The Time Series Behavior of Consumption in the CEQ PIH Model

1 Theory

Consider the Permanent Income Hypothesis model of consumption *a la* Deaton (1992) in which the utility function is quadratic,

\[ u(c) = -\frac{\alpha}{2}(c - \bar{c})^2 \]  
\[ u'(c) = -(c - \bar{c})\alpha, \]  
where \( \bar{c} \) is the ‘bliss point’ of maximum utility and the budget constraint guarantees that \( c < \bar{c} \).

If there is no uncertainty about the return factor \( R \), the first order condition at period \( t - 1 \) is

\[ u'(c_{t-1}) = R\beta E_{t-1}[u'(c_t)] \]  
\[ -\alpha(c_{t-1} - \bar{c}) = -R\beta\alpha E_{t-1}[(c_t - \bar{c})], \]  
and if we assume for convenience that \( R\beta = 1 \), this reduces to

\[ c_{t-1} = E_{t-1}[c_t] \]  
\[ c_t = c_{t-1} + \epsilon_t \]  
\[ \Delta c_t = \epsilon_t, \]  
where \( \epsilon_t \) is a standard expectational error with the usual property that \( E_s[\epsilon_t] = 0 \) \( \forall s < t \).

2 Hall and Flavin

The implication that consumption follows a random walk was first tested by Hall (1978) with a regression of the form

\[ c_t = \gamma_0 + \gamma_1 c_{t-1} + \gamma_2 z_{t-1} \]  
where the theory implies that the coefficient \( \gamma_2 \) should be equal to zero for all variables \( z_{t-1} \) whose value was known at the time that consumption in period \( t - 1 \) was decided. Hall tried a variety of plausible \( z \)'s and generally found a coefficient on lagged consumption \( \gamma_1 \) close to one, while for most \( z \)'s he found a coefficient not significantly different from zero – just as the random walk theory would predict.
However, when he chose a measure of stock market performance as his $z_{t-1}$, he did find a statistically significant coefficient $\gamma_2$.

Flavin (1981) extended the theory in Hall, and was the first person to estimate the consumption Euler equation in the form we use today:

$$\Delta c_t = \mu_0 + \mu_1 z_{t-1}. \quad (9)$$

That is, she imposed the theory’s restriction that the coefficient $\gamma_1$ should be equal to one. For deep and general econometric reasons this provides a more powerful test of the theory, and Flavin found that several variables in addition to the lagged stock market could predict consumption growth, including lagged income growth. Deaton (p. 94) reports results analogous to Flavin’s:

$$\Delta c_t = 11.39 + 0.121 \Delta y_{t-1} \quad (9.7) \quad (3.20)$$

Thus, contrary to theory, consumption growth is predictable by lagged income growth - it is ‘excessively sensitive’ to this lagged variable, since the correct sensitivity would be zero.

3 Time Aggregation

Suppose that consumption data are collected annually, with years indexed by $t$, but agents make their consumption decisions only once every six months, in periods indexed by $\tau$. Suppose that for a specific year $t$ we refer to the first six-month period as $\tau$ and the second six-month period as $\tau + 1$.

The recorded change in measured, annual consumption $c^*_t$ is related to the underlying semi-annual, unmeasured consumption changes by

$$\Delta c^*_t = c_{\tau} + c_{\tau+1} - c_{\tau-1} - c_{\tau-2} \quad (10)$$

$$\Delta c^*_t = \Delta c_{\tau} + c_{\tau+1} - c_{\tau-2} \quad (11)$$

$$\Delta c^*_t = \Delta c_{\tau} + (c_{\tau-2} + \Delta c_{\tau-1} + \Delta c_t + \Delta c_{\tau+1}) - c_{\tau-2} \quad (12)$$

$$\Delta c^*_t = \Delta c_{\tau+1} + 2\Delta c_t + \Delta c_{\tau-1}, \quad (13)$$

and the previous year’s measured income change is

$$\Delta y^*_{t-1} = \Delta y_{t-1} + 2\Delta y_{t-2} + \Delta y_{t-3}, \quad (14)$$

since if $\tau$ corresponds to $t$, $\tau - 2$ must correspond to $t - 1$.

The crucial point is that the half-year period $\tau - 1$ appears in both (13) and (14). Thus the ‘news’ about income in half-year $\tau - 1$ will affect both $\Delta c^*_t$ and $\Delta y^*_{t-1}$, so that measured consumption growth will be predictable from measured lagged income growth even though in each subperiod $\tau$ ‘true’ consumption follows a random walk.
Working (1960) is the first person to make the point that *time aggregation can cause serial correlation* in the aggregated data even if the unaggregated data follow a random walk. Though the algebra is more complex, the same insight holds for quarterly data if the decision period is a month, and more generally the point holds whenever one examines data which are averaged over a time period longer than the period in which the series follows a random walk.

However, consider measured data from period \( t-2 \):

\[
\Delta y^*_{t-2} = \Delta y_{t-3} + 2\Delta y_{t-4} + \Delta y_{t-5}. \tag{15}
\]

Note that since there is no overlap between the time periods represented in equation (15) and (13), \( \Delta c^*_{t} \) should be truly orthogonal to \( \Delta y^*_{t-2} \) (and any other information dated \( t-2 \)).

We can then see whether the ‘excess sensitivity’ finding of Flavin and others merely reflects time aggregation problems either by directly estimating \( \Delta c^*_{t} = \zeta_0 + \zeta_1 \Delta y^*_{t-2} \) and testing whether \( \zeta_1 \) is statistically significant, or by estimating \( \Delta c^*_{t} = b_0 + b_1 \Delta y^*_{t-1} \) but instrumenting for \( \Delta y^*_{t-1} \) using instruments from periods \( t-2 \) and earlier. Deaton reports the results of the latter experiment as

Thus, the ‘excess sensitivity’ results cannot be explained as resulting from time aggregation.

4 Campbell and Mankiw

Campbell and Mankiw (1989) proposed a simple explanation for the importance of lagged variables in predicting current consumption growth: They suggest that some fraction \( \lambda \) of income goes to consumers who simply set their consumption equal to their income in every period, called ‘rule of thumb’ or ‘Keynesian’ consumers. The existence of such people can potentially explain the excess sensitivity findings because the lagged variables may be statistically significant for current consumption growth merely because they are good at predicting current income growth. Campbell and Mankiw also show that if we assume a perfect foresight CRRA model then for PIH consumers the relationship between consumption growth and income growth can be rewritten in terms of changes in the logs rather than the levels of consumption and income,

\[
\Delta \log c^{\text{PIH}}_t = \rho^{-1}(\mathbb{E}_{t-1}[r_t] - \delta) \tag{16}
\]

and that if aggregate consumption is done partly by such PIH consumers and partly by rule-of-thumb consumers the equation for aggregate consumption growth is
approximately\(^1\)

\[
\Delta \log c_t \approx (1 - \lambda) \rho^{-1}(\mathbb{E}_{t-1}[r_t] - \delta) + \lambda \mathbb{E}_{t-1}[\Delta \log y_t].
\]  

(17)

They therefore estimate an equation of the form

\[
\Delta \log c_t = \gamma_0 + \gamma_1 \mathbb{E}_{t-2}[r_t] + \gamma_2 \mathbb{E}_{t-2}[\Delta \log y_t],
\]

(18)

where the fact that the expectations are dated time \(t - 2\) signifies that only information from periods \(t - 2\) and earlier is used to forecast income growth and the interest rate. They find an estimate of \(\gamma_2 \approx 0.5\), and interpret this result as an indication that roughly half of aggregate income goes to ‘rule of thumb’ consumers.

5 Flavin and the ‘Warranted’ Change in Consumption

The intertemporal budget constraint (IBC) says that the present discounted value of consumption must be less than or equal to total current assets plus the present discounted value of income,

\[
\sum_{s=t}^{\infty} \frac{c_s}{R^{s-t}} \leq b_t + \sum_{s=t}^{\infty} \frac{y_s}{R^{s-t}}
\]

(19)

\[
c_t + c_{t+1}/R + c_{t+2}/R^2 + \ldots \leq b_t + \sum_{s=t}^{\infty} \frac{y_s}{R^{s-t}}.
\]

(20)

Define human wealth \(h_t\) as

\[
h_t = \mathbb{E}_t \left[ \sum_{s=t}^{\infty} R^{t-s} y_s \right].
\]

(21)

Since this equation must hold for all possible future realizations of income, it must hold in expectation. Furthermore, if the marginal utility of consumption is always positive, the equation will hold with equality. Imposing expectations gives

\[
c_t + \mathbb{E}_t[c_{t+1}/R] + \mathbb{E}_t[c_{t+2}/R^2] + \ldots = b_t + h_t
\]

(22)

\[
c_t + \mathbb{E}_t[(c_t + \epsilon_{t+1})/R] + \mathbb{E}_t[(c_{t+1} + \epsilon_{t+2})/R^2] + \ldots = b_t + h_t
\]

\[
c_t + \mathbb{E}_t[c_t/R] + \mathbb{E}_t[c_t/R^2] + \ldots = b_t + h_t
\]

\[
c_t + c_t/R + c_t/R^2 + \ldots = b_t + h_t
\]

\[
c_t (1 + 1/R + 1/R^2 + \ldots) = b_t + h_t
\]

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Thus, consumption will be equal to the interest income on total wealth, human and nonhuman \((b_t + h_t)\). Deaton defines the expression \((r/R)(b_t + h_t)\) as permanent income.\(^2\)

Define \(b_t = R(b_{t-1} + y_{t-1} - c_{t-1})\) and substitute for \(b_t\) in equation (23) to get

\[
c_t = r(b_{t-1} + y_{t-1} - c_{t-1}) + (r/R) \mathbb{E}_t \left[ \sum_{s=t}^{\infty} y_s R^{t-s} \right] \tag{24}
\]

Now lag (23) by one period and multiply both sides by \(R\),

\[
(1 + r)c_{t-1} = r \left[ b_{t-1} + \mathbb{E}_{t-1} \left[ \sum_{s=t-1}^{\infty} R^{(t-1)-s} y_s \right] \right] \tag{25}
\]

\[
(1 + r)c_{t-1} = r \left[ b_{t-1} + y_{t-1} + \mathbb{E}_{t-1} \left[ \sum_{s=t}^{\infty} R^{t-s-1} y_s \right] \right] \tag{26}
\]

\[
(1 + r)c_{t-1} = rb_{t-1} + ry_{t-1} + (r/R) \mathbb{E}_{t-1} \left[ \sum_{s=t}^{\infty} R^{t-s} y_s \right] \tag{27}
\]

Now subtract (27) from (24) to obtain:

\[
c_t - c_{t-1} - rc_{t-1} = r(b_{t-1} + y_{t-1} - c_{t-1}) + (r/R) \mathbb{E}_t \left[ \sum_{s=t}^{\infty} R^{t-s} y_s \right]
- \left[ rb_{t-1} + ry_{t-1} + (r/R) \mathbb{E}_{t-1} \left[ \sum_{s=t}^{\infty} R^{t-s} y_s \right] \right] \tag{28}
\]

\[
c_t - c_{t-1} = \left[ (r/R) \sum_{s=t}^{\infty} R^{t-s} \left( \mathbb{E}_t[y_s] - \mathbb{E}_{t-1}[y_s] \right) \right]. \tag{29}
\]

Note that we also had an earlier expression for \(c_t - c_{t-1}\): equation (7). Thus we can

\(^2\)Deaton, here, is codifying practice in much of the immediately preceding literature; however, this definition of ‘permanent income’ differs importantly from the verbal definitions originally provided by Friedman (1957); for a discussion, and an argument that Friedman’s original definition was actually more appropriate, see Carroll (1997).
conclude that
\[
\epsilon_t = \left[ \frac{r}{R} \sum_{s=0}^{\infty} R^{t-s} (E[y_s] - E_{t-1}[y_s]) \right]. \tag{30}
\]

Not only is consumption a random walk, it is a very specific random walk: the change in consumption must be equal to the change in the expectation of permanent income.

Consider now a specific assumption about the stochastic process for income,
\[
(y_t - P) = \epsilon_t + \beta \epsilon_{t-1}. \tag{31}
\]

Given this equation, we have
\[
(E_t - E_{t-1})y_{t+k} = \begin{cases} 
\epsilon_t & \text{if } k = 0 \\
\beta \epsilon_t & \text{if } k = 1 \\
0 & \text{if } k > 1
\end{cases} \tag{32}
\]

Substituting this into (29) gives
\[
\Delta c_t = \left( \frac{r}{R} \right) (1 + \beta/R) \epsilon_t \tag{33}
\]

Now suppose that we estimate a value of $\beta = 0.5(1.04)$ and for simplicity let’s set $r = 0.04$. The standard deviation of changes in consumption will be given by
\[
\sigma_{\Delta c} = (0.04/1.04)(1 + 0.5) \sigma_\epsilon \tag{34}
\]
\[
\approx 0.06 \sigma_\epsilon. \tag{35}
\]

Thus, consumption will be much smoother than income, as it is in the data.

This example can be extended to any example of a moving average representation of income. If the income process is
\[
y_t = P + \epsilon_t + \sum_{k=1}^{\infty} \beta_k \epsilon_{t-k}, \tag{36}
\]
the change in consumption warranted by a current innovation $\epsilon_t$ to income is
\[
\Delta c_t = \left( \frac{r}{R} \right) (1 + \beta_1/R + \beta_2/R^2 + \ldots) \epsilon_t \tag{37}
\]
\[
= \left( \frac{r}{R} \right) \beta[1/R] \epsilon_t \tag{38}
\]
where $\beta[1/R]$ is the lag polynomial that defines the moving average process evaluated at the discount factor $R^{-1}$.

If income is a general ARMA process, if we write $z_t = y_t - P$ we have
\[
z_t + \alpha_1 z_{t-1} + \alpha_2 z_{t-2} + \ldots = \epsilon_t + \beta_1 \epsilon_{t-1} + \beta_2 \epsilon_{t-2} + \ldots \tag{39}
\]
which using the lag polynomial notation is:
\[
\alpha[L]z_t = \beta[L] \epsilon_t. \tag{40}
\]

If this is an ‘invertible’ lag polynomial, it can be rewritten in the form $z_t =$
\[ \alpha^{-1}[L] \beta[L] \epsilon_t \] so that if income is generated by (39) the warranted change in consumption upon receiving a shock \( \epsilon_t \) is given by

\[ \Delta c_t = \frac{(r/R) \beta[R^{-1}]}{\alpha[R^{-1}]} \epsilon_t. \]  

(41)

Thus, if we know the ‘true’ ARMA process for income, the change in consumption upon receipt of a shock \( \epsilon_t \) can be predicted and compared with the outcome.

Flavin (1981) was the first to consider using these time series formulas to test whether the change in consumption is related to the ‘warranted’ change. She proposed estimating equations like

\[ \Delta c_t = \mu + \theta \epsilon_t \]  

(42)

where the parameter \( \theta \) is the warranted change in consumption implied by equation (41).

So far we have assumed a stationary income process (that is, a process in which income always eventually returns to its mean level \( P \)). The truth is that income has grown steadily over time, so interpreted strictly the assumption of stationarity is nonsense. There are several ways of improving upon the implausible stationarity assumption. The first (pursued by Flavin) is to simply assume that there is a deterministic trend to income,

\[ y_t = \mu_0 + \mu y_{t-1} + \mu_2 t. \]  

(43)

This is not a plausible process, however, because it implies that income in the far distant future can be predicted with a high degree of accuracy simply by extending the trend into the future. A vast literature on time-series econometrics has arisen to examine problems like this, and its conclusion is that it is much more plausible to represent the process for income as difference stationary,

\[ \Delta y_t = \mu_0 + \mu_1 \Delta y_{t-1} + \epsilon_t. \]  

(44)

Consider first the simplest possible difference stationary process for income,

\[ \Delta y_t = \epsilon_t. \]  

(45)

What implications does this process have for the behavior of consumption changes?

\[ \Delta c_t = \left[ \frac{(r/R)}{\alpha[R^{-1}]} \sum_{s=t}^{\infty} R^{t+1-s} (E_t[y_s] - E_{t-1}[y_s]) \right]. \]  

(46)

Consider the expectation as of time \( t-1 \) of \( y_t \):

\[ E_{t-1}[y_t] = y_{t-1} + E_{t-1}[y_t - y_{t-1}] \]  

(47)

\[ = y_{t-1} + E_t[\epsilon_t] \]  

(48)

\[ = y_{t-1}. \]  

(49)
Similar logic shows that $\mathbb{E}_{t-1}[y_{t+1}] = \mathbb{E}_{t-1}[y_{t+2}] + \ldots = y_{t-1}$. Furthermore, the same logic shows that $\mathbb{E}_t[y_{t+1}] = \mathbb{E}_t[y_{t+2}] + \ldots = y_t$. Substituting these expressions into (46),

$$
\Delta c_t = \left[ \frac{r}{R} \sum_{s=t}^{\infty} R^{t+1-s}(y_t - y_{t-1}) \right]
= \left[ \frac{r}{R} \sum_{s=t}^{\infty} R^{t+1-s} \epsilon_t \right]
= \left[ \frac{r}{R} \left[ \frac{1}{1 - (1/R)} \right] \epsilon_t \right]
= \left[ \frac{1}{R - 1} \right] \epsilon_t
= \left[ \frac{1}{r} \right] \epsilon_t
= \epsilon_t.
$$

(50) (51) (52) (53) (54) (55)

This makes perfect intuitive sense: when income increases by $\epsilon_t$, the expectation is that it will stay at this new higher level forever, so permanent income has gone up by exactly as much as current income and therefore consumption should react to income changes one-for-one.

When an equation like (44) is estimated on US quarterly data, the coefficient on the lagged income change is typically in the neighborhood of 0.26. Consider now what this implies for the change in permanent income that is associated with a given change in current income. Suppose for simplicity that the change in income in the previous period was $\Delta y_{t-1} = 0$, and suppose that $\mu_0 = 0$. Then

$$
\mathbb{E}_t[y_{t+1}] = \mathbb{E}_t[y_t + \Delta y_{t+1}]
= \mathbb{E}_t[y_t + \mu_1 \epsilon_t + \epsilon_{t+1}]
= \mathbb{E}_t[y_t + \mu_1 \epsilon_t]
$$

(56) (57) (58)

and so on, so that the change in expected permanent income is

$$
(\mathbb{E}_t - \mathbb{E}_{t-1})(r/R)h_t = (r/R) \left[ \epsilon_t \left( \frac{1}{1 - (1/R)} \right) + \epsilon_t(\mu_1/R) \left( \frac{1}{1 - (1/R)} \right) + \epsilon_t(\mu_1^2/R^2) \left( \frac{1}{1 - (1/R)} \right) + \ldots \right]
$$

(59) (60)
\[ \epsilon_t \left( \frac{1}{1 - \mu_1/R} \right) \]
\[ \approx \left( \frac{1}{1 - 0.25} \right) \epsilon_t \]
\[ \approx 1.33 \epsilon_t \]

Now consider what this means for the standard deviation of changes in consumption in the PIH model. Since consumption is equal to permanent income, the standard deviation of changes in consumption should be equal to the standard deviation of changes in permanent income,

\[ \sigma_{\Delta c} = 1.33 \sigma_{\epsilon}. \quad (64) \]

But the standard deviation of changes in income is only \( \sigma_{\epsilon} \). Thus if income changes are positively serially correlated, the permanent income hypothesis implies that the magnitude of consumption variability should actually be greater than the variability in income, exactly the opposite of the pattern the PIH was invented to explain!

### 6 Campbell and Deaton

Campbell (1987) and Campbell and Deaton (1989) showed how to carry out the test that Flavin (1981) originally proposed: whether consumption responds the 'right amount' to news about income \( \epsilon_t \). As suggested by the simple example above, their main finding was that consumption is 'excessively smooth.'

To see how they reached that conclusion, begin with the formula for consumption in the CEQ PIH model:

\[ c_t = \frac{r}{R} \left[ b_t + \sum_{k=0}^{\infty} R^{-k} E_t[y_{t+k}] \right]. \quad (65) \]

Define saving in period \( t \) as (discounted) income minus consumption

\[ s_t = (r/R) b_t + y_t - c_t. \quad (66) \]

Substitute (65) into (66) to get

\[ s_t = \left( r/R \right) b_t + y_t - \left( r/R \right) \left[ b_t + \sum_{k=0}^{\infty} R^{-k} E_t[y_{t+k}] \right] \]
\[ = y_t \left[ 1 - \frac{r}{R} \right] - \left( r/R \right) \sum_{k=1}^{\infty} R^{-k} E_t[y_{t+k}] \]
\[ = y_t R^{-1} - \left( r/R^2 \right) E_t[y_{t+1}] - \left( r/R \right) \sum_{k=2}^{\infty} R^{-k} E_t[y_{t+k}] \]

\[ = y_t R^{-1} - \left( r/R^2 \right) E_t[y_{t+1}] - \left( r/R \right) \sum_{k=2}^{\infty} R^{-k} E_t[y_{t+k}] \]
\[ R^{-1}y_t - R^{-1}E_t[y_{t+1}] + R^{-1}E_t[y_{t+1}] - (r/R^2)E_t[y_{t+1}] - (r/R) \sum_{k=2}^{\infty} R^{-k} E_t[y_{t+k}] = 0 \]

\[ = -R^{-1}(E_t[y_{t+1} - y_t]) + (1 + r)E_t[y_{t+1}]/R^2 - (r/R^2)E_t[y_{t+1}] - (r/R) \sum_{k=2}^{\infty} R^{-k} E_t[y_{t+k}] \]

\[ = -R^{-1}E_t[\Delta y_{t+1}] + R^{-2}E_t[y_{t+1}] - (r/R) \sum_{k=2}^{\infty} R^{-k} E_t[y_{t+k}] \]

\[ = -R^{-1}E_t[\Delta y_{t+1}] + R^{-2}E_t[y_{t+1}] - (r/R) \left( R^{-2}E_t[y_{t+3}] + \sum_{k=2}^{\infty} R^{-k} E_t[y_{t+k}] \right) \]

\[ = -R^{-1}E_t[\Delta y_{t+1}] + R^{-2}E_t[y_{t+1}] - (r/R) \left( R^{-2}E_t[y_{t+3}] + \sum_{k=2}^{\infty} R^{-k-1} E_t[y_{t+k+1}] \right) \]

\[ = R^{-1} \left[ -E_t[\Delta y_{t+1}] + R^{-1}E_t[y_{t+1}] - (r/R^2)E_t[y_{t+2}] + (r/R) \sum_{k=2}^{\infty} R^{-k} E_t[y_{t+k+1}] \right] \]

Repeated substitution leads to the result that

\[ s_t = - \sum_{k=1}^{\infty} R^{-k} E_t[\Delta y_{t+k}] \quad (70) \]

Saving today is equal to the PDV of future changes in labor income. The intuition is straightforward. If income in the future were expected to be the same as today, consumption would be equal to income. If income in the future will be higher than income today (positive values of \( \Delta y_{t+n} \)) then consumption should be higher than income today and the saving rate will be negative.

Rewriting this as

\[ y_t^d - c_t = - \sum_{k=1}^{\infty} R^{-k} E_t[\Delta y_{t+k}] \quad (71) \]

\[ c_t = y_t^d + \sum_{k=1}^{\infty} R^{-k} E_t[\Delta y_{t+k}] \quad (72) \]

it is clear that consumption and disposable income are cointegrated.

Now consider the possibility that individual households know more about their future income prospects than the econometrician. Call the information set of the individuals \( I_t \) and that of the econometrician \( \Omega_t \) where \( \Omega_t \subset I_t \). Consumers will of
course make their decisions based upon their own information set:

\[ s_t = - \sum_{k=1}^{\infty} R^{-k} \mathbb{E}_t[\Delta y_{t+k} | I_t] \]  

(73)

Notice that the econometrician with information set \( \Omega_t \) would calculate that saving should be

\[ s_t^e = - \sum_{k=1}^{\infty} R^{-k} \mathbb{E}_t[\Delta y_{t+k} | \Omega_t] \]  

(74)

and thus the difference between what the econometrician would calculate saving should be and what saving actually is is:

\[ s_t - s_t^e = - \sum_{k=1}^{\infty} R^{-k} (\mathbb{E}_t[\Delta y_{t+k} | \Omega_t] - \mathbb{E}_t[\Delta y_{t+k} | I_t]) \]  

(75)

which reveals to us all of the relevant information that individuals have about the PDV of their future incomes that we cannot know from the information directly available to us! This implies that, whatever variables we are using to forecast future income growth, the saving rate is likely to provide additional forecasting power.

Thus, the Flavin approach of using only lagged values of income to forecast future values of income is inefficient, and we should at a minimum add the saving rate to the forecasting equation for income. This leads us to the following VAR:

\[
\begin{pmatrix}
\Delta y_t \\
 s_t
\end{pmatrix} =
\begin{pmatrix}
\zeta_{11} & \zeta_{12} \\
\zeta_{21} & \zeta_{22}
\end{pmatrix}
\begin{pmatrix}
\Delta y_{t-1} \\
 s_{t-1}
\end{pmatrix} +
\begin{pmatrix}
u_{1t} \\
u_{2t}
\end{pmatrix}
\]  

(76)

which can be rewritten as

\[ x_t = A x_{t-1} + u_t. \]  

(77)

Forecasts of \( x \) are formed using

\[ \mathbb{E}_t[x_{t+i}] = A^i x_t \]  

(78)

Define the vectors \( e_1 = (1, 0) \), which picks out the income change element of \( x \), and \( e_2 = (0, 1) \) which picks out the saving component of \( x \), so that

\[ \mathbb{E}_t[\Delta y_{t+i}] = e'_{1} A^i x_{t+i} \]  

(79)

\[ -s_t = \sum_{k=1}^{\infty} R^{-i} \mathbb{E}_t[\Delta y_{t+k}] \]  

(80)

\[ -e'_{2} x_t = \sum_{k=1}^{\infty} e'_{1} A^k R^{-k} x_t \]  

(81)
Now we can divide both sides of (81) by $-x_t$ to obtain

$$e_2' = -e_1' \sum_{k=1}^{\infty} A^k R^{-k}. \tag{82}$$

Recall the fact for scalars that if $\mu < 1$ then $\sum_{k=0}^{\infty} \mu^k = 1/(1 - \mu) = (1 - \mu)^{-1}$. There is an equivalent formula for matrices:

$$\sum_{k=0}^{\infty} (A/R)^k = (I - A/R)^{-1} \tag{83}$$

$$\sum_{k=1}^{\infty} (A/R)^k = (I - A/R)^{-1} - I \tag{84}$$

which can be substituted into (82) to obtain

$$e_2' = -e_1' [(I - A/R)^{-1} - I] \tag{85}$$

$$e_2'(I - A/R) = -e_1'[I - (I - A/R)] \tag{86}$$

$$e_2' = (e_2' - e_1')(A/R) \tag{87}$$

$$[0 \ 1] = [[0 \ 1] - (1 \ 0)] A/R \tag{88}$$

$$[0 \ 1] = [(-1 \ 1)] (\begin{bmatrix} \zeta_{11} & \zeta_{12} \\ \zeta_{21} & \zeta_{22} \end{bmatrix}) / R \tag{89}$$

$$0 = (-\zeta_{11} + \zeta_{21}) / R \Rightarrow \zeta_{21} = \zeta_{11} \tag{90}$$

$$1 = (-\zeta_{12} + \zeta_{22}) / R \Rightarrow \zeta_{22} = \zeta_{12} + R \tag{91}$$

So, to test the theory one estimates the equations of the VAR one by one

$$\Delta y_t = \zeta_{11}(\Delta y_{t-1}) + \zeta_{12}s_{t-1} + u_{1t} \tag{92}$$

$$s_t = \zeta_{21}(\Delta y_{t-1}) + \zeta_{22}s_{t-1} + u_{2t} \tag{93}$$

and then tests the econometric restrictions that $\zeta_{21} = \zeta_{11}$ and $\zeta_{22} = \zeta_{12} + R$.

To understand these restrictions, start with the definition of saving $s_t = (r/R)b_t + y_t - c_t$, lag one period and multiply through by $R$ to get

$$Rs_{t-1} = rb_{t-1} + R(y_{t-1} - c_{t-1}) \tag{94}$$

Now recall that the evolution of assets is given by

$$b_t = R(b_{t-1} + y_{t-1} - c_{t-1}) \tag{95}$$

$$R(y_{t-1} - c_{t-1}) = b_t - Rb_{t-1} \tag{96}$$

Use (94) to substitute into (92):

$$Rs_{t-1} = rb_{t-1} + b_t - Rb_{t-1} \tag{97}$$
But the first difference of the definition of saving says

\[
\Delta s_t = \frac{r}{R} \Delta b_t + \Delta y_t - \Delta c_t \tag{98}
\]

\[
\Delta c_t = rs_{t-1} + \Delta y_t - \Delta s_t \tag{99}
\]

\[
= \Delta y_t - s_t + Rs_{t-1}. \tag{100}
\]

Now let’s suppose that the CEQ PIH is true so that the restrictions above hold true. Substituting in equation (100) for \(\Delta y_t\) and \(s_t\) we have

\[
\Delta c_t = \frac{\zeta_{11} \Delta y_{t-1} + \zeta_{12} s_{t-1} + u_t}{\Delta y_t} \tag{101}
\]

\[
- \frac{\zeta_{11} (\Delta y_{t-1}) - (\zeta_{12} + R) s_{t-1} - u_{2t}}{s_t} + Rs_{t-1}
\]

\[
= u_{1t} - u_{2t}. \tag{102}
\]

Thus we find again that consumption follows a random walk, and the change in consumption is predictable neither by lagged saving nor the lagged income change.

Now consider the change in permanent income:

\[
\Delta y^p_t = \sum_{k=0}^{\infty} R^{-k} (E_t - E_{t-1}) \Delta y_{t+k} \tag{103}
\]

\[
= e_1' \sum_{k=0}^{\infty} (A/R)^k u_t \tag{104}
\]

\[
= e_1' (I - A/R)^{-1} u_t \tag{105}
\]

But from equation (85) we have

\[
e_2' = -e_1' [(I - A/R)^{-1} - I] \tag{106}
\]

\[
e_1' - e_2' = e_1' (I - A/R)^{-1} \tag{107}
\]

Substituting (107) into (105),

\[
\Delta y^p_t = (e_1' - e_2') u_t \tag{108}
\]

\[
= ([1 0] - [0 1]) u_t \tag{109}
\]

\[
= [1 - 1] u_t \tag{110}
\]

\[
= u_{1t} - u_{2t} \tag{111}
\]

and thus the change in permanent income is equal to the change in consumption.

Suppose now that the restrictions imposed by the CEQ PIH model are not true. In particular, suppose that in the saving equation the coefficient on lagged income growth \(\zeta_{21}\) is not \(\zeta_{11}\) as it should be but instead \(\zeta_{11} - \mu\). Then the same substitutions
used to derive $\Delta c_t$ above now lead to the result that

$$\Delta c_t = \mu(\Delta y_{t-1}) + u_{1t} - u_{2t}. \quad (112)$$

Thus, if $\zeta_{21}$ differs at all from the value it should take, it must be true that consumption today exhibits ‘excess sensitivity’ to lagged income growth. Thus excess sensitivity and excess smoothness are not different, but two ways of looking at the same phenomenon.
References


\[ \Delta c_t = 10.63 + 0.174 \ E_{t-2}[\Delta y_{t-1}] . \]

(6.83) \hspace{1cm} (2.18)