Distributive Justice in Taxation*

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A classical criterion for apportioning taxes is that all should sacrifice equally in
loss of utility. Suppose that a method of apportioning taxes is continuous and has
the following four properties: (i) the way that taxpayers split a given tax total
depends only on their own taxable incomes; (ii) an increase in the tax total implies
that everyone pays more; (iii) every incremental increase in tax is apportioned
according to taxpayers' current after-tax incomes; (iv) the ordering of taxpayers by
pre-tax income and after-tax income is the same. Then there exists a utility function
relative to which all sacrifice equally. Journal of Economic Literature Classification

1. INTRODUCTION

Distributive justice is concerned with the fair allocation of costs and
benefits among individuals. All too often the problem is one of distributing
losses rather than gains, of meting out sacrifice rather than reward. Con-
sider a bequest in which the heirs are left more than the total amount in
the estate. How should the shortfall be allocated? Suppose that landowners
in a river basin are entitled to withdraw certain quantities of water
annually from the river. On what principle should the water be rationed
among them in a dry year? If an enterprise goes bankrupt, what is the
fairest way for the creditors to absorb the losses?

In practice, the favored solution to many problems of this sort is to
allocate the losses proportionally to the claims or entitlements. Proportion-
ality is a time-honored principle. Aristotle held that it is virtually syn-
onymous with distributive justice.1 But proportional solutions are not the

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1 “What is just ... is what is proportional, and what is unjust is what violates the propor-
tion.” (Aristotle [2], Ethics, Book V.)
only plausible ones. Especially when the amounts in question are large and the claims very unequal, a good case can be made that it is fairer for the larger claimants to give up relatively more than the smaller ones. The reason is that a larger claimant feels the loss to a lesser degree; though he loses more, he can afford to do so because he is still relatively well off.

Nowhere is this "subjective sacrifice" argument more compelling than in the case of taxation, which is perhaps the most familiar example of a distribution problem involving loss.\(^2\) In fact, equal sacrifice is the traditional utilitarian argument in support of progressive (or at least nonregressive) taxation (J. S. Mill [12], Musgrave [13]). The equal sacrifice doctrine has been variously interpreted. Some authors take it to mean that total sacrifice, summed over all individuals, should be minimized. In this case taxes should be distributed so as to equalize the marginal rate of utility of all persons who pay tax (Edgeworth [8]). A more customary interpretation is that all taxpayers should suffer the same absolute loss in utility. One postulates a utility function for income \(U(x)\), where the same function is usually assumed to apply to all persons. Equal absolute sacrifice means that if \(t\) is the tax on income level \(x\), then \(U(x) - U(x - t) = c\) is a constant, that is, \(t = x - U^{-1}[U(x) - c]\) for all \(x > 0\). An equally plausible interpretation of equal sacrifice, however, is that all should suffer the same rate of loss in utility. In this case \(U(x)/U(x - t) = r\) is a constant and \(t = x - U^{-1}[U(x)/r]\) for all \(x > 0\). Depending on which of these criteria one adopts and the specific form of the utility function, various tax formulas can be derived, some of which are progressive and some not (Cohen Stuart [7], Edgeworth [9], Samuelson [16, p. 227]). Similar reasoning can be applied to the resolution of bankruptcy cases and related "entitlement" problems.

Unfortunately, while equal sacrifice is appealing as a general concept of distributive justice, it suffers from several serious difficulties. The precise form of the utility function must be specified. For lack of better information one must usually assume that the same utility function applies to all taxpayers. Then a specific criterion of equal sacrifice must be chosen. Furthermore, the whole analysis presupposes that the utility of income can be measured and that interpersonal comparisons based on it are valid.

In this paper we shall show that it is not necessary to adopt a utilitarian position in order to justify the equal sacrifice approach. In fact it is not necessary to postulate a utility function at all. Instead we shall show that equal sacrifice is a consequence of much more primitive concepts of distributive justice. If the total tax burden increases, does everyone pay more? Is the increase shared in a fair way? Does the distribution of tax fall

\(^2\) Of course, taxpayers also benefit from some of the public services funded by taxes. But since any single individual enjoys essentially the same benefits irrespective of how much tax he or she alone pays, from the individual's standpoint the tax paid can be viewed as a loss.
equitably on each subgroup of the population? Can a person be penalized for earning more income? We shall show that these principles uniquely characterize the class of equal sacrifice methods; that is, they imply the existence of a utility function relative to which all sacrifice equally. Moreover, if the relative distribution of taxes is assumed to be independent of the units in which taxable income is measured, then the tax method must equalize sacrifice relative to one of the standard utility functions from the theory of risk-bearing. Within this family the only method that treats gains and losses symmetrically is the proportional method, which equalizes absolute sacrifice relative to the classical utility function $U(x) = \ln x$.

2. Problems of Loss

We shall be concerned with the following general situation. Let $I$ be a group of individuals, each of whom is characterized by a positive real number $x_i$. In the case of taxation, $x_i$ represents $i$'s taxable income or some other measure of ability to pay. In bankruptcy cases, $x_i$ is the amount owed to creditor $i$. In the abatement of bequests, $x_i$ is the amount bequeathed to $i$. Each group of individuals will be indexed by a finite subset $I$ of the positive integers $N$. The claims or incomes characterizing the individuals in $I$ will be denoted by a strictly positive vector $x \in \mathbb{R}^I_{++}$ indexed by elements of $I$; for each $i \in I$, $x_i > 0$ is the taxable income of $i$ (or the amount of $i$'s claim, etc.). A loss problem is a pair $(x, T)$ where $x \in \mathbb{R}^I_{++}$ and $0 \leq T \leq \sum x_i$ is the amount to be allocated. An allocation method is a function $F$ defined for every loss problem $(x, T)$ on a finite set of individuals $I$, such that $t_i = F_i(x, T)$ is the allocation to $i$, $0 \leq t_i \leq x_i$, and $\sum t_i = T$. (If $I$ is empty the definitions are vacuous.)

Henceforth we shall treat taxation as the generic example of a loss problem, and speak in terms of tax methods, tax problems, and so forth. Two regularity conditions on tax methods will be assumed throughout. The first is that individuals with equal taxable incomes pay equal taxes. A method with this property is said to be symmetric. The second standing assumption is that a method is continuous on every subdomain $\{(x, T) : x \in \mathbb{R}^I_{++}, 0 \leq T \leq \sum x_i\}$.

3. Principles of Distributive Justice

A general principle of distributive justice states that an allocation that is equitable for a group of individuals should be equitable when restricted to each subgroup of individuals. To put it another way, every subgroup of individuals should feel that the way they share the amount allotted to
them is fair. In the context of taxation, the principle says that the way a group splits a given amount of tax should depend only on their own taxable incomes. A tax method \( F \) is consistent [19] if for every finite subset \( I \subseteq N \), every income distribution \( x \in \mathbb{R}^I_{++} \), and tax total \( T, 0 \leq T \leq \sum x_i \),

\[
 t = F(x, T) \Rightarrow \forall J \subseteq I \left[ t_j = F \left( x_j, \sum t_i \right) \right]. \tag{1}
\]

Consistency is a property of a wide variety of allocation methods, including the Nash bargaining solution [10, 11], the nucleolus of cooperative games [17], and many classical methods for apportioning political representation [5]. It even features in a 2500-year-old method from the Babylonian Talmud for solving bankruptcy problems [4]. (Other Talmudic solutions are discussed in [14].)

In the context of taxation, consistency is extremely natural. Indeed it is implicit in the very concept of a tax schedule. Let \( t = f(x) \) be the amount of tax owed as a function of taxable income \( x \). Faced with this schedule, a group of persons with taxable incomes \( x_1, x_2, \ldots, x_n \) would pay the amounts \( t_i = f(x_i) \). So would any subgroup of these individuals. Hence the taxes paid by any subgroup of individuals depends only on their own taxable incomes, which is exactly what consistency requires.

Of course a single tax schedule is not a method, because it only raises one amount of tax for each given distribution of taxable incomes. A natural generalization of a single tax schedule is to introduce a parameter \( \lambda \) whose value depends on the total amount of tax to be raised. A parametric tax schedule is a continuous function \( t = f(x, \lambda) \) defined for all \( x > 0 \) and all \( \lambda \) in some closed interval \( [a, b] \) of the extended reals. We require that for every \( x > 0 \), \( f \) be (weakly) monotone increasing in \( \lambda \) and that \( f(x, a) = 0 \), \( f(x, b) = x \). The appropriate value of \( \lambda \) is dictated by the tax total that needs to be levied. Namely, given a tax problem \( (x, T) \), choose \( \lambda \) so that \( \sum f(x_i, \lambda) = T \). The required \( \lambda \) may not be unique, because \( f \) may not be strictly monotonic, but given \( (x, T) \) there is only one solution \( t \) such that for some \( \lambda \) and all \( i, t_i = f(x_i, \lambda) \) and \( \sum t_i = T \).

Many historical proposals for tax methods are of the parametric type. For example, the flat tax takes the form \( t = \lambda x \), where \( 0 \leq \lambda \leq 1 \). The head tax (subject to ability to pay) is given by \( t = \min \{x, \lambda \} \), where \( 0 \leq \lambda \leq \infty \). In 1889 A. J. Cohen Stuart [7] proposed tax schedules of the form \( t = x - x^1 - \lambda \), where \( 0 \leq \lambda \leq \infty \).\(^3\) In 1901 G. Cassel [6] suggested the family \( t = x^2/(x + 1/\lambda) \), where \( 0 \leq \lambda \leq \infty \).\(^4\)

\(^3\) Cohen Stuart actually proposed the class of tax schedules \( t = x - \alpha(x/\alpha)^{1-\lambda} \) for \( x \geq \alpha \) and \( 0 \leq \lambda \leq 1 \). The parameter \( \alpha \) amounts to an "exemption." In our example we have taken \( \alpha = 1 \).

\(^4\) Cassel [6] proposed the general class of schedules \( t = r[x - \beta x/(x + \beta - \alpha)] \), where \( 0 \leq \alpha < \beta \) and \( x \geq \alpha \). The term \( \beta x/(x + \beta - \alpha) \) may be interpreted as a variable exemption that rises with income. In our example we have taken \( \alpha = 0, r = 1, \lambda = 1/\beta \).
All parametric methods are consistent for the same reason that a single
tax schedule is consistent. Conversely, it can be shown that every (con-}
tinuous, symmetric) consistent method must be parametric [19].
A method $F$ is monotonic if no one’s taxes go down when the total tax
load goes up:

$$0 \leq T < T' \leq \sum x_i \Rightarrow F(x, T) \leq F(x, T'). \quad (2)$$

Strict monotonicity requires that everyone’s taxes actually increase when
the total tax load increases:

$$0 \leq T < T' \leq \sum x_i \Rightarrow F(x, T) < F(x, T'). \quad (3)$$

By definition, all parametric methods are monotonic, but not necessarily
strictly so. Surprisingly, consistency and continuity imply monotonicity
(but not strict monotonicity) [19].

Monotonicity expresses the idea that everyone should share in a tax
increase, or at least not get a rebate. It leaves open the question of exactly

how a tax increase should be shared. There is a natural answer to this
question. Let $t = F(x, T)$ and suppose that $T < T'$. Think of the increase

$T' - T$ as a new tax added on top of the old (e.g., a surtax). The natural
basis for allocating the new tax is the taxpayers’ current ability to pay, that
is, their current after-tax incomes $x_i - t_i$. If we apply the method $F$ to the
problem $(x - t, T' - T)$, then the increment $T^* = T' - T$ should be shared
as $t^* = F(x - t, T' - T)$. The idea is that every increment of tax should be
assessed equitably relative to what taxpayers currently have.

$F$ satisfies the composition principle if for all problems $(x, T)$ and $(x, T')$,
where $T < T'$,

$$t = F(x, T) < x \Rightarrow F(x, T') - F(x, T) = F(x - t, T' - T). \quad (4)$$

Notice that $t < x$ in the hypothesis of (4), for otherwise $F(x - t, T' - T)$
would be undefined.

The effect of taxation on incentives to increase income is also relevant to
our analysis. At a minimum, we do not want a person who earns more
than another to end up with less after taxes. This will be the case, however,
unless for all problems $(x, T)$ and all $i, j$,

$$t = F(x, T) \& x_i > x_j \Rightarrow x_i - t_i \geq x_j - t_j. \quad (5)$$

If (5) holds, $F$ is said to be order-preserving. $F$ is strictly order-preserving if

$$0 \leq T < \sum x_i \& t = F(x, T) \& x_i > x_j \Rightarrow x_i - t_i > x_j - t_j. \quad (6)$$
When \( x_i = x_j \), symmetry guarantees that \( t_i = t_j \) and \( x_i - t_i = x_j - t_j \). Notice that strict order preservation cannot hold when \( T = \sum x_i \), that is, when taxes are confiscatory, because everyone is levelled down to zero.

At this point let us remark that monotonicity (not necessarily strict) and order preservation (not necessarily strict) are actually consequences of composition and consistency. The requirement of strictness, while natural enough, is an additional assumption that is needed to characterize the family of equal sacrifice methods.


Let \( U(x) \) be some hypothetical utility function for income, which we assume is continuous and strictly increasing for all \( x > 0 \). A taxation method \( F \) equalizes absolute sacrifice relative to \( U(x) \) if for all problems \( (x, T), x \in R_{++}^I \),

\[
    t = F(x, T) \iff \exists c \geq 0 \quad \forall i \in I [ U(x_i) - U(x_i - t_i) = c ]. \tag{7}
\]

Notice that for \( F \) to be everywhere defined, it must be that \( U(x) \to -\infty \) as \( x \to 0^+ \). Indeed, suppose that \( U(x) \) were bounded below. Then \( U(x_i) \) comes arbitrarily close to \( U(x_i - t_i) \) for all sufficiently small \( x_i \), by virtue of the fact that \( 0 \leq t_i \leq x_i \). But then for every positive level of sacrifice \( c \), \( U(x_i) - U(x_i - t_i) < c \) for all sufficiently small \( x_i \), so equal sacrifice at level \( c \) cannot hold for all incomes.

An alternative definition of equal sacrifice is that everyone give up the same percentage of their utility; that is, they all sacrifice at the same rate \( r = 1 - U(x - t)/U(x) \). Clearly, \( F \) equalizes absolute sacrifice relative to \( U(x) \) if and only if it equalizes the rate of loss in utility relative to \( e^{U(x)} \). Hence we may unambiguously refer to \( F \) as an equal sacrifice method without specifying whether equal absolute or equal rate of sacrifice is meant. The criterion of sacrifice is relevant to the form of the utility function but not to the form of the method.

Observe that equal sacrifice cannot hold if \( T > 0 \) and some incomes are positive while others are zero. This is because no sacrifice is possible if a person has nothing to give up. It is for this reason that we restrict attention to the case where all incomes are positive.

Theorem 1. A tax method satisfies strict monotonicity, strict order-preservation, consistency, and composition if and only if it is an equal sacrifice method.
Proof. Every equal sacrifice method is necessarily strictly monotonic. Indeed, if $T$ increases then someone must pay more tax, hence someone must sacrifice more, hence equal sacrifice implies that all must sacrifice more. Every equal sacrifice method is also strictly order-preserving. For suppose that some individual $i$ has a higher taxable income than does $j$. Then $i$'s initial utility is higher than $j$'s (since $U(x)$ is strictly increasing). Since both sacrifice the same amount of utility, $i$'s utility after-tax is higher than $j$'s, so $i$'s after-tax income must be higher too.

Consistency obviously holds for every equal sacrifice method. Next let us check composition. Suppose that a first-stage tax $t$ results in equal sacrifice $\lambda$ for everyone relative to the initial incomes $x$. Suppose further that a second-stage tax $t'$ results in an additional sacrifice of $\lambda'$ for everyone relative to the incomes $x - t > 0$. Then everyone sacrifices the same amount $\lambda + \lambda'$ under the combined tax $t + t'$.

Conversely, let $F$ be a tax method with the stated properties and we shall show that $F$ equalizes absolute sacrifice relative to some continuous, strictly increasing utility function. For every finite $I \subset N$ define a partial relation $P$ on $R^I_+$ as follows:

$$\forall x, y \in R_+^I [y \ F \ x \iff \exists T[y = x - F(x, T)]] .$$  (8)

The statement $y \ F \ x$ means: "$y$ is a distribution of after-tax incomes associated with a pre-tax distribution $x$ via the method $F$.”

The following properties of $P$ are immediate from the assumptions about $F$. The consistency principle says that $P$ is invariant under projection:

$$\forall x, y \in R_+^I \forall J \subseteq I [y \ P \ x \Rightarrow y_J \ P \ x_J].$$  (9)

(If $I$ or $J$ is empty we interpret the statement to be vacuously true.)

The composition principle says that $P$ is transitive:

$$\forall x, y, z \in R_+^I [z \ P \ y \ & \ y \ P \ x \Rightarrow z \ P \ x].$$  (10)

Given $x > 0$, $y(T) = x - F(x, T)$ traces out a continuous path in $R_+^I$ between 0 and $x$ as $T$ decreases from $\sum x_i$ to 0, and by assumption it is strictly monotone decreasing in $T$. This is the path to $x$. $y \ P \ x$ simply means that $y$ lies on the path to $x$. Transitivity means that if $z$ is on the path to $y$, and $y$ is on the path to $x$, then $z$ is on the path to $x$.

Given $x \in R_+^I$, fix some individual $i \in I$ and a value $0 \leq v \leq x_i$. Continuity and strict monotonicity imply that

$$\exists! y [y \ P \ x \ & \ y_i = v].$$  (11)
Define a binary operation $\otimes$ on the half-open interval $[0, 1]$ as follows:

$$\forall x, y \in (0, 1] \ [(x \otimes y, y) P(x, 1)]. \quad (12)$$

By (11), $x \otimes y$ exists and is unique. Observe also that $0 < x \otimes y \leq x$. By strict monotonicity $x \otimes 1 = x$, and by symmetry $1 \otimes y = y$. Hence 1 plays the role of identity. Figure 1 illustrates the relationship between $x$, $y$, and $x \otimes y$.

We claim that $\otimes$ is associative, and thus forms a semigroup on $(0, 1]$. Given $x$, $y$, $z \in (0, 1]$, consider three individuals with claims $(x \otimes y, y, 1)$. By (11) there exist numbers $u$, $u'$ such that

$$(u, u', z) P(x \otimes y, y, 1). \quad (13)$$

Consistency applied to the first and third coordinates of (13) shows that $(u, z) P (x \otimes y, 1)$ and so

$$u = (x \otimes y) \otimes z. \quad (14)$$

Consistency applied to the second and third coordinates of (13) shows that $(u', z) P (y, 1)$ and hence $u' = y \otimes z$. Finally, consistency applied to the first and second coordinates shows that $(u, u') P (x \otimes y, y)$, that is, $(u, y \otimes z) P (x \otimes y, y)$. On the other hand, by definition of $\otimes$, $(x \otimes y, y) P (x, 1)$. Combining the last two statements with the transitivity of $P$ shows that $(u, y \otimes z) P (x, 1)$. Thus $u = x \otimes (y \otimes z)$. Combined with (14) this establishes associativity.

The existence of inverses is not guaranteed. Nonetheless, $\otimes$ is reducible on both sides in the sense that, for all $x$, $x'$, $y$, $y' \in (0, 1]$,

$$x \otimes y = x \otimes y' \Rightarrow y = y' \quad (15)$$

and

$$x \otimes y = x' \otimes y \Rightarrow x = x'. \quad (16)$$

Fig. 1. The path to $(x, 1)$. 
Statement (15) follows at once from the strict monotonicity of $F$. Statement (16) is proved thus: if $x \otimes y = x' \otimes y = z$, then $(z, y) P (x, 1)$ and $(z, y) P (x', 1)$. Consider the path to $(x', x, 1)$. By (11) there exist $\alpha, \beta$ such that $(\alpha, \beta, y) P (x', x, 1)$. By consistency, $(\alpha, y) P (x', 1)$. Since $(z, y) P (x', 1)$ by assumption, and since $F$ is strictly monotonic, $z = \alpha$. Similarly, $(\beta, y) P (x, 1)$ and $(z, y) P (x, 1)$ imply $z = \beta$. Thus $(z, z, y) P (x', x, 1)$. Since individuals 1 and 2 are equal after taxes, strict order preservation implies that they must be equal before taxes; hence $x = x'$, proving (16).

In sum, the function $G(x, y) = x \otimes y$ is defined for all $x, y \in (0, 1]$, is reducible, and satisfies the associativity equation

$$G(G(x, y), z) = G(x, G(y, z)).$$

It is straightforward to verify that $G$ is continuous. By a theorem of Aczel [1, Sect. 6.2.2], $G$ must therefore have the form

$$x \otimes y = G(x, y) = U^{-1} [U(x) + U(y)], \quad (17)$$

where $U(x)$ is a continuous, strictly monotone increasing, real-valued function defined on $(0, 1]$. Since $G(1, y) = y$, (17) implies that $U(1) = 0$. Therefore $U(x) < 0$ whenever $0 < x < 1$. We assert that

$$\lim_{x \to 0^+} U(x) = -\infty. \quad (18)$$

We know that for all $x > 0$,

$$x \otimes x = G(x, x) = U^{-1} [U(x) + U(x)],$$

so

$$U(x \otimes x) = 2U(x).$$

Moreover, $0 < x \otimes x \leq x$. Therefore $x \otimes x \to 0^+$ as $x \to 0^+$ and so

$$\lim_{x \to 0^+} U(x \otimes x) = \lim_{x \to 0^+} U(x) = 2 \lim_{x \to 0^+} U(x).$$

Since for $x < 1$, $U(x)$ is strictly negative, (18) follows.

Extend $U$ so that $U(0) = -\infty$. Next, extend $U(x)$ to all $x > 1$ by defining $U(x) = -U(v)$, where $v$ is uniquely determined such that $0 < v \leq 1$ and $(1, v) P (x, 1)$. We claim that for all $x > 0$ and $y, z \geq 0$,

$$(z, y) P (x, 1) \Rightarrow U(z) = U(x) + U(y). \quad (19)$$

Strict monotonicity implies that $y = 0$ if and only if $z = 0$, so $U(z) = U(x) + U(y) = -\infty$ and (19) holds in this case. Assume then that $y, z > 0$. If $0 < x \leq 1$ then (19) follows from (17) and the definition of $G$. If $x > 1$ there are two cases to consider.
Case 1: \( x > 1, 0 < z \leq 1 \). Suppose that \((z, y) P (x, 1)\). We shall prove that \( U(z) = U(x) + U(y) \). By (11) there is a unique \( v, 0 < v \leq 1 \), such that \((1, v) P (x, 1)\). Likewise there is a unique \( w > 0 \) such that \((z, w) P (1, v)\). By transitivity, \((z, w) P (x, 1)\). By assumption \((z, y) P (x, 1)\), so strict monoticity implies that \( w = y \). Therefore \((z, y) P (1, v)\). By symmetry, \((y, z) P (v, 1)\). Since \( v, y, z \leq 1 \), \( \otimes \) is defined and \( y = v \otimes z \). From (17) it follows that \( U(y) = U(v) + U(z) \). But by choice of \( v \), \( U(v) = -U(x) \). Thus \( U(z) = U(x) + U(y) \), as was to be shown.

Case 2: \( x > 1, z > 1 \). Again suppose that \((z, y) P (x, 1)\) and we shall show that \( U(z) = U(x) + U(y) \). Consider the distribution \((z, y, 1)\). Since \( z > 1 \), there exist unique numbers \( w, v \) such that \((1, w, v) P (z, y, 1)\). We know that \( 0 < w \leq y \leq 1 \) and \( 0 < v \leq 1 \). Consistency applied to the first and third coordinates shows that \((1, v) P (z, 1)\); hence \( U(v) = -U(z) \). Consistency applied to the first and second coordinates shows that \((1, w) P (z, y)\). Since by assumption \((z, y) P (x, 1)\), transitivity implies that \((1, w) P (x, 1)\) and thus \( U(w) = -U(x) \). Finally, consistency applied to the second and third coordinates shows that \((w, v) P (y, 1)\). Since \( 0 < w, v, y \leq 1 \), \( \otimes \) is defined and \( w = y \otimes v \). Thus by (17), \( U(w) = U(y) + U(v) \). Substituting \( U(v) = -U(z) \) and \( U(w) = -U(x) \) results in \( U(z) = U(x) + U(y) \), as was to be shown.

The proof of the theorem is concluded as follows. Suppose that \( t = F(x, T), x \in \mathbb{R}_{++}^I \). Let \( y = x - t \), whence \( y P x \). Continuity and strict monotonicity imply that there is a unique point \((z, v) \geq 0 \) such that \((z, v) P (x, 1)\) and \( \sum z_i = \sum y_i \). By consistency, \( z P x \). Since \( \sum z_i = \sum y_i \), \( z = y \). Consistency applied to the \( i \)th coordinate together with the last coordinate of the expression \((y, v) P (x, 1)\) shows that \((y, v) P (x, 1)\) holds for all \( i \in I \). Thus \( y_i = x_i - t_i \geq x_i \otimes v \) for all \( i \), so by (19),

\[
t = F(x, T) \Rightarrow \exists v \geq 0 \quad \forall i \in I \left[ U(x_i - t_i) = U(x_i) + U(v) \right]. \tag{20}
\]

In other words, if \( t = F(x, T) \), then \( U(x_i) - U(x_i - t_i) = -U(v) \) is a constant, so all individuals sacrifice equally relative to the continuous, strictly increasing “utility” function \( U(x) \).

We assert that the converse of (20) also holds. Suppose that for some \( x > 0 \), some \( 0 \leq t \leq x \), and some \( v \geq 0 \), \( U(x_i) - U(x_i - t_i) = -U(v) \) for all \( i \). Let \( T = \sum t_i \) and \( t' = F(x, T) \). By the preceding we know that for some \( v' \geq 0 \) and all \( i \in I \), \( U(x_i) - U(x_i - t'_i) = -U(v') \). Since \( \sum t'_i = \sum t_i \) and \( U \) is strictly increasing, it follows that \( v' = v \) and \( t' = t \). This completes the proof of Theorem 1.
5. Scale-Invariant Equal Sacrifice Methods

Equity is relative. People tend to evaluate how fairly they are treated, not in absolute terms, but in relation to how others are treated. If $A$ has an income of $100,000 and pays a tax of $30,000, while $B$ has an income of $20,000 and pays a tax of $3000, then the perceived fairness or unfairness of this distribution does not depend to any significant extent on whether these are 1985 or 1986 dollars. The relevant fact in determining the appropriate size of the tax bite on different income levels is not how rich people are in relation to some historical standard, but how rich they are relative to each other.

To this it might be objected that there is an absolute benchmark—the "minimum subsistence" level of income—that is relevant to taxation. We answer that such a benchmark, if it can be defined objectively at all, should enter into the definition of taxable income itself. Taxable income is traditionally defined as income over and above the subsistence level, for otherwise it does not represent true ability to pay (J. S. Mill [12, Book V, Chap. II, Sect. 3], Edgeworth [9, p. 141]). If we regard taxable income, net of subsistence, as an index of relative ability to pay, then we should not insist that the relative distribution of taxes depend on anything more than the relative sizes of taxable incomes.

A method $F$ is scale-invariant (or homogeneous) if

$$\forall \theta > 0 \forall (x, T) \ [F(\theta x, \theta T) = \theta F(x, T)]. \quad (21)$$

Scale invariance, together with the conditions characterizing equal sacrifice methods, has strong implications for the form of a tax method.

**Theorem 2.** A strictly monotonic, strictly order-preserving tax method $F$ satisfies consistency, composition, and scale invariance if and only if it equalizes absolute sacrifice relative to the utility function $U(x) = \ln x$ or a utility function of form $U(x) = -x^p$, $p < 0$. In the former case $F$ is the flat tax; in the latter case it is a parametric method of form

$$t = x - [x^p + \lambda^p]^{1/p}, \quad \text{where} \quad p < 0, \lambda \in [0, \infty]. \quad (22)$$

**Proof.** By Theorem 1 there exists a strictly increasing, continuous function $U(x)$ such that for all problems $(x, T)$,

$$t = F(x, T) \iff \exists \forall i \ [U(x_i) - U(x_i - t_i) = c].$$

Combined with scale invariance this implies (see [18, Lemma]) that $U(x)$ must take either the form

$$U(x) = ax^p + b, \quad ap > 0;$$
or

\[ U(x) = a \ln x + b, \quad a > 0. \]

Suppose that \( U(x) = a \ln x + b \). Then \( t = F(x, T) \) if and only if a \( \ln(x_i) - a \ln(x_i - t_i) \) is a constant. This means that \( (x_i - t_i)/x_i \) is a constant; in other words, \( t_i \) is proportional to \( x_i \), so \( F \) is the flat tax.

Suppose that \( U(x) = ax^p + b \), where \( ap > 0 \). By the proof of Theorem 1 we know that \( U(x) \to -\infty \) as \( x \to 0^+ \). Hence both \( a, p \) must be negative. In this case \( F \) takes the form

\[ t = F(x, T) \iff \exists \forall i [ax_i^p - a(x_i - t_i)^p = -c], \]

which means that \( F \) is the parametric method \( t = x - [x^p + c/a]^{1/p} \).

The reader may check that the Cassel tax results from substituting \( p = -1 \) in (22).

The functions \( \ln x \) and \( -x^p \), where \( p < 0 \), are precisely the utility functions having a constant degree of relative risk aversion equal to one or more.\(^5\) These have become fairly standard representations of utility in the modern theory of risk-bearing \([3, 15]\). They are consistent with the notion that people invest a fixed proportion of their wealth in any particular class of risky assets, such as securities. Coincidentally, the tax methods implied by constant relative risk aversion are closely related to constant elasticity of substitution (CES) functions from production theory.

All of the methods characterized in Theorem 2 are progressive in the sense that \( x_i \leq x_j \) implies \( t_i/x_i \leq t_j/x_j \) whenever \( t = F(x, T) \). In fact the methods defined by (22) are strictly progressive. Thus we obtain the following.

**Corollary.** Every scale-invariant equal sacrifice tax method is progressive.

For each \( p < 0 \), let \( F_p \) denote the method defined by (22), which equalizes sacrifice relative to the utility function \( U(x) = -x^p \). Further, let \( F_0 \) denote the proportional method, which equalizes sacrifice relative to the utility function \( U(x) = \ln x \). It may be checked that, for every fixed income distribution \( x \) and tax total \( T \), \( F_p(x, T) \) converges to \( F_0(x, T) \) as \( p \) approaches zero. The more negative \( p \) is, the more progressive is the tax. As \( p \) approaches negative infinity, the limiting distribution has the property that everyone above a certain threshold has equal after-tax income, and everyone below the threshold pays no tax. This effect is illustrated in Table I, which shows the distribution of a fixed tax burden among five equal-sized income classes for variable levels of \( p \).

\(^5\) The coefficient of relative risk aversion of \( U(x) \) is \( -xU''(x)/U'(x) \).
TABLE I

Distribution of $30,000 in Tax for Diverse Values of $p$

<table>
<thead>
<tr>
<th>Income</th>
<th>$p = 0$</th>
<th>$p = -\frac{1}{2}$</th>
<th>$p = -1$</th>
<th>$p = -2$</th>
<th>$p = -5$</th>
<th>$p \to -\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10,000</td>
<td>2,000</td>
<td>1,153</td>
<td>653</td>
<td>201</td>
<td>5</td>
<td>0</td>
</tr>
<tr>
<td>20,000</td>
<td>4,000</td>
<td>3,145</td>
<td>2,453</td>
<td>1,475</td>
<td>314</td>
<td>0</td>
</tr>
<tr>
<td>30,000</td>
<td>6,000</td>
<td>5,624</td>
<td>5,201</td>
<td>4,394</td>
<td>2,776</td>
<td>0</td>
</tr>
<tr>
<td>40,000</td>
<td>8,000</td>
<td>8,469</td>
<td>8,742</td>
<td>8,978</td>
<td>9,098</td>
<td>10,000</td>
</tr>
<tr>
<td>50,000</td>
<td>10,000</td>
<td>11,609</td>
<td>12,951</td>
<td>14,952</td>
<td>17,807</td>
<td>20,000</td>
</tr>
</tbody>
</table>

6. Bankruptcy and the Proportional Method

In paying taxes, people think about how much they are giving up rather than what they are allowed to keep. The focus is on loss rather than gain. In other distributive problems the distinction is more blurred. In a bankruptcy proceeding, for example, the creditors sustain a loss, but in the end they also receive a payment, so the outcome can be viewed as partly positive. Every method $F(x, T)$ that allocates losses implicitly allocates gains through its dual $F^*(x, T) = x - F(x, \sum x_i - T)$. A method is self-dual if $F^* = F$, that is, if $F$ treats the positive and negative sides of the ledger symmetrically [4].

Self-duality is a strong assumption. It says that taking away is the mirror image of giving. It equates giving up a small part of what one has to getting a small part of what one is owed. From a utilitarian standpoint this is a dubious assumption. If taxing the rich at higher rates than the poor seems fair when the total tax burden is 40% of total income, then surely it is fair when the total tax burden is 60% of total income. Yet a self-dual method does not have this property: if it is progressive when the tax burden is below 50%, then it is regressive when the tax burden is above 50%. In taxation, therefore, self-duality is of doubtful validity.

In bankruptcy, where elements of gain and loss are more closely intertwined, self-duality seems much more plausible. Let us see what it implies in the presence of the properties introduced earlier. If $F$ is consistent, then clearly so is its dual. If $F$ is scale-invariant, then its dual is too. Composition does not carry over, however. In the allocation of losses, composition says that if a person has $x$ and gives up $t$, then the basis for any future assessment is the amount he has left, $x - t$. On the other hand, if a person is owed $x$ (but does not yet have it) and if $y = x - t$ represents a partial repayment of the debt (and $t$ the loss), then the natural basis for any future repayment is the outstanding debt $x - y$. These two conditions are not equivalent; composition for $F$ does not necessarily imply com-
position for $F^*$—unless of course they are self-dual. But in this case $F$ must be the proportional method.

**Theorem 3.** An allocation method $F$ is self-dual and satisfies composition if and only if it is the proportional method.

**Proof.** The proportional method is clearly self-dual and satisfies composition. Conversely, let $F$ have the latter two properties. Fix $x \in \mathbb{R}^+_+$. As in the proof of Theorem 1, define the path to $x$ to be the continuous curve traced out by the function $y(T) = x - F(x, T)$ as $T$ varies from $\sum x_i$ to 0. Write $y P x$ if $y$ is on the path to $x$.

Composition implies that $P$ is transitive. Self-duality implies that

$$y P x \iff (x - y) P x.$$ (23)

By the continuity of $F$, there exists a vector $y$ such that $y P x$ and $\sum y_i = \sum x_i/2$. By self-duality, $(x - y) P x$. Since $F$ is single-valued, it must be that $x - y = y$, that is, $y = x/2$. In short,

$$\forall x \in \mathbb{R}^+_+ (1/2)x P x.$$

Applying the same argument to $(1/2)x$ yields $(1/4)x P (1/2)x$. Since $P$ is transitive, $(1/4)x P x$. By self-duality, $(3/4)x P x$. Continuing in this fashion, conclude that for all positive integers $m, n$ such that $m \leq 2^n$, $(m/2^n)x P x$. Therefore, by continuity, the path to $x$ is a ray from the origin to $x$. As this argument holds for every $x$, $F$ is the proportional method. □

**References**


