Cointegration and the Dynamics of C, B, and Y

The intertemporal budget constraint (IBC) for an infinite-horizon representative agent’s consumption problem says that the PDV $P$ of consumption must be financed by wealth,

$$P_t(C) = B_t + H_t$$

(1)

where $B_t$ is the consumer’s nonhuman wealth, $H_t$ is human wealth,

$$H_t = P_t(Y),$$

(2)

and $O_t$ is total wealth (human and nonhuman).

Calling $R_{t+1}$ the riskless return factor for assets held between $t$ and $t + 1$, we can define the dynamic budget constraint for total wealth as

$$O_{t+1} = (O_t - C_t)R_{t+1}.$$  

(3)

CampManCRRWithTimeVaryingR shows (following Campbell and Mankiw (1989), henceforth ‘CM’) that if interest rates are stationary the IBC implies that

$$c_t - o_t \approx \phi (r_{t+1} - \Delta c_{t+1}) + \phi \phi (r_{t+2} - \Delta c_{t+2}) + \phi (c_{t+2} - o_{t+2}) + \phi k) + \phi k$$

(4)

where lower-case variables represent the logs of their upper-case equivalents and $k$, $\mu$, and $\phi$ are constants related to the preference and budget parameters.

It is hard to come up with empirical tests of this model, however, because human wealth $H_t$ is unobserved and therefore total wealth $O_t$ is unobserved. Lettau and Ludvigson (2001, 2004) (henceforth, ‘LL’) propose the following solution. Designating the growth rate of labor income from period $t$ to $t + 1$ as $G_{t+1}$, human wealth can be written

$$H_t = Y_t + Y_{t+1}/R_{t+1} + Y_{t+2}/R_{t+1}R_{t+2} \ldots$$

(5)

$$= Y_t + Y_t G_{t+1}/R_{t+1} + Y_t (G_{t+1} G_{t+2})/(R_{t+1} R_{t+2}) + \ldots$$

(6)

$$= Y_t (1 + G_{t+1}/R_{t+1} + (G_{t+1}/R_{t+1})(G_{t+2}/R_{t+2}) + \ldots)$$

(7)

$$H_{t+1} = Y_{t+1} (1 + G_{t+1}/R_{t+1} + (G_{t+1}/R_{t+1})(G_{t+2}/R_{t+2}) + \ldots)$$

(8)

$$= Y_t G_{t+1}/R_{t+1} G_{t+1}/R_{t+1} + (G_{t+1}/R_{t+1})(G_{t+2}/R_{t+2}) + \ldots)$$

(9)

$$= R_{t+1} Y_{t+1} (1 + G_{t+1}/R_{t+1} + \ldots)$$

(10)

$$H_{t+1} = R_{t+1} (H_t - Y_t).$$

(11)

But note that the form of (11) is identical to the form of (3). As a result, the same steps CM used to derive (4) can be applied to generate

$$y_t - h_t = \sum_{j=1}^{\infty} \phi^j (r_{t+j} - \Delta y_{t+j}) + \phi k/(1 - \phi),$$

(12)
which can be rewritten

\[ h_t = y_t - \sum_{j=1}^{\infty} \phi^j (r_{t+j} - \Delta y_{t+j}) + \phi k/(1 - \phi) \]

\[ \Rightarrow h_t = y_t - \sum_{j=1}^{\infty} \phi^j r_{t+j} + \phi k/(1 - \phi) \]

(14)

where for notational simplification we define the net discount rate for income as \( r_y = r_t - \Delta y_t \).

Next, LL assume that the income growth process is stationary, so that \( \lim_{j \to \infty} \mathbb{E}_t[\Delta y_{t+j}] = g \), in which case (14) can be written

\[ h_t = \kappa + y_t + z_t \]

(15)

where \( z_t \) is a stationary variable.

However, the stationarity of \( \Delta \log y_t \) is a very strong assumption; in particular, it rules out changes in the trend of productivity growth. But such changes have been quite important over the postwar period; in the US, labor productivity growth averaged about 3 percent a year from 1947 to 1973, then dropped to only about 0.9 percent a year from 1973 to 1995. Since 1995 labor productivity growth has been about 2 percent a year. So the assumption that the income growth is believed to be stationary is very questionable when actually applied to US postwar data.\(^1\)

Note that all of these results have been derived under the assumption of perfect foresight about future interest rates and growth factors for income. If either or both of these factors is stochastic, everything is more complex.

To proceed, let’s suppose that consumers behave as though their forecasts at time \( t \) are perfectly certain to come true. In this case the human wealth accumulation equation must be modified, because the substitutions that related \( H_t \) to \( H_{t+1} \) above assumed that expectations of \( G_{t+1} \ldots G_\infty \) were unchanged when moving from \( t - 1 \) to \( t \).

LL define a new rate-of-return concept corresponding to the ‘rate of return on human wealth’ that they effectively define as

\[ R^h_{t+1} = \left( \frac{H_{t+1} - Y_{t+1}}{H_t - Y_t} \right). \]

(16)

We can work out that this will be:

\[ H_{t+1} - Y_{t+1} = Y_{t+1} \mathbb{E}_{t+1} \left[ R^y_{t+2} + R^y_{t+2} R^y_{t+3} + \ldots \right] \]

\[ H_t - Y_t = Y_t \mathbb{E}_t \left[ R^y_{t+1} + R^y_{t+1} R^y_{t+2} + \ldots \right] \]

\[ \left( \frac{H_{t+1} - Y_{t+1}}{H_t - Y_t} \right) = G_{t+1} \frac{\mathbb{E}_{t+1} \left[ R^y_{t+2} + R^y_{t+2} R^y_{t+3} + \ldots \right]}{\mathbb{E}_t \left[ R^y_{t+1} + R^y_{t+1} R^y_{t+2} + \ldots \right]} \]

\[ = G_{t+1} \frac{\mathbb{E}_{t+1} \left[ R^y_{t+2} + R^y_{t+2} R^y_{t+3} + \ldots \right]}{\mathbb{E}_t \left[ R^y_{t+1} \right]} \left[ 1 + R^y_{t+2} + \ldots \right] \]

\[ \Rightarrow (19) \]

\[ \Rightarrow (20) \]

\(^1\)See Viard (1993) and Carroll, Otsuka, and Slacalek (2011) for direct evidence that professional economists’ forecasts of long-run growth in the U.S. have not been stable.
\[
\begin{align*}
&= \frac{G_{t+1}}{E_t[G_{t+1}/R_{t+1}]} \frac{(E_{t+1} - E_t) \left[ R_{t+2}^y + R_{t+2}^y R_{t+3}^y + \ldots \right] / E_t[R_{t+2}^y + \ldots] + 1}{(E_t[R_{t+2}^y + \ldots]^{-1} + 1)} \\
&\approx \frac{G_{t+1}}{E_t[G_{t+1}/R_{t+1}]} \left[ 1 + \frac{(E_{t+1} - E_t) \left[ R_{t+2}^y + \ldots \right]}{E_t[R_{t+2}^y + \ldots] - (E_t[R_{t+2}^y + \ldots]^{-1})} \right]
\end{align*}
\]

(21)

To understand this expression better, consider the case where \( G \) and \( R \) are constant over time. Then we obtain

\[
R_{t+1}^h = G,
\]

(22)

because \( Y_{t+1} = G Y_t \) and the remaining terms on the RHS of (17) and (18) are identical. Hence, the ‘return’ on human wealth is simply its growth rate, adjusted for any changes in the expected discounted growth rate of future income between periods \( t \) and \( t + 1 \). This latter component also captures the ‘human wealth effect.’

The next step is to define a rate of return on total wealth, which will be a mixture of the return on the portion of wealth invested in physical assets \( B_t + Y_t - C_t \) and the portion invested in human wealth \( H_t - Y_t \). If we define the portion of available period-\( t \) resources that is not consumed as savings \( A_t \) we have

\[
O_{t+1} = \bar{R}_{t+1} A_t
\]

(23)

\[
= R_{t+1} \left( B_t + Y_t - C_t \right) + R_{t+1}^h (H_t - Y_t)
\]

(24)

\[
\equiv A^h_t
\]

LL now define the portfolio share of savings \( \omega_t = (B_t + Y_t - C_t)/A_t \) invested in physical assets, and the portfolio share of human assets as \( (1 - \omega_t) \), allowing them to rewrite (24) as

\[
O_{t+1} = A_t (R_{t+1} \omega_t + R_{t+1}^h (1 - \omega_t)).
\]

(25)

Finally, LL make a critical assumption: that \( \omega_t \) is a stationary variable (that is, the ratio of physical wealth to human wealth is neither rising nor falling over time). Under this assumption, and using lower case variables for the log, they derive the equation

\[
c_t - \omega_t b_t - (1 - \omega_t) h_t = E_t \sum_{i=1}^{\infty} \phi^i \left\{ \omega_t r_{t+i} + (1 - \omega_t) r_{t+i}^h \right\} - \Delta c_{t+i}
\]

(26)

upon which all of their subsequent analysis is based, on the assumption that the term on the RHS is stationary.

Unfortunately, however, the theory being used implies that \( \omega_t \) should NOT be stationary. To see this consider the PIH version of the optimal consumption model with \( \sigma = 0 \). In that case the model reduces to the classic Hall (1978) framework in which the level of consumption follows a random walk. Since \( C \) is proportional to total wealth \( O_t \), it is apparent that the level of total wealth must also follow a random walk. But we know that income (and therefore human wealth) trends upward at rate \( G \) in this model. If \( H \) is rising and \( O \) is constant, it must be the case that \( \omega_t \) has been falling.

For some numerical explorations of what happens to consumption, wealth, and ratios of them to income in a model of this kind under various scenarios for paths
of growth expectations, interest rates, and other variables, see the software archive associated with this handout.
References


