Solving The Perfect Foresight CRRA Consumption Model

Consider the optimal consumption problem of a consumer with a constant relative risk aversion instantaneous utility function \( u(C) = C^{1-\rho}/(1-\rho) \). Assume that the interest rate is constant at the rate \( r \). The consumer’s optimization problem is

\[
\max_{s=t} \sum_{s=t}^{T} \beta^{s-t} u(C_s)
\]

subject to the constraints

\[
K_{t+1} = R[X_t - C_t]
\]
\[
X_{t+1} = K_{t+1} + P_{t+1}
\]

where \( P_{t+1} \) is the consumer’s ‘permanent labor income,’ which we will assume is growing steadily by a factor \( G \) from period to period:

\[
P_{t+1}/P_t = G.
\]

Bellman’s equation for this problem is

\[
V_t(X_t) = \max_{\{C_t\}} \left\{ u(C_t) + \beta V_{t+1}(R[X_t - C_t] + P_{t+1}) \right\}.
\]

The first order condition is

\[
u'(C_t) = R\beta V'_{t+1}(X_{t+1}),
\]

and the Envelope theorem tells us that

\[
V'_t(X_t) = R\beta V'_{t+1}(X_{t+1}).
\]

But note that the RHS’s of (6) and (7) are identical, implying that

\[
V'_t(X_t) = u'(C_t)
\]

and similar logic tells us that \( V'_{t+1}(X_{t+1}) = u'(C_{t+1}) \), which (substituting \( u \) for \( V \) in (7)) gives us the Euler equation for consumption

\[
u'(C_t) = R\beta u'(C_{t+1})
\]

\[
1 = R\beta \left( \frac{C_{t+1}}{C_t} \right)^{-\rho}
\]

\[
\left( \frac{C_{t+1}}{C_t} \right) = (R\beta)^{1/\rho}
\]

Thus, consumption growth is constant at growth factor \( (R\beta)^{1/\rho} \).
Now we need a couple of facts:

**Fact 1** \( \sum_{i=0}^{T} \gamma^i = \left( \frac{1 - \gamma^{T+1}}{1 - \gamma} \right) \)

**Fact 2** If \( 0 < \gamma < 1 \), then \( \sum_{i=0}^{\infty} \gamma^i = \left( \frac{1}{1 - \gamma} \right) \)

The Intertemporal Budget Constraint tells us that the present discounted value of consumption must be equal to the PDV of total resources:

\[
PDV_t(C) = K_t + PDV_t(P). \tag{12}
\]

Using Fact 1, the PDV of labor income (also called ‘human wealth’ \( H_t \)) is

\[
H_t = PDV_t(P) = \sum_{s=t}^{T} R^{-(s-t)} P_s \tag{13}
\]

\[
= P_t \sum_{s=t}^{T} R^{-(s-t)} G^{(s-t)} \tag{14}
\]

\[
= P_t \sum_{s=t}^{T} (G/R)^{(s-t)} \tag{15}
\]

\[
= P_t \left( \frac{1 - (G/R)^{T-t+1}}{1 - (G/R)} \right) \tag{16}
\]

while the PDV of consumption is

\[
PDV_t(C) = \sum_{s=t}^{T} R^{-(s-t)} C_s \tag{17}
\]

\[
= \sum_{s=t}^{T} R^{-(s-t)} C_t ((R\beta)^{1/\rho})^{s-t} \tag{18}
\]

\[
= C_t \sum_{s=t}^{T} [R^{-1}(R\beta)^{1/\rho}]^{s-t} \tag{19}
\]

\[
= C_t \left( \frac{1 - [R^{-1}(R\beta)^{1/\rho}]^{T-t+1}}{1 - [R^{-1}(R\beta)^{1/\rho}]} \right) \tag{20}
\]

Therefore we can solve the model by combining (20) and (16) using (12):

\[
C_t = \left( \frac{1 - [R^{-1}(R\beta)^{1/\rho}]^{T-t+1}}{1 - [R^{-1}(R\beta)^{1/\rho}]^{T+1}} \right) \left[ K_t + P_t \left( \frac{1 - (G/R)^{T+1}}{1 - (G/R)} \right) \right]. \tag{21}
\]

Now recall that in the infinite-horizon case \( (T = \infty) \), Fact 2 requires that for human wealth to be well-defined we need the condition

\[
G/R < 1 \tag{22}
\]

\[
G < R. \tag{23}
\]
Why is this? Because if income will grow faster than the interest rate forever, then the PDV of future income is infinite and the problem has no well-defined solution.

Similarly, in order for the PDV of consumption to be finite we must impose:

\[ R^{-1}(R\beta)^{1/\rho} < 1 \]  
\[ (R\beta)^{1/\rho} < R. \]

What this says is that the growth rate of consumption must be less than the interest rate in order for the model to have a well-defined solution. Otherwise, the PDV of future consumption is infinite, and the model does not have a well-defined solution. Note that this amounts to a requirement that there be at least a certain degree of ‘impatience.’

If these conditions do hold, then the model has a well-defined infinite horizon solution, as can be seen by realizing that if \((G/R) < 1\) then \(\lim_{T \to \infty} (G/R)^{T-t+1} = 0\) and if \(R^{-1}(R\beta)^{1/\rho} < 1\) then \(\lim_{T \to \infty} (R^{-1}(R\beta)^{1/\rho})^{T-t+1} = 0\). Substituting these zeros into (21) yields

\[ C_t = \left(1 - R^{-1}(R\beta)^{1/\rho}\right) \left[K_t + \left(\frac{P_t}{1-(G/R)}\right)\right] \]  
\[ = \left(1 - R^{-1}(R\beta)^{1/\rho}\right) \left[K_t + H_t\right] \]  
\[ = \left(\frac{R - (R\beta)^{1/\rho}}{R}\right) W_t \]

where \(W_t\) is the consumer’s ‘total wealth,’ the sum of human and nonhuman wealth.

Now consider the question ‘What is the level of \(C_t\) that will leave total wealth intact, allowing the same value of consumption in period \(t+1\) and forever after?’

The intuitive answer is that if one wants to leave one’s wealth intact, that is possible only if spending is exactly equal to the dividend and interest earnings on one’s total wealth.

Because human wealth is exactly like any other kind of wealth in this framework, it is possible to work directly with the level of total wealth \(W\). Suppose we assume the consumer will spend fraction \(\kappa\) of total wealth in each period, and we want to find the \(\kappa\) that leaves wealth intact.

\[ W_{t+1} = R[W_t - C_t] \]  
\[ \bar{W} = R[ar{W} - \kappa\bar{W}] \]  
\[ 1 = R(1 - \kappa) \]  
\[ 1/R = (1 - \kappa) \]  
\[ \kappa = 1 - 1/R \]  
\[ = \left(\frac{R - 1}{R}\right) \]  
\[ = r/R \]

Thus, the consumer can spend only the interest earnings \(r\) on their wealth, divided by the gross return \(R\). (The division occurs because we assume that...
interest is earned between periods rather than within periods; the right intuition is that if you want to preserve your wealth, you can only spend the interest on it and none of the principal.

Note that the coefficient multiplying total wealth in (28) is also divided by \( R \).

Thus, whether the consumer is spending more than his total income, exactly his total income, or less than his total income depends upon whether the numerator in (28) is greater than, equal to, or less than \( r \). If we call a consumer who is spending more than his income ‘impatient,’ the consumer will be impatient if

\[
R - (R\beta)^{1/\rho} > r \\
1 - (R\beta)^{1/\rho} > 0 \\
1 > (R\beta)^{1/\rho}.
\]

Now note that if \( R\beta = 1 \) (which is to say, the interest rate is exactly equal to the time preference rate so that they offset each other), then \( (R\beta)^{1/\rho} = 1 \) regardless of the value of \( \rho \) so that the consumer is precisely poised on the balance between patience and impatience and exactly spends his income.

The consumer will be impatient, spending more than his income, if \( R\beta < 1 \), and patient, spending less than his income, if \( R\beta > 1 \).

Equation (26) can be simplified into something a bit easier to handle by making some approximations. If \( \beta = 1/(1 + \theta) \), then we can use

**Fact 3** \( \log(1 + \epsilon) \approx \epsilon \)

and its inverse

**Fact 4** \( \exp(\epsilon) \approx 1 + \epsilon \)

to discover that

\[
\log(R\beta)^{1/\rho}/R \approx (1/\rho)(\log R + \log[1/(1 + \theta)]) - \log R = (1/\rho)(\log(1 + r) + \log 1 - \log(1 + \theta)) - \log R \approx \rho^{-1}(r - \theta) - r \\
(R\beta)^{1/\rho}/R \approx 1 + (\rho^{-1}(r - \theta) - r)
\]

Substituting this into (27) gives

\[
C_t \approx (r - \rho^{-1}(r - \theta)) W_t
\]

Now we can see again that whether the consumer is patient or impatient depends on the relationship between \( r \) and \( \theta \). Note also that if \( \rho = \infty \) then the consumer is infinitely averse to changing the level of consumption, and so once again the consumer spends exactly his income.

Now a brief note on what ‘income’ means in this model. Suppose for simplicity that the consumer had no capital assets \( K \), and suppose that income was expected to stay constant at level \( \bar{Y} \) forever. In this case human wealth would
be:

\[ H_t = \bar{Y} + \frac{\bar{Y}}{R} + \frac{\bar{Y}}{R^2} + \ldots \]  
(44)

\[ = \bar{Y}(1 + 1/R + 1/R^2 + \ldots) \]  
(45)

\[ = \bar{Y} \left( \frac{1}{1 - 1/R} \right) \]  
(46)

\[ = \bar{Y} \left( \frac{R}{R - 1} \right) \]  
(47)

\[ = \bar{Y} \left( \frac{R}{r} \right) \]  
(48)

Now recall that we found in equation (35) that the level of consumption that leaves ‘wealth’ \( W_t \) intact was

\[ C_t = \kappa W_t \]  
(49)

\[ = \kappa[K_t + H_t] \]  
(50)

\[ = \kappa\bar{Y} \left( \frac{R}{r} \right) \]  
(51)

\[ = \left( \frac{r}{R} \right) \bar{Y} \left( \frac{R}{r} \right) \]  
(52)

\[ = \bar{Y}. \]  
(53)

So in this case, spending the ‘interest income on human wealth’ corresponds to spending exactly your labor income. This seems less mysterious if you think of income \( Y_t \) as the ‘return’ on your human capital asset \( H_t \). If you ‘capitalize’ your stream of income at rate \( R \) and then spend the interest income on the capitalized stream, it stands to reason that you are spending the flow of income from that source.

Note also that in this case we can rewrite (43) as

\[ C_t \approx (r - \rho^{-1}(r - \theta)) \left[ K_t + \bar{Y} \left( \frac{R}{r} \right) \right]. \]  
(54)

Note that \( r \) appears three times in this equation, which correspond (in order) to the income effect, the substitution effect, and the human wealth effect. To see this, note that an increase in the first \( r \) basically corresponds to an increase in the payout rate on total wealth (to see this, set \( \bar{Y} = 0 \) and refer to our formula above for \( \kappa \), realizing that for small \( r, r/R \approx r \).) The second term corresponds to the substitution effect, as can be seen from its dependence on the intertemporal elasticity of substitution \( \rho^{-1} \). Finally, the \( \bar{Y}/r \) term clearly corresponds to human wealth, and therefore the sensitivity of consumption to \( r \) coming through this term corresponds to the human wealth effect.

**Renormalizing by \( P \)**

Finally, note that we can restate the whole problem by ‘dividing through’ by the level of permanent income before solving. That is, define small-letter
variables as the capital-letter equivalent, divided by the level of permanent labor income in the corresponding period, e.g. \( c_t = C_t/P_t \), and note that if \( P_{s+1} = GP_s \) \( \forall s \) then from the standpoint of time \( t \) we have that

\[
u(C_s) = \frac{C_s^{1-\rho}}{1-\rho} \quad (55)
\]

\[
= \frac{(c_s P_s)^{1-\rho}}{1-\rho} \quad (56)
\]

\[
= (P_t G^{s-t})^{1-\rho} \frac{c_s^{1-\rho}}{1-\rho} \quad (57)
\]

which means that

\[
\sum_{s=t}^{T} \beta^{(s-t)} \frac{C_s^{1-\rho}}{1-\rho} = P_t^{1-\rho} \sum_{s=t}^{T} (G^{1-\rho} \beta^{(s-t)}) \frac{c_s^{1-\rho}}{1-\rho} \quad (58)
\]

Furthermore, the accumulation equations can be rewritten by dividing both sides by \( P_{t+1} \):

\[
K_{t+1}/P_{t+1} = \frac{R[X_t - C_t]}{P_{t+1}} \quad (59)
\]

\[
k_{t+1} = \left( \frac{R[X_t - C_t]}{P_t} \right) \left( \frac{P_t}{P_{t+1}} \right) \quad (60)
\]

\[
= (R/G)[x_t - c_t] \quad (61)
\]

\[
X_{t+1} = K_{t+1} + P_{t+1} \quad (62)
\]

\[
x_{t+1} = k_{t+1} + 1. \quad (63)
\]

Now if we define \( \hat{\beta} = G^{1-\rho} \beta \) and \( \hat{R} = R/G \), and \( \hat{G} = 1 \), it is clear that the original problem can be rewritten as:

\[
\max P_t \sum_{s=t}^{T} \hat{\beta}^{s-t} u(c_s) \quad (64)
\]

subject to the constraints

\[
k_{t+1} = \hat{R}[x_t - c_t] \quad (65)
\]

\[
x_{t+1} = k_{t+1} + 1 \quad (66)
\]

and we can go through the same steps as above to find that the solution is

\[
c_t = (1 - \hat{R}^{-1}(\hat{R}\hat{\beta})^{1/\rho}) \left[ k_t + \left( \frac{1}{1 - \hat{G}/\hat{R}} \right) \right] \quad (67)
\]

subject to the ‘finite human wealth’ condition

\[
\hat{G} < \hat{R} \quad (68)
\]

\[
1 < R/G \quad (69)
\]
which is the same condition as above (23), and the ‘impatience condition’

\[
(\hat{R}\hat{\beta})^{1/\rho} < \hat{R} \tag{70}
\]

\[
\left(\frac{R}{G}\beta G^{1-\rho}\right)^{1/\rho} < \frac{R}{G} \tag{71}
\]

\[
(R\beta)^{1/\rho} < R \tag{72}
\]

which is the same as (25).

Now note that (67) can be rewritten

\[
c_t = \left(\frac{\hat{R} - (\hat{R}\hat{\beta})^{1/\rho}}{\hat{R}}\right)w_t \tag{73}
\]

where \(w_t\) is the consumer’s total wealth-to-permanent-labor-income ratio, human and nonhuman.

As before, whether \(w\) is rising or falling depends upon the relationship between \(\hat{r}\) and \(\hat{R} - (\hat{R}\hat{\beta})^{1/\rho}\). If we call a consumer who is drawing down his wealth-to-income ratio ‘impatient,’ the consumer will be impatient if

\[
R - (R\beta)^{1/\rho} > r \tag{74}
\]

\[
1 - (R\beta)^{1/\rho} > 0 \tag{75}
\]

\[
1 > (\hat{\beta})^{1/\rho} \tag{76}
\]

Now substituting the definitions of \(\hat{R}\) and \(\hat{\beta}\) we see that whether \(w\) is rising or falling depends on whether

\[
1 > \left(\frac{R}{G}\beta G^{1-\rho}\right)^{1/\rho} \tag{77}
\]

\[
1 > (R\beta G^{-\rho})^{1/\rho} \tag{78}
\]

\[
1 > G^{-1}(R\beta)^{1/\rho} \tag{79}
\]

\[
G > (R\beta)^{1/\rho} \tag{80}
\]

Thus, whether the consumer is patient or impatient in the sense of building up or drawing down a wealth-to-income ratio depends on whether the growth rate of labor income is less than, equal to, or greater than the growth rate of consumption.

To get the intuition for this, consider the case of a consumer with no nonhuman wealth, \(k_t = 0\). This consumer’s absolute level of consumption will grow at \((R\beta)^{1/\rho}\) and absolute level of income at \(G\), but the PDV of future consumption and future income are equal. Thus, if income is growing faster than consumption but has the same PDV, consumption must be starting out at a level higher than income - which is to say, if the impatience condition holds, the consumer has a high level of consumption but slow consumption growth.

A final point about the human wealth effect. For simplicity, assume that \(k_t = 0\). Then the original version of the formula informs us that the level of consumption will be given by:

\[
C_t \approx \left[r - \rho^{-1}(r - \theta)\right] \left(\frac{\bar{Y}}{1 - R/G}\right) \tag{81}
\]

\[
\approx \left[r - \rho^{-1}(r - \theta)\right] \left(\frac{\bar{Y}}{r - g}\right). \tag{82}
\]
Now suppose we choose plausible values for \((r, \theta, g, \rho) = (0.04, 0.04, 0.02, 2)\). Then (82) becomes:

\[
C_t \approx 0.04(\bar{Y}/0.02) = 2\bar{Y}.\tag{83}
\]

Now suppose the interest rate changes to \(r = 0.03\), while all other parameters remain the same. Then (82) becomes:

\[
C_t \approx 0.025(\bar{Y}/0.01) = 2.5\bar{Y}.\tag{85}
\]

The point of this analysis is that for plausible parameter values, the human wealth effect is enormously stronger than the income and substitution effects, so that we should see large drops in consumption when interest rates rise and conversely strong gains when interest rates fall. This is a summary of the main point of the paper by Summers (1981) on your syllabus; Summers derives formulas for an economy with overlapping generations of finite-lifetime consumers, but those complications do not change the basic message.

**References**