_t\phi'_t - k\left[\phi''_t\right]^2$ is positive from the definition of $\Phi_t$ in equation (6). Of course, if $\phi'''_t\phi'_t - k\left[\phi''_t\right]^2 > 0$ then we have $\frac{\phi'''_t\phi'_t}{\left[\phi''_t\right]^2} > k$. Substituting $\frac{\phi'''_t\phi'_t}{\left[\phi''_t\right]^2} > k$ for $\frac{\phi'''_t\phi'_t}{\left[\phi''_t\right]^2} \geq k$ in the appropriate places in Lemma 2 yields $\frac{V'''_tV'_t}{[V''{}_t]^2} > k$ and the Lemma is proven.

The final lemma is the strict-concavity analogue to Lemma 3.

**Lemma 6** *If* $\frac{V'''_tV'_t}{\left[V''_t\right]^2} > k$ *and* $\frac{u'''u'}{u''^2} = k$*, then the optimal consumption policy rule is strictly concave,* $c^{*''}_t(w_t) < 0$*.*

The proof is identical to the proof for Lemma 3, except for the substitution of strict for loose inequality in equations (20) and (22).

With Lemmas 4 through 6 in hand, the proof of Corollary 1 is trivial.

*Proof of Corollary 1.* Imagine there is some labor income uncertainty that is imperfectly corellated with interest rate uncertainty in period $t+1$ in some states. Then recursion using Lemmas 4 and 5 establishes that $\frac{V'''_sV'_s}{\left[V''_s\right]^2} > k$ for all $s \leq t$, and, given this, Lemma 6 establishes the strict concavity of the consumption rule for all $s \leq t$.

The second corollary addresses the case where utility is of the Constant Absolute Risk Aversion form.

**Corollary 2** *For utility functions in the HARA class with* $k = 1$ *(i.e. the CARA utility function), for any permissible income process, the optimal consumption rule is strictly concave, i.e.* $c^{*''}_t(w_t) < 0$*, whenever the future interest rate is to any degree uncertain.*

In this case, $V(w_t) = \exp(-aw_t)/(-a)$ to within a linear transformation. The determinant of the $\Phi$ matrix is now $E_t[\tilde{R}_{t+1}V'_{t+1}]E_t[\tilde{R}^3_{t+1}V'''_{t+1}] - E_t\left[\tilde{R}^2_{t+1}V''_{t+1}\right]E_t\left[\tilde{R}^2_{t+1}V''_{t+1}\right]$. But with CARA utility $V'_{t+1}V'''_{t+1} = V''_{t+1}V''_{t+1}$ so the expression will be zero only if $E_t[\tilde{R}_{t+1}]E_t[\tilde{R}^3_{t+1}] = E_t[\tilde{R}^2_{t+1}]E_t[\tilde{R}^2_{t+1}]$. This will of course hold only if $R$ is nonstochastic. The remainder of the proof is identical to that for Corollary 1.

Thus, if utility is of the CARA form, the consumption function will be strictly concave if there is any rate of return uncertainty, but, as is widely known, will be linear if the only form of uncertainty is in labor income.

The final Corollary just combines Corollaries 1 and 2.

**Corollary 3** *For utility functions in the HARA class with $k > 0$, for any permissible income process, the optimal consumption rule is strictly concave, i.e. $c_t^{*''}(w_t) < 0$, whenever future labor income and the future interest rate are both uncertain, and are not perfectly correlated with each other.*

The proof is obvious.[15]

# 3 Conclusion

Many economists from at least Keynes (1935) on have had the intuition that the consumption function is concave, with a marginal propensity to consume lower for rich consumers than for poor consumers. Nevertheless, much empirical and theoretical work has assumed, explicitly or implicitly, that the consumption function is linear. One of the most striking features of Zeldes's (1989) numerical solutions for the consumption rule under uncertainty is the introduction of concavity to consumption rules that were linear in the absence of uncertainty. Carroll (1992, 1995a) shows how important the curvature of the consumption function can be for reasoning about the behavior of consumption under uncertainty. This paper provides a solid analytical basis for concavity of the consumption rule under uncertainty.

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[15]It is also worth noting in this context that the reason consumption is unaffected by uncertainty when utility is quadratic is because $\sqrt{k} = 0$, making the off-diagonal terms of $\Phi$ zero, while the southeast diagonal element is zero because $V_{t+1}^{'''} = 0$, so all of the matrices added to get $\Phi$ are scalar multiples of each other.

## Appendix

In the introduction, we asserted that HARA utility functions or value functions could be described as the class of functions that satisfy the condition $u'''u'/u''^2 = k$. An equivalent form is $u'''u'/[u'']^2 = k$ (or $V'''/V''^2 = k$). The first integral gives

$$\log[u''] = k \log u' + A. \tag{27}$$

The second integral is

$$\begin{aligned} [u']^{1-k} &= Ax + B, & (28) \\ u' &= [Ax + B]^{1/(1-k)}. & (29) \end{aligned}$$

The third integral is

$$u = \frac{1-k}{(2-k)A}\,[Ax + B]^{\frac{2-k}{1-k}} + C. \tag{30}$$

Differentiation gives:

$$u''(x) = \frac{A}{1-k}[Ax + B]^{k/(1-k)}. \tag{31}$$

We need to require $Ax + B \geq 0$ in order to be able to raise this expression to a fractional power. Also, because we require a concave utility function for which $u'' < 0$ we must have $[A/(1-k)] < 0$ which implies that $k > 1$ if $A > 0$ and $k < 1$ if $A < 0$. Noting that linear transformations do not change the preferences represented by the utility function, the following are representatives of the HARA utility function for various values of $k < 1$.

| | | | | | | |
|---|---|---|---|---|---|---|
| If k | = | -1 | then $u(x)$ | = | $-(B-x)^{1.5}$ | |
| If k | = | 0 | then $u(x)$ | = | $-(B-x)^{2}$ | (quadratic utility) |
| If k | = | 1/3 | then $u(x)$ | = | $-(B-x)^{2.5}$ | |

and so on.

The functions for which $k > 1$ are the Decreasing Absolute Risk Aversion subset of the HARA functions. It is clear from (30) that the DARA HARA functions are just the CRRA functions and their counterparts with shifted origins.

# References


Bodkin, R. (1959): "Windfall Income and Consumption," *American Economic Review*, 49, 602–614.

Browning, M. J., and A. Lusardi (1995): "Household Saving: Micro Theories and Micro Facts," *Manuscript, Dartmouth College*.

Carroll, C. D. (1992): "The Buffer-Stock Theory of Saving: Some Macroeconomic Evidence," *Brookings Papers on Economic Activity*, 1992(2), 61–156.

——— (1995a): "Buffer-Stock Saving and the Life Cycle/Permanent Income Hypothesis," *Manuscript, Board of Governors of the Federal Reserve System, August*.

——— (1995b): "Buffer Stock Saving: Some Theory," *Manuscript, Board of Governors of the Federal Reserve System, August*.

Deaton, A. S. (1992): *Understanding Consumption*. Oxford University Press, New York.

Fama, E. F. (1970): "Multiperiod Consumption-Investment Decisions," *American Economic Review*, 60(1), 163–74.

Hall, R. E. (1978): "Stochastic Implications of the Life-Cycle/Permanent Income Hypothesis: Theory and Evidence," *Journal of Political Economy*, 96, 971–87.

Keynes, J. M. (1935): *The General Theory of Employment, Interest, and Money*. Harvest/HBJ, San Diego, New York, London, 1964.

Kimball, M. S. (1990a): "Precautionary Saving and the Marginal Propensity to Consume," *NBER Working Paper Number 3403*.

——— (1990b): "Precautionary Saving in the Small and in the Large," *Econometrica*, 58, 53–73.

——— (1993): "Standard Risk Aversion," *Econometrica*, 61(3), 589–611.

KREININ, M. E. (1961): “Windfall Income and Consumption: Additional Evidence,” *American Economic Review*, 51, 388–390.

LANDSBERGER, M. (1966): “Windfall Income and Consumption: Comment,” *American Economic Review*, 56, 534–540.

LUSARDI, A. (1992): “Permanent Income, Current Income, and Consumption: Evidence from Panel Data,” *Manuscript, Dartmouth College*.

MERTON, ROBERT C. (1969): “Lifetime Portfolio Selection under Uncertainty: The Continuous Time Case,” *Review of Economics and Statistics*, 50, 247–257.

NEAVE, EDWIN H. (1971): “Multiperiod Consumption-Investment Decisions and Risk Preference,” *Journal of Economic Theory*, 3(1), 40–53.

PRATT, J. W. (1964): “Risk Aversion in the Small and in the Large,” *Econometrica*, 32, 122–136.

SIBLEY, D. S. (1975): “Permanent and Transitory Income Effects in a Model of Optimal Consumption with Wage Income Uncertainty,” *Journal of Economic Theory*, 11, 68–82.

SOULELES, N. S. (1995): “The Response of Household Consumption to Income Tax Refunds,” *Manuscript, the Massachusetts Institute of Technology*.

ZELDES, S. P. (1989): “Optimal Consumption with Stochastic Income: Deviations from Certainty Equivalence,” *Quarterly Journal of Economics*, 104, 275–298.