Cointegration and the Ramsey Model

This handout examines implications of the Ramsey model for cointegration between consumption, income, and capital.

Consider the following Ramsey/Cass-Koopmans growth model with labor-augmenting technological progress at rate $g$:

$$\max \int_0^\infty u(C_t)e^{-\theta t}dt$$ (1)

s.t.

$$\dot{K}_t = Y_t - C_t$$ (2)

$$Y_t = K_t^\alpha(Z_tL_t)^{1-\alpha}$$

$$\dot{Z}_t = g$$

$$u(C) = \frac{C^{1-\rho}}{1-\rho}$$

and suppose there is no population growth so that we can normalize the labor force to be $L_t = 1 \forall t$. Finally, normalize the initial level of productivity to $Z_0 = 1$.

1. We start by showing that if the utility function is of the CRRA form $u(C) = C^{1-\rho}/(1 - \rho)$ the optimization problem can be rewritten as

$$\max \int_0^\infty \left( \frac{c_t^{1-\rho}}{1-\rho} \right) e^{-\nu t}dt$$ (3)

s.t.

$$\dot{k}_t = y_t - c_t - gk_t$$ (4)

$$y_t = k_t^\alpha$$ (5)

where $\nu = \theta - g(1 - \rho)$ and lower-case variables are the upper-case versions divided by $Z_t$, e.g. $c_t = C_t/Z_t$.

Start with the production function:

$$Y = K^\alpha(ZL)^{1-\alpha}$$ (7)

$$Y/ZL = (K/(ZL))^{-\alpha}$$ (8)

$$= k^\alpha.$$ (9)

The utility function is rewritten as

$$\frac{C_t^{1-\rho}}{1-\rho} = \frac{(c_tZ_t)^{1-\rho}}{1-\rho} = \frac{c_t^{1-\rho}}{1-\rho}Z_t^{1-\rho}$$ (10)

but behavior is unaffected by multiplying a utility function by a constant and so the consumer will behave identically to a consumer
whose utility function is
\[ c_t^{1-\rho} e^{g(1-\rho)t} \] (12)
so the objective function can be rewritten as specified.

For the DBC, rewrite \( \dot{K} \) as
\[ \frac{dK}{dt} = \frac{d(k_t Z_t)}{dt} \] (13)
\[ = \frac{dk_t}{dt} Z_t + k_t \frac{dZ_t}{dt} \] (14)
\[ = (\dot{k}_t + g k_t) Z_t \] (15)
so that equation (2) becomes
\[ (\dot{k}_t + g k_t) Z_t = Y_t - C_t \]
\[ \dot{k}_t + g k_t = y_t - c_t \]
\[ k_t = y_t - c_t - g k_t \] (16)
as required.

2. We now show that in such an economy, the steady-state will be given by the point where \( \dot{K}/K = \dot{C}/C = \dot{Y}/Y = g \).

The steady-state of this model will occur at the point where \( \dot{k}_t = \dot{c}_t = 0 \). But since \( k_t = K_t/Z_t \) and \( c_t = C_t/Z_t \), \( \dot{k} \) and \( \dot{c} \) can be zero only if \( K \) and \( C \) are growing at the same rate as \( z \). Thus we know that in the steady-state,
\[ \frac{\dot{C}}{C} = \frac{\dot{K}}{K} = g. \] (17)

3. We now show that the steady state interest rate must satisfy
\[ f'(k^*) = \theta + \rho g \] (18)
where \( k^* \) is the steady-state value of \( k \).

The consumption Euler equation tells us that
\[ \frac{\dot{c}}{c} = \rho^{-1}(f'(k) - g - \nu) \] (19)
\[ = \rho^{-1}(f'(k) - g - \theta + g(1-\rho)) \] (20)
\[ = \rho^{-1}(f'(k) - \theta - \rho g) \] (21)
but we know that in steady-state, \( \dot{c}/c = 0 \), implying
\[ f'(k^*) = \theta + \rho g \] (22)
Now consider the partial equilibrium perfect foresight CRRA model of consumption with constant wage growth rate \( g \). Assuming a constant real interest rate \( r \), time preference rate \( \theta \), and coefficient of relative risk aversion \( \rho \), the formula for consumption as a function of current Market wealth \( M_t \) (as opposed
to human wealth) and labor income \( N_t \) for an infinite horizon consumer is

\[
\bar{C}_t \approx (r - \rho^{-1}(r - \theta)) \left[ \left( \frac{N_t}{r - g} \right) + M_t \right] \quad (23)
\]

where the approximation becomes arbitrarily good as the time interval approaches zero (that is, as the model approaches continuous time).

4. Explain the three effects of interest rates in this model in intuitive terms, and how they are reflected in the equation.

The first place \( r \) appears reflects the income effect, as is evident from the fact that if \( \rho = \infty \) so that there is no substitution, and \( N = 0 \) so that there is no future labor income to discount and therefore no human wealth effect, the only thing that is left is the income effect.

The \( \rho^{-1} \) term reflects the substitution effect, and the \( r \) in the denominator of the human wealth expression obviously reflects the human wealth effect.

5. Now suppose that the consumer spends according to (23). Suppose further, for simplicity, that the growth rate \( g = 0 \) and the consumer’s time preference rate happens to be equal to the interest rate, \( r = \theta \). Define the ratios of consumption and wealth to labor income as

\[
\chi_t = \frac{C_t}{N_t} \quad (24) \\
\mu_t = \frac{M_t}{N_t} \quad (25)
\]

and show that assuming \( g = 0 \) and \( r = \theta \) the equation

\[
\chi_t = 1 + r \mu_t \quad (26)
\]

approximately characterizes the relationship between consumption, income, and wealth at every point in time.

For \( r = \theta \) and \( g = 0 \) equation (23) reduces to

\[
C_t = r \left[ \left( \frac{N_t}{r} \right) + M_t \right] \quad (27) \\
= N_t + r M_t \quad (28) \\
\chi_t = 1 + r \mu_t \quad (29)
\]

6. Now return to the case where \( g \) is not assumed to be zero, but suppose that the interest rate is equal to the long-run steady-state interest rate consistent with general equilibrium. We now show that in this case we will have approximately

\[
\chi_t = 1 + (r - g) \mu_t \quad (30)
\]

Note first that if (18) holds then we have that \( \rho^{-1}(r - \theta) = \rho^{-1} \rho g = g \).

In this case we have

\[
C_t = (r - g) \left[ \left( \frac{N_t}{r - g} \right) + M_t \right] \quad (31) \\
\chi_t = 1 + (r - g) \mu_t \quad (32)
\]
as required.

7. Now return to the general equilibrium model case where \( g = 0 \) and use the insights from the foregoing analysis to discuss the effects in the short-run and the long-run of a sudden and permanent increase in the time preference factor \( \theta \). In particular, show the dynamic paths for \( c_t, w_t, \) and \( r_t \) in this economy.

This is a standard problem that we covered in class. See class notes for the solution.

8. Now we want to bring together the two models, which we can do by realizing that since market wealth \( M_t \) reflects ownership of the aggregate capital stock, we can make the identification \( M_t = K_t \). However, in the general equilibrium model we normalized \( K_t \) by the level of productivity, while in the partial equilibrium model we normalized \( M_t \) by the level of labor income. We now show that if we define \( \kappa \) as the ratio of the capital stock to labor income, then we can calculate the interest rate as

\[
\alpha k^{\alpha - 1} = \left( \frac{\alpha}{(1 - \alpha)\kappa} \right).
\]

The tricky part here is constructing aggregate labor income. Though we normalized the population/workforce to one, in order to get the wage rate we need to take the derivative of output with respect to it, so reintroduce it as \( L \) and write

\[
F(K, L) = K^\alpha (ZL)^{1-\alpha}
\]

\[
F_L = (1 - \alpha) K^\alpha (ZL)^{-\alpha} z
\]

so total labor income is the population times the wage rate, which is

\[
LF_L = (1 - \alpha) K^\alpha (ZL)^{-\alpha} ZL
\]

\[
= (1 - \alpha) K^\alpha (ZL)^{1-\alpha}
\]

\[
= (1 - \alpha) Y
\]

Now we can reintroduce \( L = 1 \) and write

\[
\kappa = \left( \frac{K}{(1 - \alpha)K^\alpha z^{1-\alpha}} \right)
\]

\[
= \left( \frac{1}{1 - \alpha} \right) k^{1-\alpha}
\]

so

\[
\alpha k^{\alpha - 1} = \left( \frac{\alpha}{(1 - \alpha)\kappa} \right)
\]

as required.

9. Suppose we had statistics on two economies that were identical in every respect except the time preference factors of the representative agents in them. Both economies are in their long-term steady states. In economy \( A \), the time
preference is such that the steady-state is at $\bar{\kappa}_A = 4$, while in economy B the consumer is so impatient that $\bar{\kappa}_B = 1$. Suppose an economist came along who said that we could estimate “the marginal propensity to consume out of wealth” $\psi$ as follows. He points out that if there is such a $\psi$, we can write

$$\chi_A = 1 + \psi\kappa_A \quad (42)$$

$$\chi_B = 1 + \psi\kappa_B \quad (43)$$

so we should be able to estimate $\psi$ from

$$\psi = \frac{\chi_A - \chi_B}{\kappa_A - \kappa_B} \quad (44)$$

Assume that the production function in both economies is identical. Use your expressions for the interest rate to calculate algebraically what the economist’s estimate of $\psi$ will be. Explain why you get this result in intuitive terms. Can this measure be interpreted as a marginal propensity to consume in any sense?

In both economies we will have

$$\chi = 1 + \left(\frac{\alpha}{(1 - \alpha)\kappa}\right)\kappa, \quad (45)$$

$$= 1 + \left(\frac{\alpha}{1 - \alpha}\right) \quad (46)$$

and since we have assumed $\alpha$ is the same across the two economies this means that the $\chi$’s will be the same, so the economist’s method will estimate $\psi = 0$!

The problem is that the differences in wealth across countries are exactly counterbalanced by differences in interest rates that go in the opposite direction. This is because a country with a low wealth-to-income ratio has high interest rates, and vice versa.

This question was written as an attack on cointegration analysis, which attempts to estimate “long run MPC’s” using equations like these.
Further Discussion

Figure 1 further illustrates the point by showing the path of the ratio of consumption to labor income $c/wL$ in an example economy. The experiment is as above: The economy is initially in steady state, and then at time $t$ there is a permanent increase in the time preference rate. The figure shows that this causes a surge in consumption in the short run, but in the long run consumption falls commensurately with wages, restoring the equilibrium ratio as above.

Figure 2 shows the path of the $k/wL$ ratio in this economy. Over the sample experiment in question, this object is above its average during the period when consumption is above its average, and below when consumption is below. So a "cointegrating" regression between $c/wL$ and $k/wL$ were run in this economy, the coefficient obtained would be positive; indeed, when such a regression is run for the experiment depicted here, the coefficient on $k/wL$ is $0.024$ with a $t$-stat of $8$. But as proven above, this is not a measure of the "long-run MPC" out of anything; if the "long-run MPC" means anything, it is zero in this model.

Note further that if such a regression were performed on a time interval immediately surrounding the period when the time preference rate changes, the estimated "MPC" would be negative, because in the period immediately after $t$ consumption is above its average over the whole sample, while capital is below its average over the sample. On the other hand, if the regression were estimated over an infinite horizon period, the coefficient would be zero. Thus there is no stable meaning for the regression coefficient in this context.

In fact, further analysis shows that the same kind of problem occurs when any other parameter in the model is changed. The only exception is shocks to $k$ (or, equivalently, shocks to $z$).

The problem can be thought of as basically an endogeneity problem: $\bar{\chi}, \bar{\mu}$ and $\bar{k}$ are endogenous with respect to all the parameters of the model. Thus one cannot estimate the "effect" of variations in $k$ on $c$ when those variations in $k$ reflect changes in parameters of the model.

This discussion has implicitly been focused on the empirical literature that looks for a relationship between the consumption-to-income ratio and the capital-to-income ratio, which is the traditional approach. The alternative is the Lettau-Ludvigson framework in which what one is seeking is a cointegrating relationship between the logs of consumption, income, and wealth.

The LL framework essentially ends up assuming that there is a stable relationship of the form

$$C_t = N_t + \gamma M_t + \epsilon_t$$

for some constant MPC $\gamma$ where $\epsilon_t$ is a stationary variable. (There are a lot of complications and approximations and loglinearizations, but at the end of the day this is basically what they are doing). But notice that from (31) in steady-state we have

$$C_t = N_t + (r - g)M_t$$

The problem is that if there is a permanent change in $g$, then the error term in (47) changes in a permanent way that trends with $C_t$. Thus, $\epsilon_t$ is no longer stationary.
But if one estimates a cointegrating relationship that assumes a constant $\gamma$, it will appear that $\epsilon_t$ has predictive power, because the estimated $\gamma$ will be chosen so that in the sample as a whole, the mean value of $\epsilon_t$ is zero.

The point is illustrated with a final figure, which presents the results from the following simulation. After 10 periods at an initial steady state corresponding to a steady-state growth rate of zero, the model is hit by a random shock to $z$ that occurs once every 50 years (keeping the steady-state growth rate and all other parameters equal to zero). Finally, in period 260, there is a permanent shock to $g$ - it goes from 0 to 1 percent.

The graph shows the value of

$$\epsilon_t = C_t/Z_t - Y_t/Z_t - r^{SS}K_t/Z_t$$

where $r^{SS}$ is the steady-state interest rate associated with the original baseline parameter values. The figure illustrates the point that so long as the only shocks hitting the model are to the level of productivity, the error term is stationary (it tends back to zero). However, when the $g$ shock hits in period 260, the error term moves permanently away from zero, violating the stationarity assumption.

Note also the extreme slowness with which the error reverts to zero. This is a reflection of a more general problem with using the Ramsey model in practice: The transitional dynamics in the model are so slow that in effect we have very little ability to identify the steady-state of the model over realistic time frames for which we have empirical data.

It is useful to recast the problem in terms of what is wrong with the Lettau-Ludvigson analysis. There are two answers. The first is their assumption that the ratio of human wealth to market wealth is stationary. Neither partial nor general equilibrium theory provides a plausible rationale for this. The second is that the dynamics of the model are so slow that even if there were no important changes in parameter values, the amount of data available would not likely be able to identify the main parameters of the model.
Figure 1  $c/wL$ Ratio After Before and After Increase in $\theta$
Figure 2 $k/wL$ Ratio After Before and After Increase in $\theta$
Figure 3 \( \epsilon = c - (1 - \alpha)y - r^{SS}k \)