APPENDICES

to

Interest Rates and the Market for New Light Vehicles*

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A SUMMARY OF THE MODELS AND THE OPTIMALITY CONDITIONS

In this appendix we re-state the household's and automaker's problems and derive the first-order conditions in sections A.1 and A.2. We list the equations defining a symmetric market equilibrium (section A.3), provide details on how to reformulate the model so that key variable are in ratio form or growth rates (section A.4), demonstrate how we eliminate the labor input and productivity terms (section A.5), and list the equations describing the steady state of the market equilibrium (section A.6). In section A.7, we report the second-order conditions in the symmetric market equilibrium and discuss su cient conditions for an optimum at the steady state.

To allow for an easier comparison to the paper, note that equations that are numbered (1), (2),

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etc. are the equations that are stated in the paper. The equations that are numbered (A.1), (A.2), etc are the equtions stated solely in Appendix A.

A.1 Models of the Household

Shopping Cost Model

The representative household undertakes a two-stage optimization process. In the first stage, the household minimizes the shopping costs of purchasing automobiles. The costs of purchasing an automobile consist of both purchase costs and shopping costs. Define P_{jt} as the real price of a new automobile of type j and S_{jt} as the quantity of new automobiles of type j purchased at time t.[1] Then $P_{jt}S_{jt}$ is the real cost of purchasing new automobiles of type j at time t.

Define $(A_{jt}/A_t)\,S_{jt}$ as the total shopping cost of purchasing new automobiles of type j, where (A_{jt}/A_t) is the per unit shopping cost of purchasing new automobiles of type j , $A_{jt} = N_{jt-1}+Y_{jt}$ is the supply of new automobiles available for sale in period t by producer of type j, N_{jt-1} is the stock of inventories of new autos of type j held by the producer of type j autos at the end of period t 1, Y_{jt} is the current production of new automobiles by producer of type j , $A_t = N_{t-1} + Y_t$ is the supply of new automobiles available for sale in the industry as a whole, N_{t-1} is the stock of inventories of all new autos in the industry, Y_t is current production in the industry as a whole, and where we assume that $' < 0$. Then, $(A_{jt}/A_t)\,P_{jt}S_{jt}$ is the total real shopping cost of purchasing new automobiles of type j valued at P_{jt}.

The total real cost of purchasing new automobiles of type j, denoted by SC_t, is the sum of the purchase costs plus the shopping costs is then

$$SC_t = P_{jt}S_{jt} + \left(\frac{A_{jt}}{A_t}\right) P_{jt}S_{jt} = \left[1 + \left(\frac{A_{jt}}{A_t}\right)\right] P_{jt}S_{jt}. \tag{1}$$

In the first stage, the representative household chooses S_{jt} to minimize

$$\int_0^1 \left[1 + \left(\frac{A_{jt}}{A_t}\right)\right] P_{jt}S_{jt}dj \tag{2}$$

subject to

$$S_t = \left[\int_0^1 S_{jt}^{\frac{\varepsilon-1}{\varepsilon}} dj\right]^{\frac{\varepsilon}{\varepsilon-1}} \tag{3}$$

where $\epsilon > 1$. The Lagrangian is

[1] Specifically, P_{jt} is the nominal price of new automobiles of type j divided by the price of consumption, excluding car services.

$$\mathcal{L}^{\mathcal{SC}} = \int \left[1 + \left(\frac{A_{jt}}{A_t}\right)\right] P_{jt} S_{jt} dj + {}^{sc}_{t} \left[S_t - \left[\int_0^1 S_{jt}^{\frac{\varepsilon-1}{\varepsilon}} dj\right]^{\frac{\varepsilon}{\varepsilon-1}}\right].$$

The first-order condition is

$$\frac{\partial \mathcal{L}^{\mathcal{SC}}}{\partial S_{jt}} = \left[1 + \left(\frac{A_{jt}}{A_t}\right)\right] P_{jt} - {}^{sc}_{t} \left(\frac{S_{jt}}{S_t}\right)^{-\frac{1}{\varepsilon}} = 0 \tag{A.1}$$

where ${}^{sc}_{t}$ is the multiplier associated with aggregate sales. Solving equation (A-1) yields

$$\frac{S_{jt}}{S_t} = \left(\frac{P_{jt}}{{}^{sc}_{t}}\right)^{-\varepsilon} \left[1 + \left(\frac{A_{jt}}{A_t}\right)\right]^{-\varepsilon}. \tag{A.2}$$

Assume that shopping costs are $(A_{jt}/A_t) = (A_{jt}/A_t)^{\nu}$ 1. Then, the demand function for new automobiles, (A.2) is

$$\frac{S_{jt}}{S_t} = \left(\frac{P_{jt}}{{}^{sc}_{t}}\right)^{-\varepsilon} \left(\frac{A_{jt}}{A_t}\right)^{-\varepsilon\nu}. \tag{A.3}$$

To solve for ${}^{sc}_{t}$, take the definition of aggregate sales, equation (3), and use equation (A.3) to substitute for S_{jt} to get

$$\begin{aligned}
S_t &= \left[\int_0^1 S_{jt}^{\frac{\varepsilon-1}{\varepsilon}} dj\right]^{\frac{\varepsilon}{\varepsilon-1}} \\
&= \left[\int_0^1 \left[S_t \left(\frac{P_{jt}}{{}^{sc}_{t}}\right)^{-\varepsilon} \left(\frac{A_{jt}}{A_t}\right)^{-\varepsilon\nu}\right]^{\frac{\varepsilon-1}{\varepsilon}} dj\right]^{\frac{\varepsilon}{\varepsilon-1}} \\
&= \left[({}^{sc}_{t})^{\varepsilon-1} S_t^{\frac{\varepsilon-1}{\varepsilon}} \int_0^1 \left[P_{jt}^{-\varepsilon} \left(\frac{A_{jt}}{A_t}\right)^{-\varepsilon\nu}\right]^{\frac{\varepsilon-1}{\varepsilon}} dj\right]^{\frac{\varepsilon}{\varepsilon-1}} \\
&= ({}^{sc}_{t})^{\varepsilon} S_t \left[\int_0^1 \left[P_{jt}^{-\varepsilon} \left(\frac{A_{jt}}{A_t}\right)^{-\varepsilon\nu}\right]^{\frac{\varepsilon-1}{\varepsilon}} dj\right]^{\frac{\varepsilon}{\varepsilon-1}}.
\end{aligned} \tag{A.4}$$

Solving equation (A.4) for ${}^{sc}_{t}$ yields

$${}^{sc}_{t} = \left[\int_0^1 \left[P_{jt}^{-\varepsilon} \left(\frac{A_{jt}}{A_t}\right)^{-\varepsilon\nu}\right]^{\frac{\varepsilon-1}{\varepsilon}} dj\right]^{\frac{1}{1-\varepsilon}} = \left[\int_0^1 P_{jt}^{1-\varepsilon} \left(\frac{A_{jt}}{A_t}\right)^{\nu(1-\varepsilon)} dj\right]^{\frac{1}{1-\varepsilon}} = P_t, \tag{A.5}$$

where P_t is defined as the average real industry price level. Substituting equation (A.5) into equation (A.3), the demand function can then be written as

$$\frac{S_{jt}}{S_t} = \left(\frac{P_{jt}}{P_t}\right)^{-\varepsilon} \left(\frac{A_{jt}}{A_t}\right)^{\theta} \tag{5}$$

where $\theta = \quad \varepsilon\nu$. This is the demand function stated as equation (5) of the paper.

Utility Maximization Model

In the second-stage, the representative household is assumed to choose C_t, X_t, S_t, B_t, and D_t to maximize

$$E_o \sum_{t=0}^{\infty} \zeta^t U(C_t, X_t) \tag{6}$$

subject to

$$I_t = C_t + \quad 1 \quad \overline{\xi} \;\; P_t S_t + (r_{1t} + \mu)\, B_{t-1} \tag{7}$$

$$X_t = (1 \qquad) X_{t-1} + S_t \qquad\qquad 0 < \quad < 1 \tag{8}$$

$$B_t = (1 \quad \mu)\, B_{t-1} + D_t \qquad\qquad 0 < \mu < 1 \tag{9}$$

$$D_t = \overline{\xi} P_t S_t \qquad\qquad 0 \leq \overline{\xi} \leq 1 \tag{10}$$

where C_t denotes consumption, excluding car services, X_t is the stock of existing cars, S_t is purchases of new cars, I_t denotes real labor income, B_t represents the stock of car loans, D_t is new loans incurred to purchase automobiles, P_t is the average real price of new cars, A_t is the stock of new automobiles available for sale during period t in the industry as a whole, r_{1t} is the real interest rate on new car loans, $\overline{\xi}$ is the fraction of a new car purchase financed by a new loan, is the rate that cars depreciate, and μ is the fraction of existing loans that needs to be paid back every period.

Assume that the utility function is

$$U(C_t, X_t) = \pi_1 \ln C_t + \pi_2 \ln X_t, \tag{11}$$

where $\pi_1 > 0$ and $\pi_2 > 0$. The Lagrangian is then

$$\begin{aligned}
\mathcal{L}^{\mathcal{H}} = E_0 \Big\{ \zeta^{t+1} \Big[& \pi_1 \ln C_{t+1} + \pi_2 \ln X_{t+1} + \;{}^{h}_{1t+1} \;\; I_{t+1} \quad C_{t+1} \quad 1 \quad \overline{\xi} \;\; P_{t+1} S_{t+1} \quad (r_{1t+1} + \mu)\, B_t \\
& + \;{}^{h}_{2t+1} \left((1 \qquad) X_t + S_{t+1} \quad X_{t+1} \right) + \;{}^{h}_{3t+1} \;\; (1 \quad \mu)\, B_t + \overline{\xi} P_{t+1} S_{t+1} \quad B_{t+1} \Big] \\
& + \zeta^t \Big[\pi_1 \ln C_t + \pi_2 \ln X_t + \;{}^{h}_{1t} \;\; I_t \quad C_t \quad 1 \quad \overline{\xi} \;\; P_t S_t \quad (r_{1t} + \mu)\, B_{t-1} \\
& \qquad + \;{}^{h}_{2t} \left((1 \qquad) X_{t-1} + S_t \quad X_t \right) + \;{}^{h}_{3t} \;\; (1 \quad \mu)\, B_{t-1} + \overline{\xi} P_t S_t \quad B_t \Big] \Big\}
\end{aligned}$$

The first-order conditions are:

$$\frac{\partial \mathcal{L}^{\mathcal{H}}}{\partial C_t} = E_o \left\{ \zeta^t [\frac{\pi_1}{C_t} \quad {}^h_{1t}] \right. = 0 \tag{A.6}$$

$$\frac{\partial \mathcal{L}^{\mathcal{H}}}{\partial X_t} = E_o \left\{ \zeta^{t+1} [(1 \quad) \; {}^h_{2t+1}] + \zeta^t [\frac{\pi_2}{X_t} \quad {}^h_{2t}] \right. = 0 \tag{A.7}$$

$$\frac{\partial \mathcal{L}^{\mathcal{H}}}{\partial S_t} = E_o \left\{ \quad 1 \quad \overline{\xi} \;\; P_t \;\; {}^h_{1t} + \;\; {}^h_{2t} + \overline{\xi} P_t \;\; {}^h_{3t} \right\} = 0 \tag{A.8}$$

$$\frac{\partial \mathcal{L}^{\mathcal{H}}}{\partial B_t} = E_o \left\{ \zeta^{t+1} [\quad (r_{1t+1} + \mu) \;\; {}^h_{1t+1} + (1 \quad \mu) \;\; {}^h_{3t+1}] \quad \zeta^t \;\; {}_{3t} \right\} = 0 \tag{A.9}$$

Assuming that information is known at time t, then the first-order conditions and the constraints are:

$${}^h_{1t} = \frac{\pi_1}{C_t} \tag{12}$$

$${}^h_{2t} = \frac{\pi_2}{X_t} + \zeta(1 \quad) E_t \;\; {}^h_{2t+1} \tag{13}$$

$${}^h_{2t} = \;\; 1 \quad \overline{\xi} \;\; P_t \;\; {}^h_{1t} \quad \overline{\xi} P_t \;\; {}^h_{3t} \tag{14}$$

$${}^h_{3t} = \zeta E_t \left\{ (1 \quad \mu) \;\; {}^h_{3t+1} \quad (r_{1t+1} + \mu) \;\; {}^h_{1t+1} \right\} \tag{15}$$

$$C_t + \;\; 1 \quad \overline{\xi} \;\; P_t S_t + (r_{1t} + \mu) B_{t-1} = I_t \tag{7}$$

$$X_t = (1 \quad) X_{t-1} + S_t \quad 0 < \quad < 1 \tag{8}$$

$$B_t = (1 \quad \mu) B_{t-1} + D_t \quad 0 < \mu < 1 \tag{9}$$

$$D_t = \overline{\xi} P_t S_t \quad 0 \leq \overline{\xi} \leq 1 \tag{10}$$

These are equations (7) through (10) and (12) through (15) stated in the paper.

A.2 Model of the Firm

The representative firm, firm j, produces and sells a single durable good, namely, a type of new automobile, type j. The firm is assumed to be an integrated dealer-producer. The firm is also a monopolistic competitor that faces a stochastic downward-sloping demand curve for its product

and a variable and stochastic interest rate at which it discounts future profits. Each period, the representative firm, firm j, maximizes:

$$PV_j = E_o \sum_{t=0}^{\infty} \left[\Pi_{s=0}^{t} \beta_s\right] \quad _{jt} \tag{16}$$

where

$$\beta_s = \frac{1}{1 + r_{2s}} \tag{17}$$

$$_{jt} = \frac{P_{jt}}{P_t} S_{jt} \quad \frac{W_t}{P_t} L_{jt} \quad K_{jt} \tag{18}$$

subject to:

$$S_{jt} = \left(\frac{P_{jt}}{P_t}\right)^{-\varepsilon} \left(\frac{A_{jt}}{A_t}\right)^{\theta} S_t \quad \epsilon > 1 \;\; \theta > 0 \tag{19}$$

$$K_{jt} = \kappa_o \left(\frac{A_{jt}}{S_{jt}}\right)^{\kappa_1} N_{jt-1} \quad \kappa_0 > 0 \;\; \kappa_1 > 0 \tag{20}$$

$$A_{jt} = N_{jt-1} + Y_{jt} \tag{21}$$

$$A_{jt} = A_{jt-1} \quad S_{jt-1} + Y_{jt} \tag{22}$$

$$Y_{jt} = \quad _{jt} L_{jt}^{\alpha} \qquad 0 < \alpha < 1 \tag{23}$$

where P_{jt} is the real price firm j sets for an automobile of type j, S_{jt} is sales of automobiles of type j by firm j, L_{jt} is labor services, N_{jt} is the stock of inventories of firm j of finished automobiles at the end of the period, A_{jt} is the stock of goods available for sale during period t, Y_{jt} is the output of automobiles of type j, $_{jt}$ is labor productivity, r_{2t} is the real interest rate faced by the firm, K_{jt} is inventory storage costs, S_t, is industry sales, N_t is the industry stock of inventories of finished automobiles at the end of the period, P_t is the real industry price level, and W_t is the real wage rate.

In the auto industry, newly produced vehicles are typically shipped within days to dealers lots; hence we assume the stocks of goods for sale, A_{jt}, is the sum of residual inventories from the previous period and newly produced vehicles. Using the definition of A_{jt} inventory storage costs, equation (20), can be written as

$$K_{jt} = \kappa_o \left(\frac{A_{jt}}{S_{jt}}\right)^{\kappa_1} N_{jt-1} = \kappa_o \left(\frac{N_{jt-1} + Y_{jt}}{S_{jt}}\right)^{\kappa_1} N_{jt-1}.$$

Now, use equation (19) to eliminate price as an explicit choice variable and rewrite equation (21) to get $N_{jt-1} = A_{jt} \quad Y_{jt}$; then net revenues, equation (18), can be written as

$$_{jt} = \left(\frac{S_{jt}}{S_t}\right)^{1-\frac{1}{\varepsilon}} \left(\frac{A_{jt}}{A_t}\right)^{\frac{\theta}{\varepsilon}} S_t \quad \frac{W_t}{P_t} L_{jt} \quad \kappa_o \left(\frac{A_{jt}}{S_{jt}}\right)^{\kappa_1} (A_{jt.} \quad Y_{jt}). \tag{24}$$

The firm then chooses S_{jt}, Y_{jt}, A_{jt}, and L_{jt} to maximize equation (16) subject to equations (22) and (23) and where net revenue is now defined in equation (24) and the discount factor is again given by equation (17).

The Lagrangian is

$$\begin{aligned}
\mathcal{L}^{\mathcal{F}} = E_o \Bigg\{ \ldots \left[\Pi_{s=0}^{t+1} \beta_s\right] \Bigg[& \left(\frac{S_{jt+1}}{S_{t+1}}\right)^{1-\frac{1}{\varepsilon}} \left(\frac{A_{jt+1}}{A_{t+1}}\right)^{\frac{\theta}{\varepsilon}} S_{t+1} \quad \frac{W_{t+1}}{P_{t+1}} L_{jt+1} \\
& \kappa_o \left(\frac{A_{jt+1}}{S_{jt+1}}\right)^{\kappa_1} (A_{jt+1.} \quad Y_{jt+1}) + \quad _{1t+1} (A_{jt} \quad S_{jt} + Y_{jt+1} \quad A_{jt+1}) \\
& + \quad _{2t+1} \quad _{jt+1} L_{jt+1}^{\alpha} \quad Y_{jt+1} \Bigg] \\
+ \left[\Pi_{s=0}^{t+1} \beta_s\right] \Bigg[& \left(\frac{S_{jt}}{S_t}\right)^{1-\frac{1}{\varepsilon}} \left(\frac{A_{jt}}{A_t}\right)^{\frac{\theta}{\varepsilon}} S_t \quad \frac{W_t}{P_t} L_{jt} \quad \kappa_o \left(\frac{A_{jt}}{S_{jt}}\right)^{\kappa_1} (A_{jt.} \quad Y_{jt}) \\
& + \quad _{1t}(A_{jt-1} \quad S_{jt-1} + Y_{jt} \quad A_{jt}) + \quad _{2t}(\quad _{jt} L_{jt}^{\alpha} \quad Y_{jt}) \Bigg] \Bigg\}
\end{aligned}$$

The first-order conditions are

$$\begin{aligned}
\frac{\partial \mathcal{L}^{\mathcal{F}}}{\partial S_{jt}} = E_o \Bigg\{ & \left(\frac{\varepsilon \quad 1}{\varepsilon}\right) \left(\frac{S_{jt}}{S_t}\right)^{-\frac{1}{\varepsilon}} \left(\frac{A_{jt}}{A_t}\right)^{\frac{\theta}{\varepsilon}} \\
& + \kappa_o \kappa_1 \left(\frac{A_{jt}}{S_{jt}}\right)^{\kappa_1} \left(\frac{A_{jt.} \quad Y_{jt}}{S_{jt}}\right) \quad E_t \beta_{t+1} \quad _{1t+1}^{f} \Bigg\} = 0
\end{aligned} \tag{A.10}$$

$$\frac{\partial \mathcal{L}^{\mathcal{F}}}{\partial Y_{jt}} = E_o \left\{ \kappa_o \left(\frac{A_{jt}}{S_{jt}}\right)^{\kappa_1} + \quad _{1t}^{f} \quad _{2t}^{f} \quad = 0 \right. \tag{A.11}$$

$$\frac{\partial \mathcal{L}^F}{\partial A_{jt}} = E_o \Bigg\{ E_t \beta_{t+1} \ {}^{f}_{1t+1} + \left(\frac{\theta}{\varepsilon}\right) \left(\frac{S_{jt}}{S_t}\right)^{1-\frac{1}{\varepsilon}} \left(\frac{A_{jt}}{A_t}\right)^{\frac{\theta}{\varepsilon}-1} \left(\frac{S_t}{A_t}\right) \kappa_o \left(\frac{A_{jt}}{S_{jt}}\right)^{\kappa_1} \left[\kappa_1 \left(\frac{A_{jt}}{S_{jt}}\right)^{-1} \left(\frac{A_{jt} \quad Y_{jt}}{S_{jt}}\right) + 1\right] \quad {}^{f}_{1t} \Bigg\} = 0 \quad \text{(A.12)}$$

$$\frac{\partial \mathcal{L}^{\mathcal{F}}}{\partial L_{jt}} = E_o \Bigg\{ \alpha \lambda^{f}_{2t} \ {}_{jt} L_{jt}^{\alpha-1} \quad \frac{W_t}{P_t} \quad = 0 \quad \text{(A.13)}$$

where ${}^{f}_{1t}$ and ${}^{f}_{2t}$ are the multipliers associated with the available-supply-accumulation process and the production function, equations (22) and (23), respectively. Assume that information regarding exogenous variables is known at time t, then the first-order conditions and the constraints are

$$E_t \beta_{t+1} \ {}^{f}_{1t+1} = \left(\frac{\varepsilon \quad 1}{\varepsilon}\right) \left(\frac{S_{jt}}{S_t}\right)^{-\frac{1}{\varepsilon}} \left(\frac{A_{jt}}{A_t}\right)^{\frac{\theta}{\varepsilon}} + \kappa_o \kappa_1 \left(\frac{A_{jt}}{S_{jt}}\right)^{\kappa_1} \left(\frac{A_{jt.} \quad Y_{jt}}{S_{jt}}\right) \quad (25)$$

$${}^{f}_{1t} + \kappa_o \left(\frac{A_{jt}}{S_{jt}}\right)^{\kappa_1} = \ {}^{f}_{2t} \quad (26)$$

$${}^{f}_{1t} = E_t \beta_{t+1} \ {}^{f}_{1t+1} + \left(\frac{\theta}{\varepsilon}\right) \left(\frac{S_{jt}}{S_t}\right)^{1-\frac{1}{\varepsilon}} \left(\frac{A_{jt}}{A_t}\right)^{\frac{\theta}{\varepsilon}-1} \left(\frac{S_t}{A_t}\right) \kappa_o \left(\frac{A_{jt}}{S_{jt}}\right)^{\kappa_1} \left[\kappa_1 \left(\frac{A_{jt}}{S_{jt}}\right)^{-1} \left(\frac{A_{jt} \quad Y_{jt}}{S_{jt}}\right) + 1\right] \quad (27)$$

$$\alpha \lambda^{f}_{2t} \ {}_{jt} L_{jt}^{\alpha-1} = \frac{W_t}{P_t} \quad (28)$$

$$A_{jt} = N_{jt-1} + Y_{jt} \quad (21)$$

$$A_{jt} = A_{jt-1} \quad S_{jt-1} + Y_{jt} \quad (22)$$

These are equations (25) through (28) and (21) and (22) stated in the paper.

A.3 A Symmetric Market Equilibrium

We solve for a symmetric equilibrium. Firms make choices believing their first order conditions and constraints are given by equations (21) through (28). But since all firms are identical and

the total mass of firms in the economy is unity, in equilibrium $S_{jt} = S_t$, $A_{jt} = A_t$, $L_{jt} = L_t$, $Y_{jt} = Y_t$, and $_{jt} = _t$. Thus their optimizing behavior makes these optimality conditions become in equilibrium

$$\left(\frac{\varepsilon \quad 1}{\varepsilon}\right) + \kappa_o\kappa_1 \left(\frac{A_t}{S_t}\right)^{\kappa_1} \left(\frac{A_{t.} \quad Y_t}{S_t}\right) = E_t\beta_{t+1} \ {}^{f}_{1t+1} \tag{A.14}$$

$$\kappa_o \left(\frac{A_t}{S_t}\right)^{\kappa_1} + \ {}^{f}_{1t} = \ {}^{f}_{2t} \tag{A.15}$$

$$E_t\beta_{t+1} \ {}^{f}_{1t+1} + \ \frac{\theta}{\varepsilon} \ \left(\frac{A_t}{S_t}\right)^{-1} \quad \kappa_o \left(\frac{A_t}{S_t}\right)^{\kappa_1} \left[\kappa_1 \left(\frac{A_t}{S_t}\right)^{-1} \left(\frac{A_t - Y_t}{S_t}\right) + 1 \ = \ {}^{f}_{1t}\right. \tag{A.16}$$

$$\alpha\lambda_{2t} \ {}_tL_t^{\alpha-1} = \frac{W_t}{P_t} \tag{A.17}$$

$$A_t = A_{t-1} \quad S_{t-1} + Y_t \tag{A.18}$$

$$Y_t = \ {}_tL_t^{\alpha}. \tag{A.19}$$

For the household in market equilibrium, the optimality conditions and the constraints are

$$\frac{\pi_1}{C_t} = \ {}^{h}_{1t} \tag{12}$$

$$(1 \quad)\zeta E_t \ {}^{h}_{2t+1} + \frac{\pi_2}{X_t} = \ {}^{h}_{2t} \tag{13}$$

$${}^{h}_{2t} \quad 1 \quad \overline{\xi} \ P_t \ {}^{h}_{1t} + \overline{\xi}P_t \ {}^{h}_{3t} = 0 \tag{14}$$

$$(1 \quad \mu)\zeta E_t \ {}^{h}_{3t+1} \quad \zeta E_t \left(r_{1t+1} + \mu\right) \ {}^{h}_{1t+1} = \ {}^{h}_{3t} \tag{15}$$

$$C_t + \ 1 \quad \overline{\xi} \ P_tS_t + \left(r_{1t} + \mu\right) B_{t-1} = I_t \tag{7}$$

$$X_t = (1 \quad)X_{t-1} + S_t \tag{8}$$

$$B_t = (1 \quad \mu) B_{t-1} + \overline{\xi}P_tS_t. \tag{A.20}$$

The market equilibrium model thus contains thirteen equations in thirteen endogenous variables: S_t, Y_t, A_t, L_t, C_t, P_t, X_t, B_t, $\overset{h}{}_{1t}$, $\overset{h}{}_{2t}$, $\overset{h}{}_{3t}$, $\overset{f}{}_{1t}$, and $\overset{f}{}_{2t}$.

A.4 Ratios and Growth Rates

Our estimation approach relies on the data being stationary. Although unit sales of light vehicles and the firms' and households' interest rates are stationary, there are trends in real prices, real wages, and real disposable income. Consequently, we reformulate the model so that the relevant variables are in ratio form.

These ratios are stationary and thus guard against any statistical problems with nonstationary timeseries. Define the following ratios and growth rates:

$$R_t^{CI} = \frac{C_t}{I_t} \qquad R_t^{PSI} = \frac{P_t S_t}{I_t} \qquad R_t^{PXI} = \frac{P_t X_t}{I_t} \qquad R_t^{BI} = \frac{B_t}{I_t} \qquad R_t^{AI} = \frac{A_t}{I_t}$$

$$R_t^{AS} = \frac{A_t}{S_t} \qquad R_t^{YS} = \frac{Y_t}{S_t} \qquad LS_t = \frac{W_t L_t}{P_t Y_t} \qquad R_t^{YL} = \frac{Y_t}{L_t}$$

$$I_t = (1 + i_t)\, I_{t-1} \quad P_t = (1 + p_t)\, P_{t-1} \quad S_t = (1 + s_t)\, S_{t-1} \qquad Y_t = (1 + y_t)\, Y_{t-1}$$

$$L_t = (1 + l_t)\, L_{t-1} \qquad {}_t = (1 + \gamma_t)\ {}_{t-1} \qquad W_t = (1 + w_t)\, W_{t-1}$$

where i_t is the growth rate of real income, p_t is the growth rate of the real price of automobiles, s_t is the growth rate of real sales, y_t is the growth rate of output, l_t is the growth rate of labor, γ_t is the growth rate of labor productivity, and w_t is the growth rate of real wages.

The definition of ratios generates three additional equations that relate ratios to growth rates. In particular, observe that the ratios, R_t^{PSI}, R_t^{YS}, and LS_t can be written as

$$R_t^{PSI} = \frac{P_t S_t}{I_t} = \frac{(1 + p_t)\, P_{t-1}\, (1 + s_t)\, S_{t-1}}{(1 + i_t)\, I_{t-1}} = \frac{(1 + p_t)\, (1 + s_t)}{(1 + i_t)} R_{t-1}^{PSI} \tag{A.21}$$

$$R_t^{YS} = \frac{Y_t}{S_t} = \frac{(1 + y_t)\, Y_{t-1}}{(1 + s_t)\, S_{t-1}} = \frac{(1 + y_t)}{(1 + s_t)} R_{t-1}^{YS} \tag{A.22}$$

$$LS_t = \frac{W_t L_t}{P_t Y_t} = \frac{(1 + \omega_t)\, (1 + l_t)\, W_{t-1} L_{t-1}}{(1 + p_t)\, (1 + y_t)\, P_{t-1} Y_{t-1}} = \frac{(1 + \omega_t)\, (1 + l_t)}{(1 + p_t)\, (1 + y_t)} LS_{t-1}. \tag{A.23}$$

We now show that, using the ratios and growth rates and eliminating labor input, the model reduces to a system of fifteen equations and fifteen endogenous variables.

Using the ratios and growth rates, the equations of the model consisting of equations (A.14)-

(A.20), (7)-(8), (12)-(15), and (A.21)-(A.23) can then be written as

$$\pi_1 \quad R_t^{CI} \quad ^{-1} = \Omega_{1t} \tag{A.24}$$

$$(1 \quad)\zeta E_t \frac{(1+p_{t+1})}{(1+i_{t+1})}\Omega_{2t+1} + \pi_2 \quad R_t^{PXI} \quad ^{-1} = \Omega_{2t} \tag{A.25}$$

$$\Omega_{2t} \quad 1 \quad \overline{\xi} \quad \Omega_{1t} + \overline{\xi}\Omega_{3t} = 0 \tag{A.26}$$

$$\zeta E_t \left(\frac{1 \quad \mu}{1+i_{t+1}}\right)\Omega_{3t+1} \quad \zeta E_t \left(\frac{r_{1t+1}+\mu}{1+i_{t+1}}\right)\Omega_{1t+1} = \Omega_{3t} \tag{A.27}$$

$$R_t^{CI} + \quad 1 \quad \overline{\xi} \quad R_t^{PSI} + \left(\frac{r_{1t}+\mu}{1+i_t}\right) R_{t-1}^{BI} = 1 \tag{A.28}$$

$$(1 \quad)\frac{(1+p_t)}{(1+i_t)} R_{t-1}^{PXI} + R_t^{PSI} = R_t^{PXI} \tag{A.29}$$

$$\frac{(1 \quad \mu)}{(1+i_t)} R_{t-1}^{BI} + \overline{\xi} R_t^{PSI} = R_t^{BI} \tag{A.30}$$

$$\left(\frac{\varepsilon \quad 1}{\varepsilon}\right) + \kappa_o\kappa_1 \quad R_t^{AS} \quad ^{\kappa_1} \quad R_t^{AS} \quad R_t^{YS} = E_t\beta_{t+1} \quad {}^{f}_{1t+1} \tag{A.31}$$

$$\kappa_o \quad R_t^{AS} \quad ^{\kappa_1} + \quad {}^{f}_{1t} = \quad {}^{f}_{2t} \tag{A.32}$$

$$E_t\beta_{t+1} \quad {}^{f}_{1t+1} + \left(\frac{\theta}{\varepsilon}\right) \quad R_t^{AS} \quad ^{-1}$$

$$\kappa_o \quad R_t^{AS} \quad ^{\kappa_1}\left[\kappa_1 \quad R_t^{AS} \quad ^{-1} \quad R_t^{AS} \quad R_t^{YS} \quad + 1\right] = \quad {}^{f}_{1t} \tag{A.33}$$

$$\alpha\lambda_{2t}^{f} = LS_t \tag{A.34}$$

$$R_{t-1}^{AS} + R_t^{YS}(1+s_t) \quad 1 = R_t^{AS}(1+s_t) \tag{A.35}$$

$$y_t = \gamma_t + \alpha l_t \tag{A.36}$$

$$\frac{(1+p_t)(1+s_t)}{(1+i_t)} R_{t-1}^{PSI} = R_t^{PSI} \tag{A.37}$$

$$\frac{(1+y_t)}{(1+s_t)} R_{t-1}^{YS} = R_t^{YS} \tag{A.38}$$

$$\frac{(1+\omega_t)(1+l_t)}{(1+p_t)(1+y_t)} LS_{t-1} = LS_t \tag{A.39}$$

where $\Omega_{1t} = I_t \quad {}^{h}_{1t}$, $\Omega_{2t} = {}^{I_t} \quad {}^{h}_{2t}/P_t$, and $\Omega_{3t} = I_t \quad {}^{h}_{3t}$. There are thus sixteen equations in sixteen endogenous variables: R_t^{CI}, R_t^{PSI}, R_t^{PXI}, R_t^{BI}, R_t^{AS}, R_t^{YS}, LS_t, p_t, s_t, y_t, l_t, Ω_{1t}, Ω_{2t}, Ω_{3t}, ${}^{f}_{1t}$, and ${}^{f}_{2t}$.

A.5 Reduced Model: Eliminating Labor Input and Productivity Growth

To eliminate labor input, solve equation (A.36) for l_t to get

$$l_t = \frac{1}{\alpha}(y_t \quad \gamma_t). \tag{A.40}$$

Substitute equation (A.40) into equation (A.39) to get

$$LS_t = \frac{(1+\omega_t)\ \ 1+\frac{1}{\alpha}y_t \quad \frac{1}{\alpha}\gamma_t}{(1+p_t)\,(1+y_t)} LS_{t-1}. \tag{A.41}$$

Now, solve equation (A.41) for γ_t to get

$$\gamma_t = \alpha + y_t \quad \alpha\frac{LS_t}{LS_{t-1}}\frac{(1+p_t)\,(1+y_t)}{(1+\omega_t)}. \tag{A.42}$$

Lagging equation (A.42), multiplying the resulting equation by ρ ,and subtracting that equation from equation (A.42) yields

$$\begin{aligned}\gamma_t \quad \rho\ \gamma_{t-1} = \alpha\,(1 \quad \rho\ ) + y_t \quad \rho\ y_{t-1} \quad \alpha\frac{LS_t}{LS_{t-1}}\frac{(1+p_t)\,(1+y_t)}{(1+\omega_t)} \\ + \rho\ \ \alpha\frac{LS_{t-1}}{LS_{t-2}}\frac{(1+p_{t-1})\,(1+y_{t-1})}{(1+\omega_{t-1})}.\end{aligned} \tag{A.43}$$

But, from equation (39) in the paper,

$$\gamma_t \quad \rho\ \gamma_{t-1} = \bar{\gamma}(1 \quad \rho\ ) + \eta_t\,. \tag{A.44}$$

Substituting equation (A.44) into equation (A.43) and rearranging terms yields

$$\begin{aligned}y_t = (\overline{\gamma} \quad \alpha)\,(1 \quad \rho\ ) + \rho\ y_{t-1} + \alpha\frac{(1+p_t)\,(1+y_t)}{(1+\omega_t)}\left(\frac{LS_t}{LS_{t-1}}\right) \\ \alpha\rho\ \ \frac{(1+p_{t-1})\,(1+y_{t-1})}{(1+\omega_{t-1})}\left(\frac{LS_{t-1}}{LS_{t-2}}\right) + \eta_t\,.\end{aligned} \tag{A.45}$$

Having eliminated l_t, the model reduces to fifteen equations in fifteen endogenous variables.

A.6 Steady State: Market Equilibrium

The steady state of the model is

$$\pi_1 \left(\overline{R}^{CI}\right)^{-1} = \overline{\Omega}_1 \tag{A.46}$$

$$(1 \quad)\zeta\frac{(1+\overline{p})}{1+\overline{i}}\overline{\Omega}_2 + \pi_2 \left(\overline{R}^{PXI}\right)^{-1} = \overline{\Omega}_2 \tag{A.47}$$

$$\overline{\Omega}_2 \quad 1 \quad \overline{\xi} \;\; \overline{\Omega}_1 + \overline{\xi}\overline{\Omega}_3 = 0 \tag{A.48}$$

$$\zeta\left(\frac{1 \quad \mu}{1+\overline{i}}\right)\overline{\Omega}_3 \quad \zeta\left(\frac{\overline{r}_1+\mu}{1+\overline{i}}\right)\overline{\Omega}_1 = \overline{\Omega}_3 \tag{A.49}$$

$$\overline{R}^{CI} + \;\; 1 \quad \overline{\xi} \;\; \overline{R}^{PSI} + \left(\frac{\overline{r}_1+\mu}{1+\overline{i}}\right)\overline{R}^{BI} = 1 \tag{A.50}$$

$$(1 \quad)\frac{(1+\overline{p})}{1+\overline{i}}\overline{R}^{PXI} + \overline{R}^{PSI} = \overline{R}^{PXI} \tag{A.51}$$

$$\frac{(1 \quad \mu)}{1+\overline{i}}\overline{R}^{BI} + \overline{\xi}\overline{R}^{PSI} = \overline{R}^{BI} \tag{A.52}$$

$$\left(\frac{\varepsilon \quad 1}{\varepsilon}\right) + \kappa_o\kappa_1 \left(\overline{R}^{AS}\right)^{\kappa_1}\left(\overline{R}^{AS} \qquad \overline{R}^{YS}\right) = \overline{\beta}\,\overline{}^{f}_{1} \tag{A.53}$$

$$\kappa_o \left(\overline{R}^{AS}\right)^{\kappa_1} + \overline{}^{-f}_{1} = \overline{}^{-f}_{2} \tag{A.54}$$

$$\overline{\beta}\,\overline{}^{f}_{1} + \left(\frac{\theta}{\varepsilon}\right)\left(\overline{R}^{AS}\right)^{-1}$$

$$\kappa_o \left(\overline{R}^{AS}\right)^{\kappa_1}\left[\kappa_1 \left(\overline{R}^{AS}\right)^{-1}\left(\overline{R}^{AS} \qquad \overline{R}^{YS}\right) + 1 = \overline{}^{-f}_{1}\right. \tag{A.55}$$

$$\alpha\,\overline{}^{-f}_{2} = \overline{LS} \tag{A.56}$$

$$\overline{R}^{AS} + \overline{R}^{YS}(1+\overline{s}) \quad 1 = \overline{R}^{AS}(1+\overline{s}) \tag{A.57}$$

$$\overline{p} + \overline{s} = \overline{i} \tag{A.58}$$

$$\overline{y} = \overline{s} \tag{A.59}$$

$$\frac{\overline{\gamma} \quad \alpha}{1 \quad \alpha} + \left(\frac{\alpha}{1 \quad \alpha}\right)(\overline{p} \quad \overline{\omega}) = \bar{y} \tag{A.60}$$

where $\overline{\beta} = {}^{1}\!/_{1+\overline{r}_2}$. The steady state now consists of fifteen equations in fifteen endogenous variables: $\overline{R}^{CI}$, $\overline{R}^{PSI}$, $\overline{R}^{PXI}$, $\overline{R}^{BI}$, $\overline{R}^{AS}$, $\overline{R}^{YS}$, $\overline{LS}$, $\overline{p}$, $\overline{s}$, $\overline{y}$, $\overline{\Omega}_1$, $\overline{\Omega}_2$, $\overline{\Omega}_3$, $\overline{}^{f}_{1}$ and $\overline{}^{-f}_{2}$.

A.7 Second-Order Conditions in Symemetric Market Equilibrium

The second-order condition for a maximum is that the bordered Hessian matirx associated with the Lagrangian must be negative definite. Define

$$f^1(C_t, S_t) = C_t + \quad 1 \quad \overline{\xi} \quad P_t S_t + (r_{1t} + \mu) B_{t-1} \quad I_t \tag{A.61}$$

$$f^2(X_t, S_t) = (1 \quad)X_{t-1} + S_t \quad X_t \tag{A.62}$$

$$f^3(B_t, S_t) = (1 \quad \mu) B_{t-1} + \overline{\xi} P_t S_t \quad B_t \tag{A.63}$$

Then the relevant bordered Hessian matrix is:

$$\begin{bmatrix} 0 & 0 & 0 & \frac{\partial f^1}{\partial C_t} & \frac{\partial f^1}{\partial X_t} & \frac{\partial f^1}{\partial S_t} & \frac{\partial f^1}{\partial B_t} \\ 0 & 0 & 0 & \frac{\partial f^2}{\partial C_t} & \frac{\partial f^2}{\partial X_t} & \frac{\partial f^2}{\partial S_t} & \frac{\partial f^2}{\partial B_t} \\ 0 & 0 & 0 & \frac{\partial f^3}{\partial C_t} & \frac{\partial f^3}{\partial X_t} & \frac{\partial f^3}{\partial S_t} & \frac{\partial f^3}{\partial B_t} \\ \frac{\partial f^1}{\partial C_t} & \frac{\partial f^2}{\partial C_t} & \frac{\partial f^3}{\partial C_t} & \frac{\partial^2 \mathcal{L}^{\mathcal{H}}}{\partial C_t^2} & \frac{\partial^2 \mathcal{L}^{\mathcal{H}}}{\partial C_t \partial X_t} & \frac{\partial^2 \mathcal{L}^{\mathcal{H}}}{\partial C_t \partial S_t} & \frac{\partial^2 \mathcal{L}^{\mathcal{H}}}{\partial C_t \partial B_t} \\ \frac{\partial f^1}{\partial X_t} & \frac{\partial f^2}{\partial X_t} & \frac{\partial f^3}{\partial X_t} & \frac{\partial^2 \mathcal{L}^{\mathcal{H}}}{\partial X_t \partial C_t} & \frac{\partial^2 \mathcal{L}^{\mathcal{H}}}{\partial X_t^2} & \frac{\partial^2 \mathcal{L}^{\mathcal{H}}}{\partial X_t \partial S_t} & \frac{\partial^2 \mathcal{L}^{\mathcal{H}}}{\partial X_t \partial B_t} \\ \frac{\partial f^1}{\partial S_t} & \frac{\partial f^2}{\partial S_t} & \frac{\partial f^3}{\partial S_t} & \frac{\partial^2 \mathcal{L}^{\mathcal{H}}}{\partial S_t \partial C_t} & \frac{\partial^2 \mathcal{L}^{\mathcal{H}}}{\partial S_t \partial X_t} & \frac{\partial^2 \mathcal{L}^{\mathcal{H}}}{\partial S_t^2} & \frac{\partial^2 \mathcal{L}^{\mathcal{H}}}{\partial S_t \partial B_t} \\ \frac{\partial f^1}{\partial B_t} & \frac{\partial f^2}{\partial B_t} & \frac{\partial f^3}{\partial B_t} & \frac{\partial^2 \mathcal{L}^{\mathcal{H}}}{\partial B_t \partial C_t} & \frac{\partial^2 \mathcal{L}^{\mathcal{H}}}{\partial B_t \partial X_t} & \frac{\partial^2 \mathcal{L}^{\mathcal{H}}}{\partial B_t \partial S_t} & \frac{\partial^2 \mathcal{L}^{\mathcal{H}}}{\partial B_t^2} \end{bmatrix}$$

The second-order conditions for the household in market equilibrium stated in ratio form are

$$\frac{\partial^2 \mathcal{L}^{\mathcal{H}}}{\partial C_t^2} = \frac{I_t^2}{I_t^2} \frac{\pi_1}{C_t^2} = \frac{\pi_1}{I_t^2} \left[R_t^{CI}\right]^{-2} \tag{A.64}$$

$$\frac{\partial^2 \mathcal{L}^{\mathcal{H}}}{\partial X_t^2} = \frac{\pi_2}{X_t^2} = \frac{P_t^2}{P_t^2} \frac{I_t^2}{I_t^2} \frac{\pi_2}{X_t^2} = \pi_2 \frac{P_t^2}{I_t^2} \left[R_t^{PXI}\right]^{-2} \tag{A.65}$$

$$\frac{\partial^2 \mathcal{L}^{\mathcal{H}}}{\partial S_t^2} = 0 \tag{A.66}$$

$$\frac{\partial^2 \mathcal{L}^{\mathcal{H}}}{\partial B_t^2} = 0 \tag{A.67}$$

$$\frac{\partial^2 \mathcal{L}^{\mathcal{H}}}{\partial C_t \partial X_t} = \frac{\partial^2 \mathcal{L}^{\mathcal{H}}}{\partial C_t \partial S_t} = \frac{\partial^2 \mathcal{L}^{\mathcal{H}}}{\partial C_t \partial B_t} = 0 \tag{A.68}$$

$$\frac{\partial^2 \mathcal{L}^{\mathcal{H}}}{\partial X_t \partial C_t} = \frac{\partial^2 \mathcal{L}^{\mathcal{H}}}{\partial X_t \partial S_t} = \frac{\partial^2 \mathcal{L}^{\mathcal{H}}}{\partial X_t \partial B_t} = 0 \tag{A.69}$$

$$\frac{\partial^2 \mathcal{L}^{\mathcal{H}}}{\partial S_t \partial C_t} = \frac{\partial^2 \mathcal{L}^{\mathcal{H}}}{\partial S_t \partial X_t} = \frac{\partial^2 \mathcal{L}^{\mathcal{H}}}{\partial S_t \partial B_t} = 0 \tag{A.70}$$

$$\frac{\partial^2 \mathcal{L}^{\mathcal{H}}}{\partial B \partial C_t} = \frac{\partial^2 \mathcal{L}^{\mathcal{H}}}{\partial B \partial X_t} = \frac{\partial^2 \mathcal{L}^{\mathcal{H}}}{\partial B \partial S_t} = 0 \tag{A.71}$$

$$\frac{\partial f^1}{\partial S_t} = \quad 1 \quad \overline{\xi} \quad P_t = \quad 1 \quad \overline{\xi} \quad \left(\frac{P_t S_t}{I_t}\right)\left(\frac{A_t}{S_t}\right)\left(\frac{I_t}{A_t}\right) = \quad 1 \quad \overline{\xi} \quad R_t^{PSI} R_t^{AS} \quad R_t^{AI} \quad ^{-1} \tag{A.72}$$

$$\frac{\partial f^1}{\partial C_t} = 1 \quad \frac{\partial f^1}{\partial X_t} = \frac{\partial f^1}{\partial B_t} = 0 \tag{A.73}$$

$$\frac{\partial f^2}{\partial S_t} = 1 \quad \frac{\partial f^2}{\partial X_t} = \quad 1 \quad \frac{\partial f^2}{\partial C_t} = \frac{\partial f^2}{\partial B_t} = 0 \tag{A.74}$$

$$\frac{\partial f^3}{\partial S_t} = \overline{\xi} P_t = \overline{\xi}\left(\frac{P_t S_t}{I_t}\right)\left(\frac{A_t}{S_t}\right)\left(\frac{I_t}{A_t}\right) = \overline{\xi} R_t^{PSI} R_t^{AS} \quad R_t^{AI} \quad ^{-1} \tag{A.75}$$

$$\frac{\partial f^3}{\partial B_t} = \quad 1 \quad \frac{\partial f^3}{\partial C_t} = \frac{\partial f^3}{\partial X_t} = 0 \tag{A.76}$$

For the firm, assume in market equilibrium that $S_{jt} = S_t$, $A_{jt} = A_t$, $L_{jt} = L_t$, $Y_{jt} = Y_t$, and $_{jt} = \ _t$.and using the ratios, $R_t^{AS} = {}^{A_t}/_{S_t}$ $R_t^{YS} = {}^{Y_t}/_{S_t}$ $LS_t = {}^{W_t L_t}/_{P_t Y_t}$. The second-order condition is that the bordered Hessian matirx associated with the Lagrangian must be negative definite. Define

$$g^1(Y_{jt}, A_{jt}) \quad = \quad A_{jt-1} \quad S_{jt-1} + Y_{jt} \quad A_{jt} \tag{A.77}$$

$$g^2(Y_{jt}, L_{jt}) \quad = \quad _{jt} L_{jt}^{\alpha} \quad Y_{jt} \tag{A.78}$$

Then the relevant bordered Hessian matrix is:

$$\begin{bmatrix} 0 & 0 & \frac{\partial g^1}{\partial S_{jt}} & \frac{\partial g^1}{\partial Y_{jt}} & \frac{\partial g^1}{\partial A_{jt}} & \frac{\partial g^1}{\partial L_{jt}} \\ 0 & 0 & \frac{\partial g^2}{\partial S_{jt}} & \frac{\partial g^2}{\partial Y_{jt}} & \frac{\partial g^2}{\partial A_{jt}} & \frac{\partial g^2}{\partial L_{jt}} \\ \frac{\partial g^1}{\partial S_{jt}} & \frac{\partial g^2}{\partial S_{jt}} & \frac{\partial^2 \mathcal{L}^F}{\partial S_{jt}^2} & \frac{\partial^2 \mathcal{L}^F}{\partial S_{jt} \partial Y_{jt}} & \frac{\partial^2 \mathcal{L}^F}{\partial S_{jt} A_{jt}} & \frac{\partial^2 \mathcal{L}^F}{\partial S_{jt} \partial L_{jt}} \\ \frac{\partial g^1}{\partial Y_{jt}} & \frac{\partial g^2}{\partial Y_{jt}} & \frac{\partial^2 \mathcal{L}^F}{\partial Y_{jt} \partial S_{jt}} & \frac{\partial^2 \mathcal{L}^F}{\partial Y_{jt}^2} & \frac{\partial^2 \mathcal{L}^F}{\partial Y_{jt} A_{jt}} & \frac{\partial^2 \mathcal{L}^F}{\partial Y_{jt} L_{jt}} \\ \frac{\partial g^1}{\partial A_{jt}} & \frac{\partial g^2}{\partial A_{jt}} & \frac{\partial^2 \mathcal{L}^F}{\partial A_{jt} S_{jt}} & \frac{\partial^2 \mathcal{L}^F}{\partial A_{jt} Y_{jt}} & \frac{\partial^2 \mathcal{L}^{\mathcal{F}}}{\partial A_{jt}^2} & \frac{\partial^2 \mathcal{L}^{\mathcal{F}}}{\partial A_{jt} L_{jt}} \\ \frac{\partial g^1}{\partial L_{jt}} & \frac{\partial g^2}{\partial L_{jt}} & \frac{\partial^2 \mathcal{L}^F}{\partial L_{jt} S_{jt}} & \frac{\partial^2 \mathcal{L}^{\mathcal{F}}}{\partial L_{jt} Y_{jt}} & \frac{\partial^2 \mathcal{L}^{\mathcal{F}}}{\partial L_{jt} A_{jt}} & \frac{\partial^2 \mathcal{L}^F}{\partial L_{jt}^2} \end{bmatrix}$$

Then the second-order conditions for the firm in market equilibrium stated in ratio form are:

$$\begin{aligned}\frac{\partial^2 \mathcal{L}^{\mathcal{F}}}{\partial S_t^2} &= \frac{1}{S_t}\left[\frac{1}{\varepsilon}\left(\frac{\varepsilon \quad 1}{\varepsilon}\right) + \kappa_o\kappa_1\left(\frac{A_t}{S_t}\right)^{\kappa_1}\left[\left(\frac{A_{t.}}{S_t}\right) \quad \left(\frac{Y_t}{S_t}\right)\right](1+\kappa_1)\right. \qquad \text{(A.79)}\\ &= \frac{1}{S_t}\left[\frac{1}{\varepsilon}\left(\frac{\varepsilon \quad 1}{\varepsilon}\right) + \kappa_o\kappa_1(1+\kappa_1) \;\; R_t^{AS\;\;\kappa_1}\left[\left(R_t^{AS} \qquad R_t^{YS}\right)\right]\right.\end{aligned}$$

$$\frac{\partial^2 \mathcal{L}^{F}}{\partial Y_t^2} = 0 \qquad \text{(A.80)}$$

$$\begin{aligned}\frac{\partial^2 \mathcal{L}^{\mathcal{F}}}{\partial A_t^2} &= \frac{1}{A_t}\left[\left(\frac{\theta}{\varepsilon} \quad 1\right)\left(\frac{\theta}{\varepsilon}\right)\left(\frac{S_t}{A_t}\right) \quad \kappa_o\kappa_1\left(\frac{A_t}{S_t}\right)^{\kappa_1}\left[1+\kappa_1+\frac{Y_t}{A_t}(1 \quad \kappa_1)\right.\right. \qquad \text{(A.81)}\\ &= \frac{1}{S_t}\left(\frac{S_t}{A_t}\right)\left[\left(\frac{\theta}{\varepsilon} \quad 1\right)\left(\frac{\theta}{\varepsilon}\right)\left(\frac{S_t}{A_t}\right) \quad \kappa_o\kappa_1\left(\frac{A_t}{S_t}\right)^{\kappa_1}\left[1+\kappa_1+\left(\frac{Y_t}{S_t}\right)\left(\frac{S_t}{A_t}\right)(1 \quad \kappa_1)\right.\right.\\ &= \frac{1}{S_t} \;\; R_t^{AS\;\;-1}\left[\left(\frac{\theta}{\varepsilon} \quad 1\right)\left(\frac{\theta}{\varepsilon}\right) \;\; R_t^{AS\;\;-1} \quad \kappa_o\kappa_1 \;\; R_t^{AS\;\;\kappa_1}\left[1+\kappa_1+R_t^{YS} \;\; R_t^{AS\;\;-1}(1 \quad \kappa_1)\right]\right.\end{aligned}$$

$$\frac{\partial^2 \mathcal{L}^{\mathcal{F}}}{\partial L_t^2} = (\alpha \quad 1)\,\alpha\lambda_{2t}^f \;\; {}_tL_t^{\alpha-2} = (\alpha \quad 1)\,\alpha\lambda_{2t}^f\frac{Y_t}{L_t}\frac{1}{L_t} = (\alpha \quad 1)\,LS_tR_t^{YL}\frac{1}{L_t} \qquad \text{(A.82)}$$

where we have used the result that $\alpha\lambda_{2t}^f = {}_{W_tL_t}/_{P_tY_t} = LS_t$ from the first-order conditions.

$$\frac{\partial^2 \mathcal{L}^{\mathcal{F}}}{\partial S_t \partial Y_t} = \quad \kappa_o\kappa_1\left(\frac{A_t}{S_t}\right)^{\kappa_1}\left(\frac{1}{S_t}\right) = \quad \kappa_o\kappa_1 \;\; R_t^{AS\;\;\kappa_1}\left(\frac{1}{S_t}\right) \qquad \text{(A.83)}$$

$$\begin{aligned}\frac{\partial^2 \mathcal{L}^{\mathcal{F}}}{\partial S_t \partial A_t} &= \left(\frac{1}{A_t}\right)\left[\frac{\theta}{\varepsilon}\left(\frac{\varepsilon \quad 1}{\varepsilon}\right) + \kappa_o\kappa_1\left(\frac{A_t}{S_t}\right)^{\kappa_1+1}\left[\kappa_1\left(\frac{A_t}{S_t}\right)^{-1}\left(\frac{A_{t.} \quad Y_t}{A_t}\right)+1\right]\right] \qquad \text{(A.84)}\\ &= \frac{1}{S_t}\left(\frac{S_t}{A_t}\right)\left[\frac{\theta}{\varepsilon}\left(\frac{\varepsilon \quad 1}{\varepsilon}\right) + \kappa_o\kappa_1\left(\frac{A_t}{S_t}\right)^{\kappa_1+1}\left[\kappa_1\left(\frac{A_t}{S_t}\right)^{-1}\left(1 \quad \frac{Y_t}{A_t}\right)+1\right]\right]\\ &= \frac{1}{S_t}\left(\frac{S_t}{A_t}\right)\left[\frac{\theta}{\varepsilon}\left(\frac{\varepsilon \quad 1}{\varepsilon}\right) + \kappa_o\kappa_1\left(\frac{A_t}{S_t}\right)^{\kappa_1+1}\left[\kappa_1\left(\frac{A_t}{S_t}\right)^{-1}\left(1 \quad \frac{Y_t}{S_t}\frac{S_t}{A_t}\right)+1\right]\right]\\ &= \frac{1}{S_t} \;\; R_t^{AS\;\;-1}\left[\frac{\theta}{\varepsilon}\left(\frac{\varepsilon \quad 1}{\varepsilon}\right) + \kappa_o\kappa_1 \;\; R_t^{AS\;\;\kappa_1+1}\left[\kappa_1 \;\; R_t^{AS\;\;-1}\left(1 \quad R_t^{YS} \;\; R_t^{AS\;\;-1}\right)+1\right]\right.\end{aligned}$$

$$\frac{\partial^2 \mathcal{L}^{\mathcal{F}}}{\partial S_t \partial L_t} = 0 \qquad \text{(A.85)}$$

$$\frac{\partial^2 \mathcal{L}^{\mathcal{F}}}{\partial Y_t A_t} = \kappa_1\kappa_o\left(\frac{A_t}{S_t}\right)^{\kappa_1-1}\left(\frac{1}{S_t}\right) = \kappa_1\kappa_o \;\; R_t^{AS\;\;\kappa_1-1}\left(\frac{1}{S_t}\right) \qquad \text{(A.86)}$$

$$\frac{\partial^2 \mathcal{L}^{\mathcal{F}}}{\partial Y_t L_t} = 0 \tag{A.87}$$

$$\frac{\partial^2 \mathcal{L}^{\mathcal{F}}}{\partial A_t L_t} = 0 \tag{A.88}$$

$$\frac{\partial g^1}{\partial Y_t} = 1 \quad \frac{\partial g^1}{\partial A_t} = \quad 1 \quad \frac{\partial g^1}{\partial S_t} = \frac{\partial g^1}{\partial L_t} = 0 \tag{A.89}$$

$$\frac{\partial g^2}{\partial L_{jt}} = \alpha \;\; {}_{jt} L_{jt}^{\alpha-1} = \alpha \frac{Y_t}{L_t} = \alpha R_t^{YL} > 0 \quad \frac{\partial g^2}{\partial Y_{jt}} = \quad 1 \quad \frac{\partial g^2}{\partial S_{jt}} = \frac{\partial g^2}{\partial A_{jt}} = 0 \tag{A.90}$$

The second-order conditions for the household in steady state are then

$$\frac{\partial^2 \mathcal{L}^{\mathcal{H}}}{\partial C_t^2} = \quad \frac{\pi_1}{\overline{I}^2} \left[\bar{R}^{CI}\right]^{-2} \tag{A.91}$$

$$\frac{\partial^2 \mathcal{L}^{\mathcal{H}}}{\partial X_t^2} = \quad \pi_2 \frac{\overline{P}^2}{\overline{I}^2} \left[\bar{R}^{PXI}\right]^{-2} \tag{A.92}$$

$$\frac{\partial^2 \mathcal{L}^{\mathcal{H}}}{\partial S_t^2} = 0 \tag{A.93}$$

$$\frac{\partial^2 \mathcal{L}^{\mathcal{H}}}{\partial B_t^2} = 0 \tag{A.94}$$

$$\frac{\partial^2 \mathcal{L}^{\mathcal{H}}}{\partial C_t \partial X_t} = \frac{\partial^2 \mathcal{L}^{\mathcal{H}}}{\partial C_t \partial S_t} = \frac{\partial^2 \mathcal{L}^{\mathcal{H}}}{\partial C_t \partial B_t} = 0 \tag{A.95}$$

$$\frac{\partial^2 \mathcal{L}^{\mathcal{H}}}{\partial X_t \partial C_t} = \frac{\partial^2 \mathcal{L}^{\mathcal{H}}}{\partial X_t \partial S_t} = \frac{\partial^2 \mathcal{L}^{\mathcal{H}}}{\partial X_t \partial B_t} = 0 \tag{A.96}$$

$$\frac{\partial^2 \mathcal{L}^{\mathcal{H}}}{\partial S_t \partial C_t} = \frac{\partial^2 \mathcal{L}^{\mathcal{H}}}{\partial S_t \partial X_t} = \frac{\partial^2 \mathcal{L}^{\mathcal{H}}}{\partial S_t \partial B_t} = 0 \tag{A.97}$$

$$\frac{\partial^2 \mathcal{L}^{\mathcal{H}}}{\partial B \partial C_t} = \frac{\partial^2 \mathcal{L}^{\mathcal{H}}}{\partial B \partial X_t} = \frac{\partial^2 \mathcal{L}^{\mathcal{H}}}{\partial B \partial S_t} = 0 \tag{A.98}$$

$$\frac{\partial f^1}{\partial S_t} = \;\; 1 \quad \bar{\xi} \;\; \overline{R}^{PSI} \overline{R}^{AS} \left(\overline{R}^{AI}\right)^{-1} \tag{A.99}$$

$$\frac{\partial f^1}{\partial C_t} = 1 \qquad \frac{\partial f^1}{\partial X_t} = \frac{\partial f^1}{\partial B_t} = 0 \tag{A.100}$$

$$\frac{\partial f^2}{\partial S_t} = 1 \quad \frac{\partial f^2}{\partial X_t} = \quad 1 \quad \frac{\partial f^2}{\partial C_t} = \frac{\partial f^2}{\partial B_t} = 0 \tag{A.101}$$

$$\frac{\partial f^3}{\partial S_t} = \overline{\xi}\overline{R}^{PSI}\overline{R}^{AS}\left(\overline{R}^{AI}\right)^{-1} \quad \frac{\partial f^3}{\partial B_t} = \quad 1 \quad \frac{\partial f^3}{\partial C_t} = \frac{\partial f^3}{\partial X_t} = 0 \tag{A.102}$$

For the firm, the second-order conditions in steady state are:

$$\frac{\partial^2 \mathcal{L}^{\mathcal{F}}}{\partial S_t^2} = \quad \frac{1}{\overline{\overline{S}}}\left[\frac{1}{\varepsilon}\left(\frac{\varepsilon \quad 1}{\varepsilon}\right) + \kappa_o\kappa_1\left(\overline{R}^{AS}\right)^{\kappa_1}\left[\left(\overline{R}^{AS}\right) \quad \left(\overline{R}^{YS}\right)\right](1+\kappa_1)\right. \tag{A.103}$$

$$\frac{\partial^2 \mathcal{L}^{F}}{\partial Y_t^2} = 0 \tag{A.104}$$

$$\frac{\partial^2 \mathcal{L}^{\mathcal{F}}}{\partial A_t^2} = \frac{1}{\overline{\overline{S}}}\left(\overline{R}^{AS}\right)^{-1}\left[\left(\frac{\theta}{\varepsilon} \quad 1\right)\left(\frac{\theta}{\varepsilon}\right)\left(\overline{R}^{AS}\right)^{-1}\right.$$
$$\left.\kappa_o\kappa_1\left(\overline{R}^{AS}\right)^{\kappa_1}\left[1+\kappa_1+\overline{R}^{YS}\left(\overline{R}^{AS}\right)^{-1}(1 \quad \kappa_1)\right]\right] \tag{A.105}$$

$$\frac{\partial^2 \mathcal{L}^{\mathcal{F}}}{\partial L_t^2} = (\alpha \quad 1)\,\overline{LS\overline{R}}^{YL}\frac{1}{\overline{\overline{L}}} \tag{A.106}$$

$$\frac{\partial^2 \mathcal{L}^{\mathcal{F}}}{\partial S_t \partial Y_t} = \quad \kappa_o\kappa_1\left(\overline{R}^{AS}\right)^{\kappa_1}\left(\frac{1}{\overline{\overline{S}}}\right) \tag{A.107}$$

$$\frac{\partial^2 \mathcal{L}^{\mathcal{F}}}{\partial S_t \partial A_t} = \left(\frac{1}{\overline{\overline{S}}}\right)\left(\overline{R}^{AS}\right)^{-1}\left[\frac{\theta}{\varepsilon}\left(\frac{\varepsilon \quad 1}{\varepsilon}\right)\right.$$
$$\left.+\,\kappa_o\kappa_1\left(\overline{R}^{AS}\right)^{\kappa_{1+1}}\left[\kappa_1\left(\overline{R}^{AS}\right)^{-1}\left(1 \quad \overline{R}^{YS}\left(\overline{R}^{AS}\right)^{-1}\right)+1\right]\right. \tag{A.108}$$

$$\frac{\partial^2 \mathcal{L}^{\mathcal{F}}}{\partial S_t \partial L_t} = 0 \tag{A.109}$$

$$\frac{\partial^2 \mathcal{L}^{\mathcal{F}}}{\partial Y_t A_t} = \kappa_1\kappa_o\left(\overline{R}^{AS}\right)^{\kappa_1 \quad 1}\left(\frac{1}{\overline{\overline{S}}}\right) \tag{A.110}$$

$$\frac{\partial^2 \mathcal{L}^{\mathcal{F}}}{\partial Y_t L_t} = 0 \tag{A.111}$$

$$\frac{\partial^2 \mathcal{L}^{\mathcal{F}}}{\partial A_t L_t} = 0 \tag{A.112}$$

$$\frac{\partial g^1}{\partial Y_t} = 1 \qquad \frac{\partial g^1}{\partial A_t} = \quad 1 \qquad \frac{\partial g^1}{\partial S_t} = \frac{\partial g^1}{\partial L_t} = 0 \tag{A.113}$$

$$\frac{\partial g^2}{\partial L_{jt}} = \alpha \overline{R}^{YL} > 0 \qquad \frac{\partial g^2}{\partial Y_{jt}} = \quad 1 \qquad \frac{\partial g^2}{\partial S_{jt}} = \frac{\partial g^2}{\partial A_{jt}} = 0 \tag{A.114}$$

Because of the presence of the trending variables, S_t, P_t, B_t, I_t, and C_t, the bordered Hessians are not stationary even with the model in ratio form. Thus we evaluated the bordered Hessians using the estimated parameters, the relevant steady-state ratios, and values for the trending variables that are consistent with the steady state. In these cases, the bordered Hessian matrices were negative definite, satisfying a su cent condition for an interior maximum.

B DATA DETAILS

In appendix B we provide more information on the construction of the data, followed by analysis of some of the key variables.

B.1 Data Construction

To construct the timeseries of domestic sales, domestic output, domestic inventories, and average price for light trucks, some work is required. The BEA publishes domestic light trucks sales from 1972 onward, but not domestic production nor domestic inventories. From 1985 onward, Ward's Automotive provides detailed data on light truck inventories for the United States. Although this measure includes some foreign light truck inventories, we use this measure as an approximation of domestic inventories. The key assumption behind our approximation is that changes in Ward's inventories reflect mainly changes in domestic inventories, a reasonable conjecture given that light trucks are dominated by U.S. manufacturers.

Because Ward's data reaches back only to 1985, we use a different technique to approximate domestic inventories from 1972 to 1985. We assume that the days' supply figures for light trucks and automobiles are equal over this period. Days' supply is a measure of the number of days vehicles can be sold at the current rate out of the current stock of inventories. This statistic is often used in the industry as a gauge of whether automakers are holding too many or too few vehicles in inventory. Hence, we are assuming that automakers choose to target the same days' supply figures for automobiles and light trucks. Indeed, from 1985 onward, days' supply for automobiles and light trucks has a correlation of 0.69. An advantage to this approach is that automotive days' supply will pick up macroeconomic shocks and allow us to incorporate them into

our domestic light truck inventory measure. With light truck domestic sales and our estimate of days' supply, we can back out domestic light truck inventories from 1972 to 1985. Finally, we use the time series of domestic sales and inventories to back out domestic light truck production.[2]

We construct the average price for light trucks using quarterly data on personal consumption expenditures and private investment.[3] Before 1987, only investment in trucks is published. We approximate the level of light truck investment from 1972 to 1986 by assuming that light truck investment has the same growth rate as total truck investment over this time period. We then divide the sum of the personal consumption expenditures and private investment by unit sales to arrive at an average price. This average price is for all light trucks, but we assume it is a good approximation of domestic light trucks based on the small market share of foreign light trucks. Because we computed average prices for light trucks at the quarterly frequency, we use linear interpolation to construct a monthly series.

B.2 Additional Analysis of Sales, Production, Inventories, and Price

In looking at the data on the growth rates of sales, production, inventories, and real price, we see the well-known stylized facts that the growth rates of sales and output have a high correlation of 0.64 and are quite volatile over our sample period, with sales being less volatile than output (e.g., see Bresnahan and Ramey 1994). This volatility is illustrated in figure B.1. Comparing sales and output growth with real price growth in figure B.2 highlights the finding that the percentage changes in real prices are substantially smaller than sales and output. Over the sample period, real prices are somewhat positively correlated with sales and output, with correlation coe cients of 0.26 and 0.35, respectively.

Given this paper's focus on interest rates, we consider how prices, sales, and output fluctuate with interest rates in the data. Over our sample period, we find that the percentage changes in firms' real interest rate and in real prices are positively correlated (the correlation coe cient is 0.52), a pattern illustrated in figure B.2. Given the high correlation between firms' and the households' interest rates, it is not a surprise to find the same positive correlation between the percentage changes in households' real interest rate and in real prices. Finally we also find that the percentage changes in firms' interest rate are positively correlated with the percentage changes in sales and output, with correlation coe cients of 0.22 and 0.16, respectively.

In addition to the growth rates of sales, output, and real prices, the model makes predictions about the ratios of output to sales and of available supply to sales. Available supply is equal to

[2]We checked our inferred measure of domestic light truck production against U.S. light truck assemblies. As expected, the correlation between these two series is a high 0.976. Furthermore, domestic production is typically higher than U.S. assemblies; the average difference between monthly domestic light truck production and U.S. assemblies is more than 44,000 units.

[3]Because government investment in light trucks is small and, for most of the time periods we examine, not published separately from medium and heavy truck investment, we do not include it in our average price calculations.

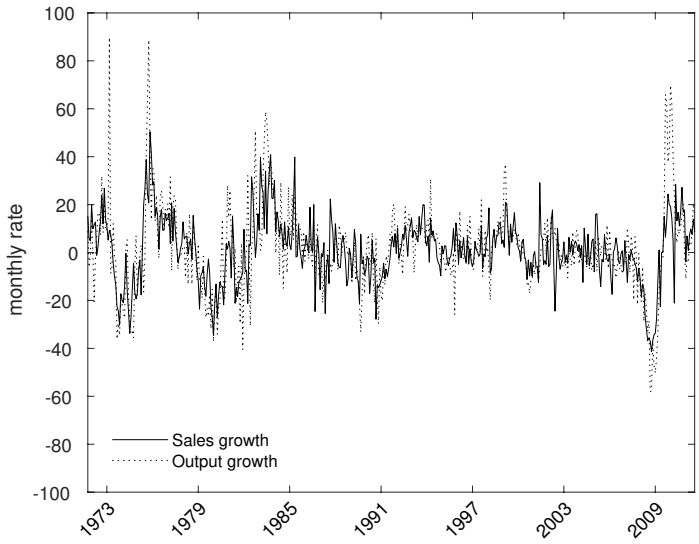

Figure B.1: Growth Rates of Sales and Output, 1972-2011

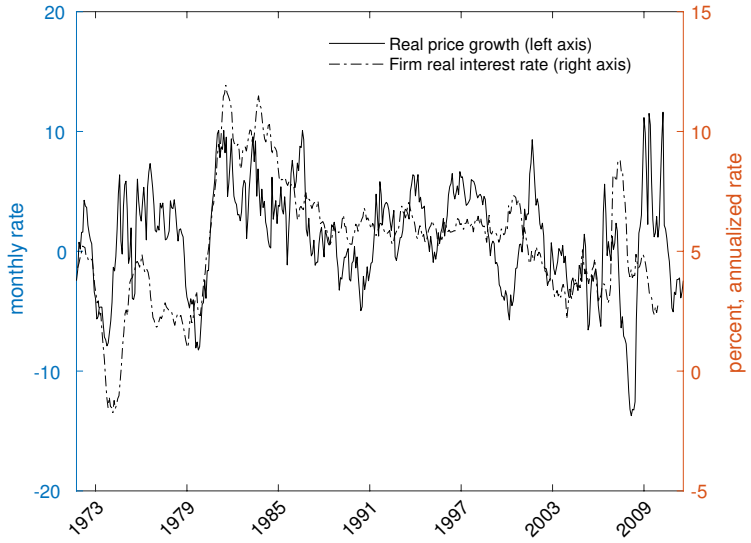

Figure B.2: Firm's Real Interest Rates and Real Prices, 1972-2011

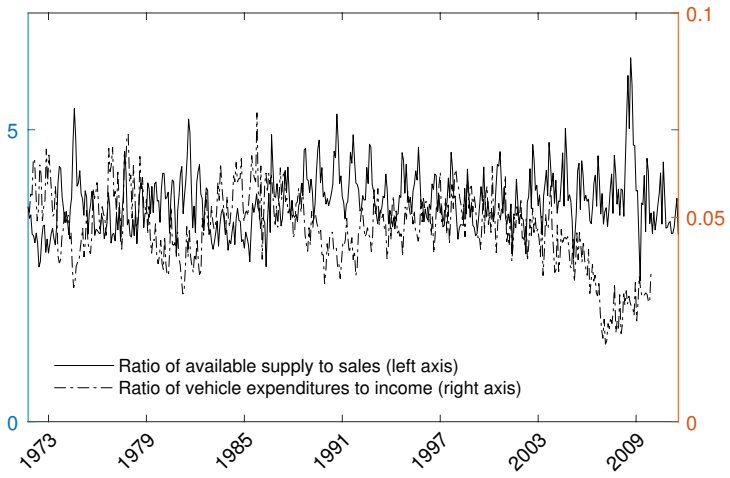

Figure B.3: Ratios of Available Supply to Sales and Light Motor Vehicle Expenditure to Income, 1972-2011

the stock of inventories at the beginning of the month plus that month's output, and the ratio of available supply to sales is commonly used in the industry as a gauge of how well production and sales are aligned. Over the sample period, we find that both ratios are roughly constant, although quite volatile.

B.3 Additional Analysis of Personal Disposable Income and Consumption

Turning to the ratio of light motor vehicle expenditure over income, we find that it fluctuates around 0.05 for most of our sample but starts a gradual decline to 0.03 starting around 2000 (see the dotted line in figure B.3). Not surprisingly, the ratio of non-motor vehicle expenditure to income is roughly the mirror image of the ratio of light vehicle expenditure to income.

In our model, available supply, measured as the beginning-of-period stock of inventories plus current production, plays a large role in both the firm's and the household's problem. In equilibrium, changes in available supply have a number of direct and indirect impacts on prices and sales. To gain a rough sense of how available supply, sales, prices, and income fluctuate together, we consider the comovement between the ratios of available supply to sales and of light vehicle expenditures and income. As illustrated in figure B.3, these two ratios are negatively correlated, with a correlation coe cient of -0.61; although part of the correlation may be mechanical as automobile sales appears in the denominator of one ratio and the numerator of the other. The negative correlation suggests that a main force in the model is that manufacturers use inventories to smooth production, as available supply tends to grow relative to sales when households reduce the amount of income they spend on motor vehicles.

C ADDITIONAL FIGURES

The figure C.1 illustrates the differences in the persistence of innovations to the interest rates. All innovations are chosen to create a 100 basis point increase in the household or firm's interest rate relative to steady-state in month 1. The evolution of the rates after the shock illustrate the difference degrees of persistence for three cases: an idiosyncratic shock to the household's rate, an idiosyncratic shock to the firm's rate, a shock to the common component of interest rates.

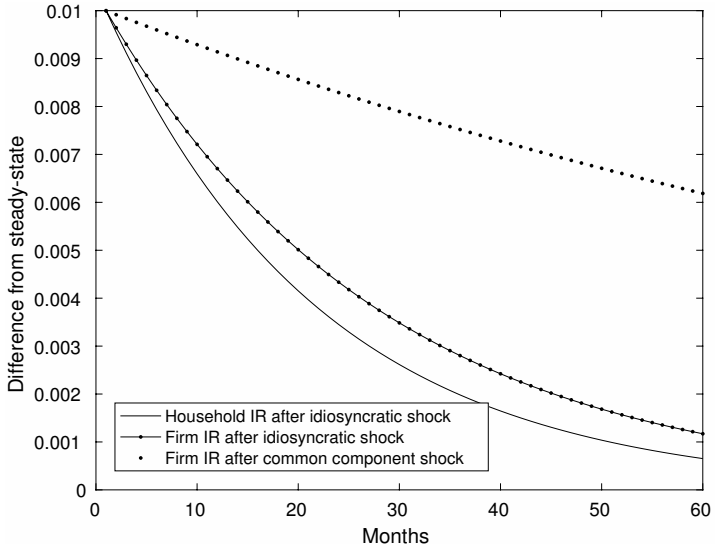

Figure C.1: Persistence of Interest Rate Shocks

References

Bresnahan, Timothy and Valerie Ramey. 1994. "Output Fluctuations at the Plant Level." Quarterly Journal of Economics 109(3), 593-624.