Ramsey Growth in Discrete and Continuous Time

This handout solves a continuous-time version of the Ramsey/Cass-Koopmans (RCK) model using the Hamiltonian method, and shows the relationship between that method and the discrete-time approach.

The problem is to choose a path of consumption per capita c from the present moment (arbitrarily called time 0) into the infinite future, $\{c\}_0^\infty$, that solves the problem

$$\max_{\{c\}_0^\infty} \int_0^\infty \mathrm{u}(c)e^{-\vartheta t} \tag{1}$$

subject to

$$\begin{aligned} \dot{k} &= \mathrm{f}(k) - c - (\xi + \delta)k \\ k &> 0 \ \forall \ t \end{aligned} \tag{2}$$

where ϑ is the time preference rate, ξ is the population growth rate, and δ is the depreciation rate. (In continuous time, we think of all variables as implicitly being a function of time, but it is cumbersome to write, e.g., $c(t)$ everywhere, so the time argument is omitted; we are also thinking of the initial value of capital at date 0 as being a 'given' in the problem, so that $k(0) = \bullet$ for some specific value of $\bullet$).

To emphasize the similarity between the continuous-time and the discrete-time solutions where we have typically used the roman V to denote value, for the continuous-time problem we define 'curly' value as a function of the initial level of capital as $\mathcal{V}(k)$.

The current-value (discounted) Hamiltonian is

$$\mathcal{H}(k, c, \lambda) = \mathrm{u}(c) + (\mathrm{f}(k) - c - (\xi + \delta)k)\lambda \tag{3}$$

where k is the state variable, c is the control variable, and λ is the costate variable.

λ is the continuous-time equivalent of a Lagrange multiplier, so its value should be equivalent to the value of relaxing the corresponding constraint by an infinitesimal amount. But the constraint in question is the capital-accumulation constraint. Thus λ should be equal to the value of having a tiny bit more capital, $\frac{\mathcal{V}(k+\Delta k)-\mathcal{V}(k)}{\Delta k}$. In other words, you can think of $\lambda = \mathcal{V}'(k)$.

The first necessary Hamiltonian condition for optimality is

$$\begin{aligned} \frac{\partial \mathcal{H}}{\partial c} &= 0 \\ \mathrm{u}'(c) &= \lambda \\ \mathrm{u}'(c) &= \mathcal{V}'(k). \end{aligned} \tag{4}$$

Note the similarity between (4) and the result we usually obtain by using the Envelope theorem in the discrete-time problem,

$$\mathrm{u}'(c_t) = \mathrm{V}'(k_t). \tag{5}$$

Thus, you can use the intuition you (should have) developed by now about why the marginal utility of consumption should be equal to the marginal value of extra resources to understand this Hamiltonian optimality condition.

The second necessary condition is

$$\begin{aligned}\dot{\lambda} &= \vartheta\lambda - \lambda(\mathrm{f}'(k) - (\xi + \delta)) \\ \left(\frac{\dot{\lambda}}{\lambda}\right) &= \vartheta - (\mathrm{f}'(k) - (\xi + \delta))\end{aligned} \tag{6}$$

which expresses the growth rate of λ at an annual rate (because the interest rate r and time preference rate ϑ are measured at an annual rate).

To interpret this in terms of our discrete-time model, begin with the condition

$$\mathrm{V}'(k_t) = \mathsf{R}\beta\mathrm{V}'(k_{t+1}). \tag{7}$$

The final necessary condition is just that the accumulation equation for capital is satisfied,

$$\dot{k} = \mathrm{f}(k) - c - (\xi + \delta)k. \tag{8}$$

This is the continuous-time equivalent of what we have previously called the Dynamic Budget Constraint.

Up to now in this course we haven't thought very much about what the time period is. Generally, we have expressed things in terms of yearly rates, so that for example we might choose $\mathsf{R} = 1.04$ and $\beta = 1/(1 + \vartheta) = 1/(1.04)$ to represent an interest rate of 4 percent and a discount rate of 4 percent.

One of the attractive features of the time-consistent model we have been using is that it generates self-similar behavior as the time interval is changed. Thus if we wanted to solve a quarterly version of the model we would choose $\mathsf{R} = 1.01$ and $\beta = 1/1.01$ and it would imply consumption of almost exactly $1/4$ of the amount implied by the annual model, so that four quarters of such behavior would aggregate to the prediction of the annual model.

To put this in the most general form, suppose R and β correspond to 'annual rate' values and we want to divide the year into m periods. Then the appropriate interest rate and discount factor on a per-period basis would be $\mathsf{R}^{1/m}$ and $\beta^{1/m}$. Thus the discrete-time equation could be rewritten

$$\mathrm{V}'(k_t) = \mathsf{R}^{1/m}\beta^{1/m}\mathrm{V}'(k_{t+1}) \tag{9}$$

where the time interval is now $1/m$th of a year (e.g. if m=52, we're talking weekly, so that period $t + 1$ is one week after period t). Now we can use our old friend, the fact that $e^z \approx 1 + z$, to note that this is approximately

$$\begin{aligned}
\mathrm{V}'(k_t) &\approx [e^{\mathsf{r}}]^{1/m}[e^{-\vartheta}]^{1/m}\mathrm{V}'(k_{t+1}) \\
&= e^{(1/m)\mathsf{r}}e^{-(1/m)\vartheta}\mathrm{V}'(k_{t+1}) \\
&= e^{(1/m)(\mathsf{r}-\vartheta)}\mathrm{V}'(k_{t+1}) \\
&= e^{(1/m)(\mathsf{r}-\vartheta)}(\mathrm{V}'(k_t) + \Delta\mathrm{V}'(k_{t+1})) \\
1 &= e^{(1/m)(\mathsf{r}-\vartheta)}\left(\frac{\mathrm{V}'(k_t) + \Delta\mathrm{V}'(k_{t+1})}{\mathrm{V}'(k_t)}\right) \\
e^{(1/m)(\vartheta-\mathsf{r})} &= \left(\frac{\mathrm{V}'(k_t) + \Delta\mathrm{V}'(k_{t+1})}{\mathrm{V}'(k_t)}\right) \\
\mathrm{V}'(k_t)(\underbrace{e^{(1/m)(\vartheta-\mathsf{r})} - 1}_{\approx 1+(1/m)(\vartheta-\mathsf{r})-1}) &= \Delta\mathrm{V}'(k_{t+1}) \\
\frac{\Delta\mathrm{V}'(k_{t+1})}{\mathrm{V}'(k_t)} &\approx (1/m)(\vartheta - \mathsf{r}) \\
\frac{m\Delta\mathrm{V}'(k_{t+1})}{\mathrm{V}'(k_t)} &\approx (\vartheta - \mathsf{r}).
\end{aligned} \tag{10}$$

We defined the interest rate and time preference rate on an annual basis, but the time interval between t and $t+1$ is only $(1/m)$th of a year. Thus $m\Delta\mathrm{V}'(k_{t+1})$ expresses the speed of change in $\mathrm{V}'(k_t)$ at an annual rate.

Now, note that since the effective interest rate in this model is $\mathrm{f}'(k) - (\xi + \delta)$, equation (10) is basically the same as (6) since $\lambda = \mathcal{V}'(k)$ and $m\Delta\mathrm{V}'(k_{t+1}) = (d/dt)\mathcal{V}'(k) = \dot{\lambda}$. Hence, the second optimality condition in the Hamiltonian optimization method is basically equivalent to the condition $\mathrm{V}'(k_t) = \mathsf{R}\beta\mathrm{V}'(k_{t+1})$ from the discrete-time optimization method!

The final required condition (the transversality constraint) is

$$\lim_{t\to\infty} \lambda k e^{-\vartheta t} = 0 \tag{11}$$

The translation of this into the discrete-time model is

$$\lim_{t\to\infty} \beta^t \mathrm{u}'(c_t)k_t = 0. \tag{12}$$

Consider the simple model with a constant gross interest rate R and CRRA utility. In that model, recall that $c_{t+1} = (\mathsf{R}\beta)^{1/\rho}c_t$. Thus considered from time zero (12) becomes

$$\begin{aligned}
\lim_{t\to\infty} \beta^t(c_0((\mathsf{R}\beta)^{1/\rho})^t)^{-\rho}k_t &= 0 \\
&= c_0^{-\rho}\beta^t[(\mathsf{R}\beta)^{t/\rho}]^{-\rho}k_t \\
&= c_0^{-\rho}\beta^t\beta^{-t}\mathsf{R}^{-t}k_t \\
&= c_0^{-\rho}\mathsf{R}^{-t}k_t \\
\to \lim_{t\to\infty} \mathsf{R}^{-t}k_t &= 0.
\end{aligned} \tag{13}$$

What this says is that you cannot behave in such a way that you expect k_t to grow faster than the interest rate forever.[1] This is the infinite-horizon version of the intertemporal budget constraint. Among the infinite number of time paths of c and k that will satisfy the first order conditions above, only one will also satisfy this transversality constraint - because all the others imply a violation of the intertemporal budget constraint.

Now differentiate (**??**) with respect to time

$$\dot{c}\mathrm{u}''(c) = \dot{\lambda} \tag{14}$$

and substitute this into equation (6) to get

$$\begin{aligned}\frac{\dot{c}\mathrm{u}''(c)}{\mathrm{u}'(c)} &= (\vartheta - (\mathrm{f}'(k) - (\xi + \delta))) \\ \dot{c} &= -\frac{\mathrm{u}'(c)}{\mathrm{u}''(c)}(\mathrm{f}'(k) - (\xi + \delta) - \vartheta)\end{aligned} \tag{15}$$

using the fact derived earlier that for a CRRA utility function $\mathrm{u}(c) = c^{1-\rho}/(1-\rho)$, $-\mathrm{u}''(c)c/\mathrm{u}'(c) = \rho$,, this becomes

$$\begin{aligned} &= (c/\rho)(\mathrm{f}'(k) - (\xi + \delta) - \vartheta) \\ \dot{c}/c &= \rho^{-1}(\mathrm{f}'(k) - (\xi + \delta) - \vartheta)\end{aligned} \tag{16}$$

[1]Note that this also rules out negative k_t values that grow faster than the interest rate.