

# The Lucas Growth Model

Lucas (1988) presents a growth model in which output is generated via a production function of the form

$$Y = AK^\alpha(\ell hL)^{1-\alpha} \quad (1)$$

where  $Y$ ,  $A$ , and  $K$  are as usually defined and  $0 < \alpha < 1$ , where  $\ell$  is defined as the proportion of total labor time spent working, and  $h$  is what Lucas calls the stock of ‘human capital.’

The production function can be rewritten in per-capita terms as

$$y = Ak^\alpha(\ell h)^{1-\alpha} \quad (2)$$

which is a constant returns to scale production function in  $k$  and  $\ell h$ .

Capital accumulation proceeds via the usual differential equation,

$$\dot{k} = y - c - (\xi + \delta)k, \quad (3)$$

while  $h$  accumulates according to

$$\begin{aligned} \dot{h} &= \phi h(1 - \ell) \\ \dot{h}/h &= \phi(1 - \ell). \end{aligned} \quad (4)$$

## 1 Discussion

Before analyzing the model, an aside. Mankiw (1995) has persuasively argued for defining ‘knowledge’ as the sum total of technological and scientific discoveries (what is written in textbooks, scholarly journals, websites, and the like), and defining ‘human capital’ as the stock of knowledge that has been transmitted from those sources into human brains via studying.

Recall Rebelo (1991)’s key insight about endogenous growth models: In order to produce perpetual growth, there must be a factor or a combination of factors that can be accumulated indefinitely without diminishing returns. Mankiw points out that since lifetimes are finite, there is a maximum limit to the amount of human capital that an individual can accumulate. Thus, while increasing human capital (with more years of schooling, for example) may be able to extend the length of the transition period in a growth model, human capital accumulation cannot be the source of perpetual growth. It is more plausible to think that scientific knowledge can be accumulated indefinitely (though presumably there is some limit even to the accumulation of knowledge). These considerations suggest that models of endogenous growth should focus more on understanding the process of fundamental research and technological development than on human capital accumulation as Mankiw defines it.

With this distinction in mind, there are (at least) two interpretations of the Lucas model. One is at the aggregate level. Here we can think of  $\ell$  as the fraction of the population engaged in useful work to produce goods and services, while proportion  $1 - \ell$

is not working in conventional boring jobs that require asking customers questions like “Would you like fries with that?” but instead is producing ‘knowledge’ by conducting scientific and technological research.

The other interpretation is at the level of an individual agent. Such an agent can be thought of as operating his or her own production function of the form in (2), where  $(1 - \ell)$  is now interpreted as the proportion of the time this individual spends studying and  $\ell$  is the time spent working.

From the point of view of Mankiw’s distinction, it is hard to interpret Lucas’s model as being either about human capital accumulation or about knowledge. It can’t be about human capital because  $h$  can be accumulated without bound, and without diminishing returns, neither of which makes sense for an individual. It can’t be about generalized knowledge, because the optimization problem reflects the return for an individual, while only a trivial proportion of total knowledge (in Mankiw’s sense) is contributed by any single individual.

Given these considerations, it probably makes more sense to think of the model as a tool for normative than for positive analysis.

## 2 The Solow Version

We analyze first the ‘Solow’ version of the model, in which the saving rate is exogenously fixed at  $s$ . Thus the capital accumulation equation becomes

$$\begin{aligned}\dot{k} &= sy - (\xi + \delta)k \\ \dot{k}/k &= s(y/k) - (\xi + \delta) \\ &= sk^{\alpha-1}(\ell h)^{1-\alpha} - (\xi + \delta) \\ &= s(k/h)^{\alpha-1}\ell^{1-\alpha} - (\xi + \delta).\end{aligned}\tag{5}$$

This equation tells us that the steady-state growth rate in this model (if one exists) requires a constant ratio of  $k$  to  $h$ . Thus,  $k$  and  $h$  must be growing at the same rate in equilibrium.

Further insight can be obtained by defining  $\hat{A} = A\ell^{1-\alpha}$  and rewriting the per-capita production function as

$$y = \hat{A}k^\alpha h^{1-\alpha}.\tag{6}$$

If we define a measure of ‘broad capital’ as the combination of physical capital and human capital,

$$\kappa \equiv k^\alpha h^{1-\alpha},\tag{7}$$

the model becomes

$$y = \hat{A}\kappa.\tag{8}$$

So if  $\ell$  is constant and if  $k$  and  $h$  are growing at the same rate, then the exponent on ‘broad capital’ is 1, and we are effectively back at the usual Rebelo  $AK$  model.

The key assumption that permits this to work is the accumulation equation for human capital, which is itself like an *AK* model, in the sense that the exponent on human capital in the accumulation equation for human capital is one. Human capital can be accumulated without bound and without diminishing returns.

### 3 The Ramsey/Cass-Koopmans Version

Lucas does not examine the Solow version of the model with a constant saving rate, but instead the version in which a social planner solves for the optimal perfect foresight paths of the two state variables  $k$  and  $h$ . It's not worth going through the math here; I'll just present the conclusion, which is that the steady-state growth rate is

$$\dot{c}/c = \rho^{-1}(\phi - \theta). \quad (9)$$

Note that this confirms the crucial role of the CRS accumulation equation for human capital: The key parameter that corresponds to the interest rate is  $\phi$ , the parameter that determines the efficiency of human capital accumulation in equation (4).

Lucas also solves a version of the model in which there is an externality to human capital. The idea here is that each person is more productive if they are surrounded by other people with high levels of human capital. Specifically, in this version of the model the individual's production function is

$$y_i = Ak_i^\alpha (\ell_i h_i)^{1-\alpha} \bar{h}^\psi \quad (10)$$

where  $\bar{h}$  is average human wealth in the population (and the other variables reflect the values for the individual).

Working through the decentralized solution, Lucas shows that the steady-state growth rate of human capital for an individual consumer will be

$$\gamma_h = \left( \frac{\rho^{-1}(\phi - \theta)}{1 + \psi(1 - 1/\rho)/(1 - \alpha)} \right). \quad (11)$$

Since every individual is assumed to be identical, the growth rate of aggregate human capital (and everything else) is the same as the rate for this individual.

It is easy to see that if there is no externality to human capital accumulation (that is, if  $\psi = 0$ ), this solution is identical to (9). If there is an externality, its effect depends on whether  $1/\rho$  is greater than, equal to, or less than 1. This is because the effect depends on whether the externality causes the saving rate to rise or to fall (since in endogenous growth models, saving is the source of all growth). The  $\bar{h}$  externality is like an increase in the interest rate, and thus its effect will be determined by the balance between the income and substitution effects. We know that for  $\rho = 1$  the income and substitution effects exactly offset each other, leaving consumption unchanged in response to an increase in the interest rate, which is why (11) collapses to (9) for  $\rho = 1$ . If consumers are very willing to cut current consumption in exchange for higher future consumption (that is, if the intertemporal elasticity of substitution ( $1/\rho$ ) is greater than 1), then the externality boosts saving and therefore growth. If consumers have an intertemporal elasticity of less

than one, the income effect outweighs the substitution effect, saving falls, and growth is slower.

Lucas also shows that this decentralized solution is suboptimal, because individual consumers do not obtain the full benefits to society of increasing their own stock of knowledge. Devoting more time to  $h_i$  accumulation they increase  $\bar{h}$ , which benefits all others in the economy in addition to themselves. Lucas shows, unsurprisingly, that the socially optimal solution requires greater investment in human capital accumulation than is obtained in the decentralized model. He also derives an optimal subsidy to human capital accumulation that corrects the externality and induces households to invest the socially optimal amount in human capital.

## References

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