1 The Problem

Consider the problem of an individual who cares about consumption relative to a “habit stock” determined by past consumption, and who takes into account the effect of current consumption on the future habit stock.\footnote{The problem can be thought of in two ways: either the representative household cares directly about its own past consumption, or atomistic households care about how their consumption compares to a lagged average ‘standard of living’ and a social planner takes account of the negative externality that each household’s consumption has on all the other households. A related paper (Carroll, Overland, and Weil (1997)), shows that behavior is qualitatively similar in a model with atomistic households in which the externality is not taken into account by decision-makers (sometimes called a model with ‘external habits’).} The utility function is

$$U(c, h) = \frac{(c/h)^{1-\rho}}{1-\rho}$$

where \(h\) is the stock of habits, \(c\) is the instantaneous flow of consumption, \(\rho\) is the coefficient of relative risk aversion, and \(\gamma\) indexes the importance of habits. If \(\gamma = 0\) then only the absolute level of consumption is important (the standard CRRA model), while if \(\gamma = 1\), then consumption relative to the habit stock is all that matters. For values of \(\gamma\) between zero and one, both the absolute and the relative levels are important. For example, if \(\gamma = .5\), then a person with consumption of 2 and habit stock of 1 would have the same utility as a person with both consumption and habit stock equal to 4. Finally, we assume \(0 \leq \gamma < 1\) and \(\rho > 1\).

The stock of habits evolves according to

$$\dot{h} = \lambda (c - h).$$

Thus, the habit stock is a weighted average of past consumption, with the parameter \(\lambda\) determining the relative weights of consumption at different times. We assume \(0 \leq \lambda\). The larger is \(\lambda\), the more important is consumption in the recent past. If \(\lambda = .1\), for example, then the half-life with which habits would adjust toward a permanent change in \(c\) is approximately 7 years (because \(e^{-\lambda t} = 0.5\) for \(t \approx 6.93\)). If \(\lambda = .3\), then the half-life is a bit over two years.\footnote{Models in which habit formation is used to explain the equity premium rely on a high value of \(\lambda\), so that the habit stock remains close to the current level of consumption. For example in Constantinides (1990), the values of \(\lambda\) considered range as high as .6. Similarly, in Abel (1990), the habit stock is equal to the previous year’s consumption. By contrast, in the growth context examined here, we think that lower values of \(\lambda\) are appropriate, so that transitional dynamics are stretched over a substantial period of time. Our baseline assumption for the numerical exercises below will be \(\lambda = .2\).}
any given instant of time, because the habit stock is effectively fixed at any point in
time. Thus it remains appropriate to call the parameter $\rho$ the coefficient of relative risk
aversion. However, because habits can and do move over finite intervals in response
to consumption choices, it will no longer be true that the intertemporal elasticity of
substitution over time is equal to the inverse of the coefficient of relative risk aversion.

The production function is

$$y = Ak.$$  \hspace{1cm} (3)

We assume that capital depreciates at rate $\delta \geq 0$. The capital stock thus evolves
according to

$$\dot{k} = (A - \delta)k - c.$$  \hspace{1cm} (4)

The individual maximizes a discounted, infinite stream of utility:

$$\int_0^\infty U(c, h)e^{-\theta t}dt,$$  \hspace{1cm} (5)

and the current value Hamiltonian is

$$H = U(c, h) + \psi[(A - \delta)k - c] + \mu \lambda (c - h).$$  \hspace{1cm} (6)

Carroll, Overland, and Weil (1997) present the full solution to this problem with
equations of motion relating consumption, the capital stock, and the habit stock. In
the steady state, $c$, $k$, and $h$ all grow at the same rate. Here we analyze the problem
in terms of three ratios, $c/h$, $\dot{c}/c$, and $k/h$ that are constant in steady state. Dynamics
arise from departures of these “state-like variables” from their steady state values, as

The equations of motion are

$$\begin{bmatrix} \dot{\frac{c}{h}} \end{bmatrix} = \frac{c}{h} \left( \frac{\dot{c}}{c} - \lambda \left( \frac{c}{h} - 1 \right) \right).$$  \hspace{1cm} (7)

$$\begin{bmatrix} \dot{\frac{c}{h}} \end{bmatrix} = \alpha_0 + \alpha_1 \begin{bmatrix} \frac{c}{h} \end{bmatrix}$$

$$+ \alpha_2 \begin{bmatrix} \frac{c}{h} \end{bmatrix} + \alpha_3 \begin{bmatrix} \frac{c}{h} \end{bmatrix}$$

$$+ \alpha_4 \left( \frac{\dot{c}}{c} \right)^2 + \alpha_5 \left( \frac{c}{h} \right)^2$$

$$\begin{bmatrix} \dot{\frac{k}{h}} \end{bmatrix} = \frac{k}{h} \left( A - \delta - \lambda \left( \frac{c}{h} - 1 \right) \right) - \frac{c}{h}. \hspace{1cm} (9)$$

Equation (8) shows the change in the rate of consumption growth as a function
of the level of consumption growth and the ratio of consumption to the habit stock
(the coefficients $\alpha_0 \ldots \alpha_5$ are functions of the taste and technology parameters; see
the appendix (A.1) for the explicit version of the equation of motion for consumption
as a direct function of taste and technology parameters). Note that this differs from the usual Euler condition that emerges from a Ramsey model in that the second time derivative of consumption is involved. In intuitive terms, this result arises because the consumer’s utility is now affected by the growth rate of consumption (through the effect of that growth rate on \(c/h^\gamma\)) as well as the level of consumption, so the temporal evolution of the growth rate must satisfy an optimality condition, just as in the Ramsey model the temporal evolution of the level of consumption must satisfy an optimality condition. Intuitively, habit-forming consumers will desire to smooth consumption growth rates for essentially the same reasons that CRRA consumers desire to smooth levels of consumption.

2 Steady State

Setting the three dynamic equations equal to zero determines the steady state of the model,

\[
\left(\frac{\dot{c}}{c}\right) = \frac{A - \delta - \theta}{\gamma(1 - \rho) + \rho}, \tag{10}
\]

\[
\left(\frac{c}{h}\right) = 1 + \frac{1}{\lambda} \left(\frac{A - \delta - \theta}{\gamma(1 - \rho) + \rho}\right), \tag{11}
\]

\[
= 1 + \frac{1}{\lambda} \left(\frac{\dot{c}}{c}\right), \tag{12}
\]

\[
\frac{k}{h} = \frac{1}{\lambda} \left[\frac{\lambda(\gamma(1 - \rho) + \rho) + (A - \delta - \theta)}{(A - \delta)[(1 - \rho)\gamma + \rho - 1] + \theta}\right]. \tag{13}
\]

Equation (12) indicates that the rate at which the habit stock catches up with consumption, \(\lambda\), affects the steady state ratio of consumption to habit stock in an intuitive way: with a higher \(\lambda\) and thus a faster catchup of the habit stock, the ratio of consumption to the habit stock gets closer to one.

Equation (10) shows the effect of the parameters on the steady state growth rate of consumption, which is also the steady state growth rate of capital, output, and the habit stock. Note that \(\lambda\) does not affect the steady-state growth rate (although we show in our companion paper that the value of \(\lambda\) does affect transitional dynamics). However, the other habit parameter, \(\gamma\), which captures the extent to which consumers care about how consumption compares to habits, has an important effect on the steady-state growth rate. Higher values of \(\gamma\) will lead to a higher growth rate of consumption in the steady state (recall that earlier we assumed that \(\rho > 1\)).

One way to interpret this result is to think of habits as increasing the infinite-horizon value of the intertemporal elasticity of substitution. Indeed, the intertemporal elasticity of substitution in consumption is defined as the response of consumption growth to interest rates.

Since the interest rate in this model is \(A - \delta\), equation (10) implies that the infinite-horizon intertemporal elasticity of substitution in this model is \(1/(\gamma(1 - \rho) + \rho)\), which
for \( \rho > 1 \) and \( 0 < \gamma < 1 \) is strictly greater than the inverse of the coefficient of relative risk aversion \( 1/\rho \). However, if we were to calculate the intertemporal elasticity of substitution with respect to temporary changes in the interest rate, we would discover that as the interval of the temporary change in the interest rate approaches zero, the intertemporal elasticity of substitution approaches \( 1/\rho \).

The reason for the discrepancy between the short-horizon and the long-horizon elasticities is that over a sufficiently short interval the habit stock is effectively fixed, while over a sufficiently long interval the habit stock is effectively perfectly flexible. Intuitively, the gain or loss in utility associated with a given increase or decrease in consumption over a long horizon will be diminished by the associated movement in the habit stock; this reduction in the effective curvature of the utility function constitutes an increase in the effective intertemporal elasticity.

Another way to interpret the consumer’s problem can be seen if we substitute the steady-state relationship between \( c, h, \) and growth into the expression \( c/h^\gamma \) (the object which is raised to the power \( 1 - \rho \) to generate utility). From (12) we know that, designating the steady-state growth rate of the economy as \( g \), in the steady-state

\[
h = \frac{c}{1 + g/\lambda}.
\]

This implies that

\[
ch^{-\gamma} = c\left[\frac{c}{1 + g/\lambda}\right]^{-\gamma} = c^{1-\gamma}(1 + g/\lambda)^{\gamma},
\]

so we can think of the consumer as maximizing the utility from a geometrically weighted average of the level of consumption and (a linear function of) the growth rate of consumption. If the weight on habits is \( \gamma = 0 \) this expression just collapses to \( c \) and the consumer is maximizing utility from the level of consumption; if \( \gamma = 1 \) the consumer is maximizing only the utility which derives from the growth of consumption and the level is unimportant.

We now take up the question of how allowing for habit formation changes the response of the economy to exogenous changes to productivity. We show that allowing for habit formation can substantially change both the quantitative and qualitative response of saving to such events. We consider changes in growth that are due to variation in the parameter \( A \), which measures productivity.

From equation (10)

\[
\frac{dg}{dA} = \frac{1}{\gamma(1 - \rho) + \rho}.
\]

The gross saving rate is

\[
s = \frac{y - c}{y} = \frac{AK - c}{AK}.
\]
In steady-state, income, capital, and consumption all grow at the same rate $g$. With an $AK$ production function with depreciation rate $\delta$, gross saving must be enough to make the capital stock grow at rate $g$ after depreciation:

$$s = \frac{(g + \delta)K}{AK} = \frac{(g + \delta)}{A}. \quad (19)$$

Differentiating this expression with respect to $g$ yields

$$\frac{ds}{dg} = \frac{A - (g + \delta)\frac{dA}{dg}}{A^2}. \quad (20)$$

The sign of $ds/dg$ depends only on the numerator of this expression. Inverting equation (16) and substituting it into the numerator, we obtain the result that $ds/dg$ is positive if

$$A - (g + \delta)(\gamma(1 - \rho) + \rho) > 0.$$  

Finally, using equation (10) to substitute for $g$, we can re-write the condition as

$$A - \left(\frac{A - \delta - \theta}{\gamma(1 - \rho) + \rho} + \delta(\gamma(1 - \rho) + \rho)\right) > 0$$

which reduces to

$$\rho < 1 + \frac{\theta}{\delta(1 - \gamma)}. \quad (21)$$

In the baseline model where habits do not matter ($\gamma = 0$), if $\theta = \delta$, for example, the relation between saving and growth is positive only if the instantaneous coefficient of relative risk aversion, $\rho$, is less than two.\(^3\) Most evidence, however, suggests an instantaneous coefficient of relative risk aversion considerably greater than two.\(^4\) Note that habit formation (a choice of $0 < \gamma < 1$) increases the range of parameter values for which increases in the growth rate of output due to increases in the productivity parameter $A$ are associated with a higher saving rate. For example, if $\theta = \delta$ and $\gamma = .75$, then $\frac{ds}{dg} > 0$ so long as $\rho < 5$.

There are two ways to interpret the fact that habits make the relationship between saving and growth more positive. The first is that this is a consequence of the corresponding increase in the infinite-horizon intertemporal elasticity of substitution: habits make consumers more willing to postpone consumption in response to an increase in

\(^3\)It turns out that this result is not unique to the endogenous growth model. In the Appendix we show that equation (21), with $\gamma$ set to zero, must also hold in the Cass-Koopmans-Ramsey model if that model is to generate a positive steady-state relationship between saving and growth.

\(^4\)Note that the choice of $\theta = \delta$ almost certainly understates the problem for the standard model, because in typical parameterizations $\theta$ is usually assumed to be considerably smaller than $\delta$. For example, if $\theta = .03$ and $\delta = .09$ (relatively conventional choices), then the coefficient of relative risk aversion must be less than $4/3$ in order for the relationship between saving and growth to be positive.
interest rates, and thus make the saving response to $A$ stronger. The second interpretation derives from the earlier observation that introducing habits is like putting growth in the utility function. Increasing the value of $A$ makes it possible for consumers to achieve higher steady-state growth rates. Since habit-forming consumers care directly about the growth rate of consumption, they will take advantage of a higher $A$ partly to boost the steady-state growth rate (by increasing the saving rate). Regardless of the interpretation, it is clear that raising the level of habit formation can qualitatively change the relation between growth and saving in the steady state.

3 Dynamics

In order to examine transition dynamics in our model, we derive policy functions tracing out the relationship between the state variable $k/h$ and the optimal values of the control variable $c/h$. Similarly, we can trace the relationship between $k/h$ and any transformation of the control variable along the optimal path. This amounts to graphing the optimal policy functions relating the state variable to each of the policy variables in question.

Figures 1 and 2 depict policy functions for the main variables of interest for several different values of $\gamma$, the parameter that determines importance of habits in utility, and $\rho$, the coefficient of relative risk aversion. For each value of $\gamma$, the value of $\rho$ is chosen to keep the steady-state growth rate the same. The dots represent equally spaced points in time as the system evolves toward the shared steady state (where the three policy functions intersect). The largest dots correspond to policy functions for the case where habits are weakest, $\gamma = .25$, the medium-sized dots correspond to a medium degree of habit formation, $\gamma = .5$, and the smallest dots correspond to $\gamma = .75$.

The first set of figures shows that an economy that starts out ‘rich,’ in the sense of having a capital-to-habit-stock ratio above the steady state, will initially have both a higher-than-steady-state ratio of consumption to the habit stock and a higher-than-steady-state level of consumption growth. But the second set of figures shows that such a ‘rich’ economy will also have a higher-than-steady-state saving rate, implying a lower-than-steady-state ratio of consumption to capital (and therefore income). Thus, compared to the steady-state ratios, consumption is high relative to habits but low relative to income.

The intuition for this pattern is simple: habits tend to pull consumption toward the level of the habit stock and away from the steady-state ratio of consumption to income. If habits are low relative to capital ($k/h$ is high), then consumption will be low relative to capital (the saving rate will be above the steady-state level). Another way to put this is that some of the economy’s good fortune is taken advantage of via a high level of consumption relative to habits, but growth-loving consumers also use

---

As can be seen in equations (10) and (13), the steady-state values of all of the “state like” variables depend on $\gamma$ and $\rho$ only through the term $\gamma(1 - \rho) + \rho$. We examine pairs of $\gamma$ and $\rho$ that hold the value of this term at 3, which is the value consistent with a coefficient of relative risk aversion of 3 if $\gamma$ were equal to zero.
some of their good fortune to achieve an extended period of above-steady-state growth by saving more than the steady-state amount.

Comparing policy functions for different values of $\gamma$, several points stand out. First, for a given level of $k/h$ which is above its steady state level, an economy in which the influence of habits on consumption is stronger will have a lower level of consumption (higher saving rate), and higher growth rates of consumption and output. This is because when habits matter more, the pull on consumption toward the habit stock is stronger. It is also clear from the spacing of the dots representing points in time that an economy with a lower degree of habit formation moves more rapidly toward the steady state for any given initial value of $k/h$. This is unsurprising because we know that in an economy with no habit formation at all there is no pull of $c$ toward $h$; thus the gap between $c$ and $h$ is larger and so $h$ will adapt to $c$ faster.

3.1 The Dynamic Response to an Unanticipated Drop in Capital

To examine how habit formation affects co-movements of saving and growth, we consider the following experiment. The economy is in steady state, with $A$ chosen such that the growth rate of output is 2 percent per year. In year 0, 10 percent of the capital stock is destroyed. Figure 3 shows the evolution of output growth and the saving rate following the shock. As shown above, so long as $A$ remains unchanged, the steady state saving rate and growth rates will also be unaffected. However, the degree of habit formation importantly affects the transitional dynamics.

We consider pairs of values of $\gamma$ and $\rho$ which hold the long run saving rate and growth rates constant. These are the same values used in Figures 1 and 2: $(\gamma = .25, \rho = \frac{11}{3})$, $(\gamma = .5, \rho = 5)$, and $(\gamma = .75, \rho = 9)$.

For a high value of $\gamma$, the immediate effect of the destruction of the capital stock is to greatly reduce the saving rate. The reduction in saving is the result of the desire to maintain consumption relative to the existing habit stock. The low level of saving, in turn, reduces the growth rate of output. Both saving and growth return only slowly to their steady state levels. Not only is the effect on saving larger when $\gamma$ is larger, but it is also more persistent. When $\gamma = .75$, saving returns halfway to its steady state level after 11.6 years. For $\gamma = .5$ the half-life of is 6.8 years, and for $\gamma = .25$ it is 5.2 years. In the limit, when $\gamma$ is zero, there is no effect of the drop in capital on either saving or growth. Of course, output will decline by 10 percent when the capital is destroyed. Thus, viewed over a time period that encompasses the drop, growth will fall and saving will be constant if $\gamma = 0$, while both growth and saving will fall if $\gamma > 0$. Note also that the persistent changes in growth observed when $\gamma > 0$ will be reflected in permanent differences in the level of output once countries have reached their steady state growth rates.
3.2 The Dynamic Response to a Change in $A$

Now consider the effect of an unanticipated but permanent change in $A$ such that the steady-state growth rate of the economy moves from 1 percent to 2 percent per year. We choose parameters such that this change in productivity has a negative long-run effect on the saving rate. Figure 4 shows the paths of output growth and the saving rate following the change for economies with different levels of $\gamma$.

The figure shows that allowing for habit formation can make the short run effect of a change in $A$ differ from the long run effect. Even though saving falls in the long run, it remains above its long run level during the transition to the new steady state, and even rises temporarily above its initial level under two of the three assumptions about the strength of habits. Growth also rises above its long-run level during the transition. The more powerful are habits, the larger and longer-lived are these transitional effects.

These experiments complement the discussion above which showed that allowing for habit formation substantially expands the set of parameter values for which increases in $A$ will result in long-run increases in both growth and saving. Here we have shown that an increase in $A$ will result in short-run increases in both growth and saving for an even larger set of parameter values.
References


Figure 1: Policy Functions for $c/h$ and $\dot{c}/c$

Note: The dots represent equally spaced points in time as the system evolves toward the shared steady state. Thus a larger gap between dots indicates that the system has moved a greater distance in a fixed amount of time.
Figure 2: Policy Functions for $s/y$ and $\dot{k}/k$
Figure 3: Dynamics Following Destruction of 10 Percent of $k$
Figure 4: Dynamics Following A Positive Shock to $A$