You are expected to answer all parts of all questions. If you cannot solve part of a question, do not give up. The exam is written so that you should be able to answer later parts even if you are stumped by earlier parts.

Write all answers on the exam itself; if you run out of room, use the back of the previous page.
Part I

Productivity Growth and Dynamic Inefficiency in the OLG Model.

Consider a Diamond (1965) OLG economy like the one in the handout OLGModel, assuming logarithmic utility and a Cobb-Douglas aggregate production function,

\[ Y_t = F(K_t, P_t L_t) \]  

where \( P_t \) is a measure of labor productivity that grows by

\[ P_{t+1} = G P_t \]  

from period to period. Assume that population growth is zero (\( \Xi = 1 \); for convenience normalize the population at \( L_\tau = 1 \forall \tau \)), and assume that productivity growth has occurred at the rate \( g = G - 1 \) forever.

One unit of the quantity \( PL \) is called an ‘efficiency unit’ of labor: It reflects a unit of labor input to the production process.

1. Assume that \( F(K, PL) \) is a Constant Returns to Scale function, and show how to rewrite the capital accumulation equation

\[ K_{t+1} = A_{1,t} \]  

in per-efficiency-unit terms as

\[ k_{t+1} = a_{1,t}/G \]  

**Answer:**

In an OLG economy, aggregate capital in period \( t + 1 \) is equal to the savings from period \( t \):

\[ K_{t+1} = A_{1,t} \]  

\[ K_{t+1} = a_{1,t} P_t \]  

\[ \left( \frac{K_{t+1}}{P_t} \right) = a_{1,t} \]  

\[ \left( \frac{K_{t+1}}{P_{t+1}} \right) \left( \frac{P_{t+1}}{P_t} \right) = a_{1,t} \]  

\[ k_{t+1} = a_{1,t}/G_{t+1} \]  

2. Show that under these assumptions, the process for aggregate \( k \) dynamics is

\[ k_{t+1} = \left( \frac{(1 - \varepsilon)\beta}{G_{t+1}(1 + \beta)} \right) k_t^\varepsilon \]  

**Answer:**
\[ a_{1,t} \text{ can be found from:} \]

\[
C_{1,t} = W_{1,t} + \overbrace{W_{2,t+1}/R_{t+1}}^{=0} / (1 + \beta) \quad \text{(11)}
\]

\[
c_{1,t} = \frac{w_t}{1 + \beta} \quad \text{(12)}
\]

\[
a_{1,t} = w_{1,t} - c_{1,t} = w_{1,t}(1 - 1/(1 + \beta)) \quad \text{(13)}
\]

\[
= w_{1,t} \beta / (1 + \beta)) \quad \text{(14)}
\]

\[
= (1 - \varepsilon) k_t^\varepsilon \left( \frac{\beta}{1 + \beta} \right) \quad \text{(15)}
\]

Thus (9) and (16) can be combined to yield:

\[
=k_{t+1}^{c_{t+1}/G_{t+1}} = k_t^\varepsilon \left[ \frac{(1 - \varepsilon) \beta}{G_{t+1}(1 + \beta)} \right] \quad \text{(17)}
\]

as required.

3. Derive the steady-state level of \( k_t \) that the economy achieves if the rate of productivity growth is constant at \( G_t = G \forall t \).

Answer:

The steady-state will be the place where \( k_{t+1} = k_t = \bar{k} \). Substituting into equation (10):

\[
\bar{k} = \bar{k}^\varepsilon \left[ \frac{(1 - \varepsilon) \beta}{G(1 + \beta)} \right] \quad \text{(18)}
\]

\[
\bar{k}^{1-\varepsilon} = \left[ \frac{(1 - \varepsilon) \beta}{G(1 + \beta)} \right] \quad \text{(19)}
\]

\[
\bar{k} = \left[ \frac{(1 - \varepsilon) \beta}{G(1 + \beta)} \right]^{1/(1-\varepsilon)} \quad \text{(20)}
\]

Now suppose that the economy had been growing at this constant rate \( G \) since the beginning of time, but all of a sudden at the beginning of period \( t \) everybody learns that henceforth and forever more, productivity will grow at a faster rate than before, \( \hat{G} > G \).

4. Define the new steady-state as \( \tilde{k} \). Will this be larger or smaller than the original steady state \( \bar{k} \)? Explain your answer.

Answer:

Since \((1 - \varepsilon) > 0\), equation (20) implies that a larger value of \( G \) implies a lower steady-state capital stock per efficiency unit. The reason is that
with faster productivity growth, the efficiency units of labor provided by the young generation are larger relative to the size of the capital stock saved by the previous generation, so the ratio of capital to efficiency units of labor is smaller.

5. Next, use a diagram to show how the $k_{t+1}(k_t)$ curve changes when the new growth rate takes effect, and show the dynamic adjustment process for the capital stock toward its new steady-state, assuming that the economy was at its original steady state leading up to period $t$.

Answer:

Defining the original steady-state capital stock as $\bar{k}$ and the new steady-state capital stock as $\hat{k}$, the convergence process looks as indicated in figure 1.

6. Define an index of aggregate consumption per efficiency unit of labor in period $t$ as $\chi_t = c_{1,t} + c_{2,t}/G$, and derive a formula for the sustainable level of $\chi$ associated with a given level of $k$.

Answer:
$$K_{t+1} = K_t + K_t^e P_t^{1-\varepsilon} - C_{1,t} - C_{2,t}$$  \hspace{1cm} (21)

$$\left(\frac{K_{t+1}}{P_t}\right) = k_t + k_t^e - c_{1,t} - \frac{c_{2,t} P_{t-1}}{P_t}$$  \hspace{1cm} (22)

$$\left(\frac{K_{t+1} P_{t+1}}{P_t}\right) = k_t + k_t^e - c_{1,t} - c_{2,t}/G$$  \hspace{1cm} (23)

$$k_{t+1} G = k_t + k_t^e + \chi_t$$  \hspace{1cm} (24)

The sustainable level of $\chi$ is the level $\bar{\chi}$ such that $k_{t+1} = k_t = k$:

$$1 + g)k = k + k^e - \bar{\chi}$$  \hspace{1cm} (25)

$$\bar{\chi} = k^e - gk.$$  \hspace{1cm} (26)

7. Derive the conditions under which a marginal increase in the productivity growth rate $g$ will result in an increase in the steady-state level of $\chi$, and explain in words why this result holds. (You can leave the term $\partial \bar{k}/\partial g$ unevaluated in your answer, using only what we know about this term from above).

**Answer:**

There are two effects of an increase in $g$. First, for a given $k$, the sustainable amount of $\chi(k)$ declines, because the faster productivity growth means that to keep capital per efficiency unit constant the economy must save more (each efficiency unit of labor must be supplied with its own capital; faster growth of efficiency units therefore requires faster growth of capital). Second, with a faster $g$ the endogenous saving rate and steady-state capital-per-capita $\bar{k}$ will change. Whether steady-state consumption per capita rises or falls depends on the balance between these two things.

Steady-state $\chi$ can be written as $\bar{\chi}(\bar{k})$. We are interested in

$$\left(\frac{d\bar{\chi}}{dg}\right) = \left(\frac{\partial \bar{\chi}}{\partial g}\right) + \left(\frac{\partial \bar{\chi}}{\partial \bar{k}}\right) \left(\frac{\partial \bar{k}}{\partial g}\right)$$  \hspace{1cm} (27)

$$= -\bar{k} + \left(\frac{\partial \bar{\chi}}{\partial \bar{k}}\right) \left(\frac{\partial \bar{k}}{\partial g}\right)$$  \hspace{1cm} (28)

But we know (from above) that $\partial \bar{k}/\partial g$ is negative; since $-\bar{k} < 0$, (28) can possibly be positive only if

$$\left(\frac{\partial \bar{\chi}}{\partial \bar{k}}\right) < 0$$  \hspace{1cm} (29)

$$\varepsilon \bar{k}^{\varepsilon - 1} - g < 0$$  \hspace{1cm} (30)

$$\bar{r} < g.$$  \hspace{1cm} (31)

where $\bar{r} = f'(\bar{k}) = \varepsilon \bar{k}^{\varepsilon - 1}$. This is just the dynamic efficiency condition.
In words: We know from above that a higher value of $g$ will decrease the steady-state capital stock per efficiency unit. We know from our analysis in class that the only circumstance in which a decrease in capital per efficiency unit will directly result in an increase in consumption per efficiency unit is if the dynamic efficiency condition fails to hold. So in order for there to be any hope of an increase in $g$ increasing $\bar{\chi}$, the economy must start out as being dynamically inefficient.

However, dynamic inefficiency is not enough - the second term in (28) must be larger than $\bar{k}$ in order to offset the negative effect of faster $g$ on $\bar{\chi}$. The economy must be sufficiently dynamically inefficient that the increase in the raw marginal product of capital that comes from lower $\bar{k}$ more than offsets the capital-dilution effect from the requirement to equip the new efficiency units of labor with capital. In math, faster growth increases consumption per efficiency unit when

$$ \left( \frac{d\bar{\chi}}{dg} \right) > 0 \quad (32) $$

$$ -\bar{k} + (\bar{r} - g) \left( \frac{\partial \bar{k}}{\partial g} \right) > 0 \quad (33) $$

$$ (\bar{r} - g) \left( \frac{\partial \bar{k}}{\partial g} \right) > \bar{k}. \quad (34) $$
Part II

Dynamic Inefficiency and the Great Recession.

Consider a Diamond (1965) OLG economy like the one in the handout OLGModel, with no population growth and no depreciation, but where technological progress causes wage rates to rise by a factor $G$ from one young generation to the next.

1. On a diagram showing the relationship between the sustainable level of consumption per capita $\chi_t = c_{1,t} + c_{2,t}/G$ and the level of capital per capita $k$, indicate the level of $k$ beyond which the economy is in a state of dynamic inefficiency. Explain why the terminology makes sense: That is, explain why points in the region of dynamic inefficiency are “inefficient” in some “dynamic” sense. Is the economy also inefficient in a static (one-period) sense?

Answer:

The figure requested can be found in OLGModel.

The OLG economy is statically efficient; that is, any reallocation of resources within a given period $t$ would fail a test of Pareto efficiency evaluated at date $t$ because the economy is perfectly competitive and satisfies all the other usual conditions for Pareto optimality. However, a reallocation between periods (that is, a dynamic reallocation) could improve Pareto efficiency when evaluated with respect to each generation’s lifetime utility.

2. Consider the following quote from Brad deLong’s web page in 2011:

[T]here is ... an overwhelming case for borrow-and-spend right now. Why? Because the thirty-year Treasury inflation-indexed security rate at 1.62% per year is lower than the expected long-run growth rate of the real economy right now of close to 3% per year. ... If the economy ever gets itself into a situation in which risk-adjusted long-run interest rates are lower than the risk-adjusted expected long-run growth rate of the economy, it is dynamically inefficient—and government should borrow and spend and keep borrowing and spending until at least it drives long-term interest rates up to and above the risk-adjusted expected long-run growth rate.

Discuss the conditions under which deLong’s argument is a proper application of the theory of dynamic inefficiency in the Diamond (1965) model. Specifically, address these questions:

a) Does the interest rate on riskfree long-term Treasury bonds correspond to the interest rate that is used in the theory? Why or why not?

Answer:

One of the drawbacks of the theory as presented in class was that the economy we studied is characterized by only one interest rate,
whereas in the real world there are many different interest rates corresponding to different kinds of investments. The treasury rate is a (nearly) riskfree rate (unless you worry that the government might actually default on its debts). But the interest rate that is relevant in the model is the rate that corresponds to the marginal product of an additional unit of capital. Arguably, the appropriate rate therefore would be some weighted average of the short-term rates associated with a mix of investments that represent the average mixture in the economy as a whole. {However, short-term corporate bond rates for} JY → {However, short-term corporate bond rates reflect primarily the premium required by investors in the presence of default risk, whereas the interest rate on risk-free long-term Treasury bonds reflects the expected underlying average return to capital over the long run. In the context of the model, one period corresponds to a generation, therefore the interest rate on risk-free long-term Treasury bonds seem to be more relevant. }

b) Is deLong right in comparing the real interest rate to the overall growth rate of the economy, rather than (say) per-capita income growth? Defend your answer intuitively and analytically.

Answer:

The mathematical analysis of the question corresponds to the derivations in GenAcctsAndGov, in which the effects of productivity growth and of population growth are essentially the same; this is because either productivity growth or labor force growth “dilutes” the capital stock; maintaining the same ratio of capital to effective labor incurs a penalty of $1 - \Xi G$. A particularly perceptive student might note that in the OLG model the “risk-adjusted expected long run growth rate of the economy” might not correspond exactly to $\Xi G$ because of capital accumulation or decumulation. Only one point was deducted for failure to notice this subtle point – which most economists would say is also inconsequential because over a period as long as 30 years, the contribution of capital accumulation or decumulation to economic growth is not likely to be large in an advanced economy like that of the U.S.

Interest rates are endogenous to saving decisions in the OLG model. For the rest of this question, assume instead that interest rates are determined by the global capital market, and can therefore be taken as exogenous to the U.S. economy.

3. The 30 year real interest rate that deLong quotes is very low compared to previous historical experience. Suppose that the period of low interest rates turns out to apply only for one 30-year generation: The returns $R_{t+1}$ earned between period $t$ and $t + 1$ will be low. Furthermore, suppose that wage growth continues to be $G$ in every period. Call the generation that is old when the shock hits generation
$t-1$ (so that the generation that is young in the shock period is generation $t$ and so on). Use the generational accounting framework outlined in class to figure out which generation or generations are hurt by the crisis, and why.

**Answer:**

The mathematical analysis of the question corresponds to the derivations in GenAcctsAndGov, in which the effects of productivity growth and of population growth are essentially the same; this is because either productivity growth or labor force growth “dilutes” the capital stock; maintaining the same ratio of capital to effective labor incurs a penalty of $1 - \Xi$. A particularly perceptive student might note that in the OLG model the “risk-adjusted expected long run growth rate of the economy” might not correspond exactly to $\Xi$ because of capital accumulation or decumulation. Only one point was deducted for failure to notice this subtle point – which most economists would say is also inconsequential because over a period as long as 30 years, the contribution of capital accumulation or decumulation to economic growth is not likely to be large in an advanced economy like that of the U.S. \rightarrow \{In this scenario, generation $t-1$ is unaffected. Generation $t$ is hurt because their savings would have a low return. Generation $t+1$ is unaffected as generation $t-1$ because the low interest rate does not apply to their savings.\}

4. Now consider a different crisis, in which there is no effect on interest rates, but the pattern of income growth is affected. $G_t = 1 < G$ where $G$ is the normal rate of growth. But between $t$ and $t+1$, the growth rate is $G^2$ so that the level of income of the generation that is young at $t+1$ is the same as was expected before the crisis (this may be realistic: after the Great Depression ended, wage growth was fast enough in the subsequent 3 decades to restore U.S. wage rates to their pre-1929 trend). Describe the effects of this crisis on the generational accounts of the various generations.

**Answer:**

In this scenario, generation $t-1$ is unaffected by the crisis. Generation $t$ suffers because their wage rate is lower than it would have been without the crisis. But wage growth between $t$ and $t+1$ leaves generation $t+1$ in exactly the same circumstances they would have been in had the “wage crisis” not occurred. So generation $t$ is the only one that is affected.

5. Assuming that the lessons learned from the previous exercises are correct, briefly describe implications about how the recent crisis, which affected both wage growth and interest rates, might differently affect different generations.

**Answer:**

In the crisis that we have experienced, wage growth has been low and long-run interest rates have dropped. This means that the generation
that is “young” at $t$ suffers the worst of the crisis, because both the slow wage growth and the low returns on their savings affect them directly. If, after the crisis is over, wages grow fast enough so that the originally anticipated wage level is restored by the time generation $t + 1$ comes on the stage, then the generation that is young at $t$ is the only generation to suffer. Congratulations to you, my students, who are members of generation $t$!
References