A Tractable Model of Precautionary Saving

Consider an economy with a wage rate of $P_t$ that is growing by a constant factor of $G$ every period:

$$P_{t+1} = GP_t. \quad (1)$$

Now consider the optimization problem of a permanently unemployed (state ‘u’) consumer who has a CRRA instantaneous utility function $u(C) = C^{1-\rho}/(1-\rho)$. Assume that the interest factor is constant at the rate $R$. The consumer’s Bellman equation is

$$V_u(X_t) = \max_{\{C_t\}} \{u(C_t) + \beta V_u(X_{t+1})\}, \quad (2)$$

with a DBC that can be written:

$$S_t = X_t - C_t \quad (3)$$

$$K_{t+1} = RS_t \quad (4)$$

$$X_{t+1} = K_{t+1} + \epsilon_{t+1} P_{t+1} \quad (5)$$

$$\epsilon_{t+1} = \epsilon_t = 0, \quad (6)$$

where $\epsilon$ is a variable that indicates what proportion of the aggregate wage is received as income by the unemployed. For simplicity, we are assuming that the unemployed receive zero income, as indicated by (6), which also implies that a consumer who is unemployed in period $t$ will remain unemployed forever after.

We learned earlier (see handout PerfForesightCRRAModel) that the solution to this consumer’s problem is

$$C_t = \left(1 - R^{-1}(R^\hat{\beta})^{1/\rho}\right) K_t \quad (7)$$

$$C_t = \kappa K_t, \quad (8)$$

where the second equation implicitly defines $\kappa = \left(1 - R^{-1}(R^\hat{\beta})^{1/\rho}\right)$.

Now consider the problem of a consumer who is employed (state ‘e’) and therefore earns a full wage $P_t$ (i.e. $\epsilon_t = 1$) as well as capital income, but faces a constant risk $\mu$ of becoming permanently unemployed, thereby losing all of his labor income forever. Thus becoming unemployed is equivalent to making a transition in which this consumer becomes identical to the consumer whose problem was analyzed above. The employed consumer’s utility function is identical to the one for the consumer above; the only differences are that the consumer now maximizes expected utility,

1This handout is a simplification of some results in Carroll (1997) based in the framework proposed in Toche (2000).
and that the budget constraint depends on whether the consumer is employed or not.

\[ V_e(X_t) = \max_{\{C_t\}} u(C_t) + \beta E_t[V(\tilde{X}_{t+1})] \quad (9) \]

such that

\[ \epsilon_{t+1} = \begin{cases} 
1 & \text{with probability } (1 - \mu) \\
0 & \text{with probability } \mu 
\end{cases} \quad (10) \]

and

\[ X_{t+1} = R[X_t - C_t] + P_{t+1}\epsilon_{t+1}, \quad (11) \]

where \( V_e \) indicates that this is the value function for an employed consumer, and \( E_t[V(\tilde{X}_{t+1})] \) is the mathematical expectation of the value function for next period. By choosing a different time preference factor, \( \beta \) rather than \( \hat{\beta} \), we allow for the possibility that the ‘retired/unemployed’ consumers may have a different degree of patience or may face a different (presumably higher) mortality risk.\(^2\)

It turns out that the same methods we have applied to solve the perfect foresight model (FOC wrt \( C \), envelope theorem) can be applied here, with the only change being the need to keep the expectations operator in place:

\[ u'(C_t) = R\beta E_t \left[ V'(\tilde{X}_{t+1}) \right] \quad (12) \]

\[ V'_e(X_t) = R\beta E_t \left[ V'(\tilde{X}_{t+1}) \right] \quad (13) \]

\[ u'(C_t) = R\beta E_t \left[ u'(\tilde{C}_{t+1}) \right] \quad (14) \]

\[ 1 = R\beta E_t \left[ \left( \frac{\tilde{C}_{t+1}}{C_t} \right)^{-\rho} \right]. \quad (15) \]

Now let’s define lower-case variables as the upper-case equivalent divided by the level of permanent wage income: \( c_t = C_t / P_t \) and rewrite the consumption Euler equation as

\[ 1 = R\beta E_t \left[ \left( \frac{c_{t+1} P_{t+1}}{c_t P_t} \right)^{-\rho} \right] \quad (16) \]

\[ = G^{-\rho} R\beta E_t \left[ \left( \frac{\tilde{c}_{t+1}}{c_t} \right)^{-\rho} \right]. \quad (17) \]

Suppose we designate the consumption that will take place in period \( t \) if the consumer is employed as \( c_t^e \) and if unemployed \( c_t^u \). Then the mathematical expectations

\(^2\)The \( \sim \) over the \( X \) indicates that the value of \( X_{t+1} \) is unknown at the time the expectation is being taken, because only one of the two possible transition equations for \( X \) will actually occur, depending on the consumer’s employment status next period.
operator in equation (17) can be written explicitly as

\[ 1 = G^{-\rho} R \beta E_t \left[ \left( \frac{c_{t+1}}{c_t} \right)^{-\rho} \right] \]  

(18)

\[ = G^{-\rho} R \beta \left\{ (1 - \mu) \left( \frac{c_{t+1}^e}{c_t^e} \right)^{-\rho} + \mu \left( \frac{c_{t+1}^u}{c_t^u} \right)^{-\rho} \right\} \]  

(19)

\[ = G^{-\rho} R \beta \left( \frac{c_{t+1}^e}{c_t^e} \right)^{-\rho} \left\{ (1 - \mu) + \mu \left[ \left( \frac{c_{t+1}^u}{c_t^u} \right)^{-\rho} - 1 \right] \right\} \]  

(20)

\[ \left( \frac{c_{t+1}^e}{c_t^e} \right)^{\rho} = G^{-\rho} R \beta \left\{ 1 + \mu \left[ \left( \frac{c_{t+1}^u}{c_t^u} \right)^{\rho} - 1 \right] \right\} \]  

(21)

\[ \left( \frac{c_{t+1}^e}{c_t^e} \right)^{\rho} = G^{-1}(R \beta)^{1/\rho} \left\{ 1 + \mu \left[ \left( \frac{c_{t+1}^u}{c_t^u} \right)^{\rho} - 1 \right] \right\}^{1/\rho} . \]  

(22)

To understand this expression, we will temporarily make a rather bad approximation. Define \( \nabla c_{t+1}^e = \left( \frac{c_{t+1}^e - c_{t+1}^u}{c_{t+1}^u} \right) \) (which is basically the percentage by which one expects that consumption would be greater next period if one is employed compared to if one is unemployed). Now notice that if we use the approximation that \( \log[1 + \zeta] = \zeta \) (which is the bad approximation since it holds well only if \( \zeta \) is small and we will use it for large \( \zeta \)) the expression in braces in (23) can be rewritten

\[ \left\{ 1 + \mu \left[ \left( \frac{c_{t+1}^u}{c_t^u} \right)^{\rho} - 1 \right] \right\}^{1/\rho} = \left\{ 1 + \mu \left[ \left( \frac{c_{t+1}^u + c_{t+1}^e - c_{t+1}^u}{c_{t+1}^u} \right)^{\rho} - 1 \right] \right\}^{1/\rho} \]  

(24)

\[ = \left\{ 1 + \mu \left[ (1 + \nabla c_{t+1}^e)^{\rho} - 1 \right] \right\}^{1/\rho} \]  

(25)

\[ \approx \left\{ 1 + \mu \left[ 1 + \rho \nabla c_{t+1}^e - 1 \right] \right\}^{1/\rho} \]  

(26)

\[ = \left\{ 1 + \rho \mu \nabla c_{t+1}^e \right\}^{1/\rho} \]  

(27)

\[ \approx 1 + \rho \mu \nabla c_{t+1}^e . \]  

(28)

which means we can rewrite (23) as

\[ \left( \frac{c_{t+1}^e}{c_t^e} \right)^{\rho} \approx G^{-1}(R \beta)^{1/\rho} \left( 1 + \mu \nabla c_{t+1}^e \right) . \]  

(29)

Now since consumption if employed next period \( c_{t+1}^e \) is surely greater than consumption if unemployed \( c_{t+1}^u, \nabla c_{t+1}^e \) is certainly a positive number. But notice that \( G^{-1}(R \beta)^{1/\rho} \), the first term in (29), is simply the value that \( c_{t+1}^e/c_t^e \) would take in a perfect foresight world. Thus, roughly speaking, uncertainty boosts consumption growth for consumers who remain employed by an amount proportional to the amount by which consumption will fall if they lose their jobs. If there is little fall in
consumption upon job loss, this precautionary term is small; if job loss would cause a large drop in consumption, the precautionary term is large.

Now recall that the PDV of consumption must be equal to the PDV of future income. Faster consumption growth with the same PDV must correspond to a lower initial consumption level. Thus, we have shown that the introduction of a risk of becoming unemployed \( \mu \) reduces the level of consumption but increases expected consumption growth.

In order to do a phase-diagram analysis of this model, we must find the loci that correspond to steady-states for \( c^e \) and \( x^e \). The steady-state for \( c^e \), if one exists, will obviously be at some point where \( c^e_{t+1} = c^e_t \). Substituting this and equation (8) into (23) (note that we have divided both sides of (8) by \( P_t \)), we have

\[
1 = G^{-\rho} R \beta \left\{ 1 + \mu \left[ \left( \frac{c^e_{t+1}}{x^u_{t+1}} \right)^\rho - 1 \right] \right\} \quad (30)
\]

\[
G^\rho (R \beta)^{-1} = 1 - \mu + \mu \left( \frac{c^e}{x^u} \right)^\rho \quad (31)
\]

\[
\frac{\left( G^\rho (R \beta)^{-1} + \mu - 1 \right)}{\mu} = \left( \frac{c^e}{x^u} \right)^\rho \quad (32)
\]

\[
c^e = \kappa x^u \left( \frac{G^\rho (R \beta)^{-1} + \mu - 1}{\mu} \right)^{1/\rho} \quad (33)
\]

\[
c^e = \kappa (x^e - 1) \left( \frac{G^\rho (R \beta)^{-1} + \mu - 1}{\mu} \right)^{1/\rho} \quad (34)
\]

where the last line follows because \( x^u_{t+1} = X^u_{t+1}/P_{t+1} = (X^e_{t+1} - P_{t+1})/P_{t+1} = x^e - 1 \).

Note that in order for this equation to make any sense, the numerator of the fraction must be a positive number; that is, we need the condition:

\[
G^\rho (R \beta)^{-1} + \mu - 1 > 0 \quad (35)
\]

\[
G^\rho > (R \beta)(1 - \mu) \quad (36)
\]

\[
G > (R \beta)^{1/\rho}(1 - \mu)^{1/\rho} \quad (37)
\]

Now recall that the growth rate of consumption in the problem without uncertainty was \((R \beta)^{1/\rho}\) (which is obvious from (37) if we set \( \mu = 0 \)). Recall that when we studied the perfect foresight version of this problem we concluded that the consumer would be ‘impatient’ in the sense of spending too much to keep \( x \) constant if

\[
G > (R \beta)^{1/\rho} \quad (38)
\]

It turns out that in order for this problem to have a steady-state, we need consumers to be impatient, because recall that patient consumers always spend less than the amount needed to keep their wealth-to-permanent-labor-income ratio constant, and thus \( x \) heads to infinity for patient consumers and clearly there will be no steady-state. But if we assume that (38) holds, so that consumers are impatient enough
not to accumulate infinite wealth in the certainty context, then clearly (37) will hold because \((1 - \mu)^{1/\rho} < 1\).

Now we need to determine the \(x_{t+1}^e = x_t^e\) locus (also referred to as the \(\Delta x^e = 0\) locus):

\[
x^e = \frac{R}{G}(x^e - c^e) + 1 \quad (39)
\]

\[
(G/R)(x^e - 1) = x^e - c^e \quad (40)
\]

\[
c^e = x^e - (G/R)(x^e - 1) \quad (41)
\]

\[
= x^e(1 - (G/R)) + (G/R). \quad (42)
\]

The steady-state level of \(x^e\) and \(c^e\) will of course be given by the values of these two variables at which both (42) and (34) hold true. This is just a set of two equations and two unknowns, and with some tedious algebra can be solved explicitly. The appendix shows that if \(\rho = 1\) the steady-state level of \(x^e\) can be (crudely) approximated by

\[
\bar{x}^e \approx 1 + \frac{\mu}{\theta(g + \theta - r + \mu)} \quad (43)
\]

We will always be choosing parameter values such that \(\theta(g + \theta - r + \mu) > 1\).

From this equation we can see that the steady-state or ‘target’ wealth-to-income ratio is increased by 1) an increase in interest rates; 2) a decline in the time preference rate; or 3) a decline in the growth rate. To see the effect of \(\mu\), note that

\[
\frac{d}{d\mu} \left( \frac{\mu}{\theta(g + \theta - r + \mu)} \right) = \frac{\theta(g + \theta - r + \mu) - \mu\theta}{(\theta(g + \theta - r + \mu))^2} \quad (44)
\]

\[
= \frac{\theta(g + \theta - r)}{(\theta(g + \theta - r + \mu))^2}, \quad (45)
\]

and since both numerator and denominator are positive, it is clear that \(\bar{x}^e\) is increasing in the unemployment risk \(\mu\) as well.

The phase diagram for this problem is depicted in figure 1.

Several points in this figure are worth noting. First, the \(\Delta x^e = 0\) locus, given in (42), indicates, for a given level of \(x^e\), how much consumption \(c^e\) would be exactly the right amount to leave exactly the same amount of \(x^e\). Thus, any point below the \(\Delta x^e\) line will constitute consuming less than this amount, so wealth will rise. Conversely for points above \(x^e\).

The intuition for the \(\Delta c^e\) locus (which comes from (34)) is a bit subtler. Recall that expected consumption growth depends on the amount by which consumption will fall if the bad state is realized. For a given level of resources, if actual consumption when employed is less than the break-even amount, then the \(c^e/c^n\) ratio is smaller, and thus consumption growth is smaller. Since \(c^e\) growth was zero along the \(\Delta c^e = 0\) locus, lower than zero means negative \(c^e\) changes. Hence the arrows of motion are downward-pointing below the \(\Delta c^e = 0\) locus and upward-pointing above it.
The next figure shows the optimal consumption function \( c(x) \). This is actually just the lower part of the stable arm in the phase diagram. (Think about why). Also plotted are the 45 degree line along which \( c(x) = x \) and the linearized consumption function that passes through the steady-state.

Note that \( c(x) \) is \textit{concave}. That is, the marginal propensity to consume is higher at low levels of wealth. The reason for this is that the intensity of the precautionary motive increases as resources \( x \) decline, because the consequences of becoming unemployed with a low amount of resources are very painful. The MPC is high at low levels of \( x \) because at low levels of \( x \) the reduction in the intensity of the precautionary motive as \( x \) rises is quite large. The reduction in the precautionary motive translates into an increase in consumption, so at low levels of \( x \) even a modest boost to \( x \) can give a substantial boost to \( c \).

This point is clearest as \( x \) approaches zero. Note that the consumption function always remains below the 45 degree line. The reason is that if the consumer were to spend all his resources in period \( t \), \( c_t = x_t \), then if he were to become unemployed next period he would have \( x_{t+1}^u = (R/G)[x_t - c_t] = 0 \) which would yield negative infinite utility. Thus the consumer will never spend all of his resources - he will always leave at least a little bit for next period in case of disaster.

The next figure illustrates some of the same points in a different way. It depicts

\footnote{Carroll and Kimball (1996) prove that the consumption function must be concave for almost all plausible assumptions about risk and utility functions.}
the growth rate of consumption as a function of $x$. The baseline case is depicted using the solid curves.

Recall our impatience condition (38) and note that it can be approximated by:

$$G > (R\beta)^{1/\rho}$$
$$\log G > \rho^{-1}(\log R - \log \beta)$$
$$g > \rho^{-1}(r - \theta).$$

Now note that in the figure we have $g > \rho^{-1}(r - \theta)$, indicating that the consumers in question are indeed impatient as required.

Now recall (23):

$$\left( \frac{c_{t+1}^e}{c_t^e} \right) = G^{-1}(R\beta)^{1/\rho} \left\{ 1 + \mu \left[ \left( \frac{c_{t+1}^u}{c_t^u} \right)^{\rho} - 1 \right] \right\}^{1/\rho}$$
$$\left( \frac{C_{t+1}^e}{C_t^e} \right) = (R\beta)^{1/\rho} \left\{ 1 + \mu \left[ \left( \frac{c_{t+1}^e}{c_t^e} \right)^{\rho} - 1 \right] \right\}^{1/\rho}$$
$$\Delta \log C_{t+1}^e \approx \rho^{-1}(r - \theta) + \mu \nabla c_{t+1}^e,$$

where the last line uses the same (dubious) approximations used to obtain (29).

Thus consumption growth is equal to what it would be in the absence of uncertainty, plus a precautionary term. Furthermore, it is clear that the precautionary contribution will grow without bound as $x_t \to 0$ because $c_{t+1}^u = \kappa x_{t+1}^u = $
\( \kappa(R/G)(x_t - c(x_t)) \) approaches zero as \( x_t \) does. Sure enough, figure 3 shows that as \( x \) gets low, expected consumption growth gets very large.

Figure 3: Growth Relationships

Next, note that the point where the expected consumption growth figure meets the expected income growth line is labelled \( \bar{x}^e \). This is because the place where consumption growth is exactly equal to income growth is exactly at the target value of \( x^e \).

We are now in a position to get an intuitive understanding of how this model works, and why there is a target wealth ratio. On the one hand, consumers are impatient. This prevents their wealth-to-income ratio from heading off to infinity. On the other hand, these consumers have a precautionary motive that intensifies more and more as the level of wealth gets lower and lower. At some point the precautionary motive gets strong enough so that it exactly counterbalances the consumer’s impatience. This point where impatience matches prudence defines the target wealth-to-income ratio.

Now consider the results of increasing the interest rate to \( \hat{r} > r \). Obviously the perfect foresight consumption growth locus will shift up to \( \rho^{-1}(\hat{r} - \theta) \), as will the expected consumption growth locus. But we have not changed the expected growth rate of income. It is clear from the figure, therefore, that the new target level of cash-on-hand \( \hat{x} \) will be greater than the original target. That is to say, an increase in interest rates increases the target level of wealth, just as would be expected on intuitive grounds.

Now, a crucial insight. Figures 3 and 4 show that, so long as consumers are
impatient, the steady state growth rate of consumption will be equal to the steady-
state growth rate of income,

$$\Delta \log C_{e}^{t+1} \approx g.$$  \hfill (52)

Yet the approximate Euler equation for consumption growth

$$\Delta \log C_{e}^{t+1} \approx \rho^{-1}(r - \theta) + \mu \nabla C_{e}^{t+1},$$

does not contain any term explicitly involving income growth. How can we reconcile
these two expressions for consumption growth? Only by realizing that the size of
the precautionary term $\mu \nabla C_{e}^{t+1}$ is endogenous: it depends on $g$. Indeed, we can solve
(52) and (53) to determine that in steady-state we must have

$$\mu \nabla C_{e}^{t+1}(\bar{x}^e) = g - \rho^{-1}(r - \theta).$$

(54)

We can use this equation to understand the relationship between parameters and
steady-state levels of wealth, by noting that $\nabla C_{e}^{t+1}$ is a downward-sloping function
of $x^e$ (see figure 3) again). The reason is that at low levels of current wealth, much of
the spending of employed consumers is financed by their current income. If they lose
that income, they will have no choice but to cut consumption drastically, implying
a large value of $\nabla C_{e}^{t+1}$.

For example, an increase in the growth rate of income implies that the RHS of
equation (54) increases. To induce a matching increase in the LHS, it is necessary
to choose a lower value of $\bar{x}^e$. This is the human wealth effect in this framework: Consumers who anticipate faster income growth will hold less wealth.

The fact that consumption growth equals income growth in the steady-state poses severe problems for empirical attempts to estimate the Euler equation. To see why, suppose we had a collection of countries indexed by $i$ that are identical in all respects except that they have had different interest rates $r_i$. Then in the spirit of Hall (1988), one might be tempted to estimate an equation:

$$\Delta \log C_i = \alpha_0 + \alpha_1 r_i + \epsilon_i,$$

and to interpret the coefficient estimate on $r_i$ as an indication of the value of $\rho^{-1}$.

But suppose that all of these countries contained impatient consumers and were in their steady-states where $\Delta \log C_i = g_i$. Suppose further, for simplicity (this is not a necessary condition), that all countries had the same steady-state income growth rate. Then the regression equation would return the estimates

$$\alpha_0 = g,$$  \hspace{1cm} (56)
$$\alpha_1 = 0.$$  \hspace{1cm} (57)

The econometric problem here is that there is an omitted variable from the regression specification, the $\mu \nabla C^e_{i+1}$ term, which is (perfectly) correlated with the included variable $r$. Thus, Euler equation estimation cannot be expected to return an unbiased estimate of $\rho^{-1}$. For much more on this problem, see Carroll (2001).

A final experiment is depicted in figure 5. This depicts the result of decreasing the time preference rate. Note that in the preceding figure analyzing the effect of changing the interest rate, the effects are indistinguishable from a change in the time preference rate, so we already know the path of growth over time: There is an immediate sharp increase in growth, which gradually declines back over time toward its steady-state rate.

Here we depict the effect on the level of consumption (or rather the consumption/income ratio) by showing each successive point in time as a dot on the figure. Starting at time 0 from the steady-state level of consumption, the decrease in the time preference rate (an increase in patience) causes an instantaneous drop in the level of consumption. It is from this lower level of consumption that consumption growth is subsequently faster.

Eventually consumption approaches its new, higher equilibrium ratio to permanent income at a new, higher level of equilibrium $x$. This higher level of consumption is financed in the long run by the higher interest income earned on the higher level of wealth.

Note again that the equilibrium steady-state growth rate of consumption is still equal to the growth rate of income. This means that the higher level of wealth in equilibrium ends up being precisely enough to reduce the precautionary term by an amount that exactly offsets the fact that the $-\rho^{-1} \theta$ term in the Euler equation is now smaller.
Figure 5: Effect of Lower $\theta$ On Consumption Function

\[
\text{Orig SS} \quad \xrightarrow{\quad} \quad \text{New SS}
\]

\[
\text{Orig } c(x) \quad \xrightarrow{\quad} \quad \text{New } c(x)
\]
Appendix

1 The Exact Formula for \( \bar{x}^e \)

The steady-state value of \( x^e \) will be where both (34) and (42) hold:

\[
\kappa (\bar{x}^e - 1) \left( \frac{G^\rho (R\beta)^{-1} + \mu - 1}{\mu} \right)^{1/\rho} = \bar{x}^e (1 - (G/R)) + (G/R) \quad (58)
\]

\[
\bar{x}^e \left( \kappa \left( \frac{G^\rho (R\beta)^{-1} + \mu - 1}{\mu} \right)^{1/\rho} + (G/R - 1) \right) = (G/R) + \kappa \left( \frac{G^\rho (R\beta)^{-1} + \mu - 1}{\mu} \right)^{1/\rho} \quad (59)
\]

\[
\bar{x}^e = \left\{ \begin{array}{l}
(G/R) + \kappa \left( \frac{G^\rho (R\beta)^{-1} + \mu - 1}{\mu} \right)^{1/\rho} \\
\kappa \left( \frac{G^\rho (R\beta)^{-1} + \mu - 1}{\mu} \right)^{1/\rho} + (G/R - 1)
\end{array} \right\}
\]

2 The Approximate Formula for \( \bar{x}^e \)

Note that the \( c^e_{t+1} = c^e_t \) locus defined in (23) can be approximated by:

\[
c^e \approx \kappa (x^e - 1) \left( \frac{(1 + \rho g)(1 + \theta - r) + \mu - 1}{\mu} \right)^{1/\rho} \quad (60)
\]

\[
= \kappa (x^e - 1) \left( \frac{\rho g + \theta - r + \mu}{\mu} \right)^{1/\rho} \quad (61)
\]

and that the \( x^e_{t+1} = x^e_t \) locus can be approximated by:

\[
\bar{c}^e \approx \bar{x}^e (r - g) + 1 + g - r \quad (62)
\]

Combining the approximations to find \( \bar{x}^e \) gives us:

\[
(r - g)\bar{x}^e + 1 + g - r \approx \kappa (\bar{x}^e - 1) \left( \frac{\rho g + \theta - r + \mu}{\mu} \right)^{1/\rho} \quad (63)
\]

\[
\bar{x}^e (r - g - \kappa \left( \frac{\rho g + \theta - r + \mu}{\mu} \right)^{1/\rho}) = - \left[ 1 + g - r + \kappa \left( \frac{\rho g + \theta - r + \mu}{\mu} \right)^{1/\rho} \right]
\]

\[
\bar{x}^e \approx \frac{- \left[ 1 + g - r + \kappa \left( \frac{\rho g + \theta - r + \mu}{\mu} \right)^{1/\rho} \right]}{(r - g - \kappa \left( \frac{\rho g + \theta - r + \mu}{\mu} \right)^{1/\rho})} \quad (64)
\]
Now simplify the last line to

\[
\bar{x}^e \approx \frac{1 + g - r + \kappa \left( \frac{g+\theta-r+\mu}{\mu} \right)^{1/\rho}}{\kappa \left( \frac{g-r+\theta+\mu}{\mu} \right)^{1/\rho} - (r - g)}
\]

\[
= \frac{(1 + g - r) \left[ \kappa \left( \frac{g+\theta-r+\mu}{\mu} \right)^{1/\rho} \right]^{-1} + 1}{1 - (r - g) \left[ \kappa \left( \frac{g+r+\theta+\mu}{\mu} \right)^{1/\rho} \right]^{-1}} \quad (65)
\]

\[
\approx 1 + \left( \frac{1 + g - r}{\kappa} \right) \left( \frac{g + \theta - r + \mu}{\mu} \right)^{-1/\rho} + \left( \frac{r - g}{\kappa} \right) \left( \frac{g + \theta - r + \mu}{\mu} \right)^{-1/\rho} \quad (66)
\]

\[
\approx 1 + \left( \frac{1}{\kappa} \right) \left( \frac{\mu}{g + \theta - r + \mu} \right)^{1/\rho} \quad (67)
\]

where we get from (65) to (66) by dividing numerator and denominator by the first term in the denominator, and from (66) to (67) by using the fact that \((1+\epsilon)/(1-\zeta) \approx 1 + \epsilon + \zeta\) if \(\epsilon\) and \(\zeta\) are 'small'.

Now choose \(\rho = 1\) which implies \(\kappa = \theta\) and simplify the last line to

\[
\bar{x}^e \approx 1 + \left( \frac{\mu}{\theta(g + \theta - r + \mu)} \right) \quad (69)
\]

### 3 Numerical Solution

To solve the model by reverse shooting, we need \(c_t^e\) as a function of \(c_{t+1}^e\). Starting with (22):

\[
\left( \frac{c_{t+1}^e}{c_t^e} \right) = G^{-1}(R\beta)^{1/\rho} \left\{ 1 + \mu \left[ \left( \frac{c_{t+1}^e}{c_{t+1}^e} \right)^{\rho} - 1 \right] \right\}^{1/\rho} \quad (70)
\]

\[
c_t^e = \left( G^{-1}(R\beta)^{1/\rho} \left\{ 1 + \mu \left[ \left( \frac{c_{t+1}^e}{\kappa(x_{t+1}^e - 1)} \right)^{\rho} - 1 \right] \right\}^{1/\rho} \right) \quad (71)
\]

\[
= c_{t+1}^e \left( G(R\beta)^{-1/\rho} \left\{ 1 + \mu \left[ \left( \frac{c_{t+1}^e}{\kappa(x_{t+1}^e - 1)} \right)^{\rho} - 1 \right] \right\}^{-1/\rho} \right) \quad (72)
\]

The reverse shooting equation for \(x_t^e\) is

\[
x_{t+1}^e = (R/G)(x_t^e - c_t^e) + 1 \quad (73)
\]

\[
x_t^e = (G/R)(x_{t+1}^e - 1) + c_t^e \quad (74)
\]
References


