The Two-Period Optimal Consumption Problem

This handout summarizes the solution to an optimal consumption problem for a consumer who lives for two periods.

Written in its most general form, the household’s utility function is

\[ V(c_1, c_2) \]

and we assume that the derivatives with respect to the first and second arguments are positive,

\[ V_1, V_2 > 0, \]

and the second derivatives are negative,

\[ V_{11}, V_{22} < 0. \]

The consumer begins the first period with assets of \( k_1 \) and income of \( y_1 \). Total resources are divided between consumption and saving:

\[ k_1 + y_1 = c_1 + s_1, \]
\[ s_1 = k_1 + y_1 - c_1. \]

Assets at the beginning of period 2 are equal to end-of-first-period savings \( s_1 \), accumulated by a gross interest factor \( R = (1 + r) \):

\[ k_2 = Rs_1. \]

This is the dynamic budget constraint or DBC for this problem. The other kind of budget constraint that must be satisfied is over the course of the entire lifetime; because it in principle encompasses a long span of time (rather than just the gap connecting two periods), we call this the intertemporal budget constraint (IBC):

\[ c_1 + c_2/R \leq y_1 + y_2/R + k_1. \]

For various purposes in this class, it will be important to keep track of human wealth \( h_t \), which is defined as the present discounted value of future labor income,

\[ h_t = PDV_t(y) \]
\[ h_1 = y_1 + y_2/R. \]

Because we have assumed (in (1)) that an additional unit of consumption always yields a positive amount of extra utility, we can reach our first conclusion (as opposed to assumption) in the model: The IBC will hold with equality, because if it did not utility could be increased by increasing consumption in one or both periods. Thus, the IBC can be rewritten as

\[ c_1 + c_2/R = h_1 + k_1. \]
The general form that the IBC will take in all our uses of it will be that the present discounted value of lifetime spending must equal the present discounted value of lifetime resources:

\[ PDV_t(c) = PDV_t(y) + k_t. \] (10)

We know that once the consumer has reached the second period of life, he will consume all available resources:

\[ c_2 = k_2 + y_2. \] (11)

Substituting in the definition of \( k_2 \) means that our problem can now be stated as:

\[
\max_{\{c_1,c_2\}} V(c_1,c_2) \tag{12}
\]

s.t.

\[ c_2 = R[k_1 + y_1 - c_1] + y_2. \] (13)

Now we can write the problem as a Kuhn-Tucker multiplier problem, where the maximand is:

\[ V(c_1,c_2) + \lambda(c_2 - R[k_1 + y_1 - c_1] + y_2). \] (14)

The first order conditions are:

\[ V_1 + R\lambda = 0 \] (15)
\[ V_2 = -\lambda \] (16)

and substituting (16) into (15) we get

\[ V_1 = RV_2. \] (17)

This is the same condition you get when deciding between two commodities at a point in time, where we can now think of \( R \) as the intertemporal price (how much of good 2 do I get in exchange for giving up a unit of good 1).

Now suppose that the consumer’s utility is time-separable, and the felicity function (felicity is the utility obtained in a single period of a multi-period problem) is the same in both periods of life, so that

\[ V(c_1,c_2) = u(c_1) + \beta u(c_2) \] (18)

where \( \beta \) is a time preference factor used to determine how the consumer trades off utility in period 1 against utility in period 2.

From our assumptions (1) and (2) we know that the felicity function must satisfy

\[ u^\prime(.) > 0 \] (19)
\[ u^\prime\prime(.) < 0, \] (20)
and since the felicity functions are the same in both periods we have that
\[ V_1(c_1, c_2) = u'(c_1) \quad (21) \]
\[ V_2(c_1, c_2) = \beta u'(c_2). \quad (22) \]

Substituting these equations into (17) yields the Euler equation for consumption:
\[ u'(c_1) = R\beta u'(c_2). \quad (23) \]

The Euler equation is a central result in consumption theory, and will be used over and over again as the course progresses. It is therefore worth studying carefully to be sure you understand it thoroughly.

To help obtain the intuition for why the Euler equation is necessary for optimality, consider the following thought experiment. Designate \( c^*_1 \) and \( c^*_2 \) as the optimal levels of consumption in this problem, the levels that solve the maximization problem. Thus, the highest attainable utility is
\[ u(c^*_1) + \beta u(c^*_2). \quad (24) \]

Now consider reducing consumption by some small amount \( \epsilon \) in period 1, investing that \( \epsilon \) so that it grows to \( R\epsilon \) in period 2, and then consuming it in period 2. What happens to utility?

Taking first-order Taylor expansions, the levels of first-period and second-period utility are now
\[ u(c_1^* - \epsilon) \approx u(c_1^*) - u'(c_1^*)\epsilon \quad (25) \]
\[ u(c_2^* + R\epsilon) \approx u(c_2^*) + u'(c_2^*)R\epsilon. \quad (26) \]

Now the difference between the maximum possible utility and the new situation is given by
\[ u(c_1^*) + \beta u(c_2^*) - [u(c_1^*) - u'(c_1^*)\epsilon + \beta (u(c_2^*) + u'(c_2^*)R\epsilon)] \approx u'(c_1^*)\epsilon - \beta u'(c_2^*)R\epsilon \quad (27) \]

But it must be the case that (27) is equal to zero. To see why, suppose it were a negative number. That would say that moving from the original situation with \( \{c_1, c_2\} = \{c_1^*, c_2^*\} \) to the new situation with \( \{c_1, c_2\} = \{c_1^* - \epsilon, c_2^* + R\epsilon\} \) resulted in an increase in utility. But we assumed that \( c_1^*, c_2^* \) were already the utility-maximizing choices, which clearly could not be true if adjusting \( c_1^* \) downward by \( \epsilon \) and \( c_2^* \) upward by \( R\epsilon \) increased utility. Similarly, if the expression were positive, then utility could be increased by doing the opposite (i.e. increasing consumption in period 1 by \( \epsilon \) and reducing it in period 2 by \( R\epsilon \)). Thus, in either case if the expression is not zero, we have a contradiction to the assumption that \( c_1^* \) and \( c_2^* \) are the utility-maximizing levels.

To make any further progress on this topic, it is necessary to make more specific assumptions about the structure of the utility function. The most common assumption is that utility takes the Constant Relative Risk Aversion form,
\[ u(c) = \left( \frac{c^{1-\rho}}{1-\rho} \right), \quad (28) \]
with marginal utility

\[ u'(c) = c^{-\rho}. \]  

Consider equation (23) with CRRA utility,

\[ c_1^{-\rho} = R\beta c_2^{-\rho}, \]  

\[ c_2/c_1 = (R\beta)^{1/\rho} \]  

\[ c_2 = (R\beta)^{1/\rho} c_1. \]

Now note that this equation allows us to calculate the intertemporal elasticity of substitution as the change in the ratio of the logs of \( c_2 \) to \( c_1 \) as a function of the log change in the intertemporal price \( R \):

\[ \left( \frac{d}{d \log R} \right) \log \left( \frac{c_2}{c_1} \right) = \left( \frac{d}{d \log R} \right) \log (R\beta)^{1/\rho} \]

\[ = \rho^{-1}. \]  

Next note that from (32) we can calculate the PDV of lifetime consumption from the perspective of the first period of life as

\[ PDV_i(c) = c_1 + R^{-1}c_2 \]

\[ = c_1 \left( 1 + R^{-1}(R\beta)^{1/\rho} \right). \]  

Now we can use the intertemporal budget constraint:

\[ PDV_i(c) = k_i + PDV_i(y) \]

\[ c_1 \left( 1 + R^{-1}(R\beta)^{1/\rho} \right) = k_1 + y_1 + R^{-1}y_2 \]

\[ k_1 = \left( \frac{k_1 + h_1}{1 + R^{-1}(R\beta)^{1/\rho}} \right). \]

Thus, we have solved the two-period life cycle saving problem for the consumption function relating the level of consumption to all of the parameters of the problem.

A common assumption (for simplicity, not realism) is that \( \rho = 1 \), which is equivalent to assuming that the utility function is logarithmic:

\[ \lim_{\rho \to 1} \left( \frac{c^{1-\rho}}{1-\rho} \right) = \log c. \]  

In this case it turns out that we can simply substitute \( \rho = 1 \) into the solution for consumption, obtaining

\[ c_1 = \left( \frac{k_1 + h_1}{1 + \beta} \right). \]