Appendices B and C to accompany

Maccini, Moore, and Schaller (2014) "Inventory Behavior with Permanent Sales Shocks."

APPENDIX B. Derivations and Calibration Method.

This appendix is organized as follows. Section I shows the log-linearization of the Euler equation. Section II derives the decision rule. Section III shows how to derive the cointegrating regression from the Euler equation. Section IV gives the derivation of the analytical conditional variance ratio. Section V shows how we calibrate the model using the cointegrating regression. Sections I through IV are organized in a *Proposition-Proof* format. This is done to make it easy for the reader to locate the derivation of interest. In the text of the paper results are not stated as formal propositions.

I. Log-Linear Euler Equation.

Proposition 1: *Log-linearizing the optimality conditions around steady-state values yields*

$$\begin{aligned} E_{t-1}\Big\{&(\theta_1-1)\theta_1\overline{J}\left[\ln Y_t-\overline{\beta}\ln Y_{t+1}\right]+\theta_2\theta_1\overline{J}\left[\ln W_t-\overline{\beta}\ln W_{t+1}\right] \\ &+\overline{\beta}(\delta_2-1)\delta_2\overline{\psi}\left[\ln N_t-\ln X_{t+1}\right]+\theta_1\overline{J}r_{t+1}+\theta_1\overline{J}\tilde{u}_{t+1}^A+c\Big\}=0 \end{aligned} \tag{10}$$

Proof: The representative firm is assumed to minimize the present discounted value of expected total costs,

$$E_0\sum_{t=1}^{\infty}\left[\prod_{j=1}^{t}\beta_j\right]C_t,$$

where $E_0=E\{.|\Omega_0\}$, and

$$C_t=PC_t+HC_t=A_tY_t^{\theta_1}W_t^{\theta_2}+\delta_1\left(\frac{N_{t-1}}{X_t}\right)^{\delta_2}X_t+\delta_3N_{t-1},$$

subject to the inventory accumulation equation,

$$N_t-N_{t-1}=Y_t-X_t. \tag{4}$$

and to a non-negativity constraint on the stock of inventories,

$$N_t\geq 0\ . \tag{5}$$

Applying the Law of Iterated Expectations, and assuming that the non-negativity constraint on inventories is not binding so that $N_t > 0$, the optimality conditions for this optimization problem reduce to:

$$E_{t-1}\left\{\beta_t\left[\theta_1 A_t Y_t^{\theta_1-1} W_t^{\theta_2} - \xi_t^1\right]\right\} = 0 \qquad (6)$$

$$E_{t-1}\left\{\beta_t\left[\beta_{t+1}\left(\delta_2\delta_1\left(\frac{N_t}{X_{t+1}}\right)^{\delta_2-1} + \delta_3 - \xi_{t+1}^1\right) + \xi_t^1\right]\right\} = 0 \qquad (7')$$

$$E_{t-1}\left\{\beta_t\left[N_t - N_{t-1} - Y_t + X_t\right]\right\} = 0 \qquad (9)$$

where ξ_t^1 is the lagrange multiplier associated with (4).

Define the inventory-sales ratio as $R_{Nt} = \frac{N_t}{X_t}$ and the output-sales ratio as $R_{Yt} = \frac{Y_t}{X_t}$, let lower case letters represent the growth rates of upper case letters so that, for example, $\frac{X_t}{X_{t-1}} = 1 + x_t$, and use the approximation $\frac{1}{1+x_t} \approx 1 - x_t$. Then, define $J_t = \frac{A_t Y_t^{\theta_1} W_t^{\theta_2}}{Y_t}$ as average production cost so that marginal production cost is $\theta_1 J_t$, define

$$\psi_t = \delta_1\left(\frac{N_t}{X_{t+1}}\right)^{\delta_2-1} = \delta_1\left(\frac{N_t}{X_t}\frac{X_t}{X_{t+1}}\right)^{\delta_2-1} = \delta_1\left(\frac{N_t}{X_t}\frac{1}{1+x_{t+1}}\right)^{\delta_2-1} \approx \delta_1\left[R_{Nt}\left(1-x_{t+1}\right)\right]^{\delta_2-1}$$

as average stockout avoidance costs so that marginal stockout avoidance costs are $\delta_2\psi_t$, Then, the optimality conditions, (6) and ($7'$), can be written as

$$E_{t-1}\left\{\beta_t\left[\theta_1 J_t - \xi_t^1\right]\right\} = 0 \qquad \text{(A-1)}$$

$$E_{t-1}\left\{\beta_t\left[\beta_{t+1}\left(\delta_2\delta_1\left(R_{Nt}\left(1-x_{t+1}\right)\right)^{\delta_2-1} + \delta_3 - \xi_{t+1}^1\right) + \xi_t^1\right]\right\} = 0 \qquad \text{(A-2)}$$

Then, divide (9) by X_{t-1} and re-arrange to obtain

$$E_{t-1}\left\{\beta_t\left[\frac{N_t}{X_t}\frac{X_t}{X_{t-1}}-\frac{N_{t-1}}{X_{t-1}}-\frac{Y_t}{X_t}\frac{X_t}{X_{t-1}}+\frac{X_t}{X_{t-1}}\right]\right\}=0$$

or, using the appropriate definitions,

$$E_{t-1}\left\{\beta_t\left[R_{Nt}\left(1+x_t\right)-R_{Nt-1}-R_{Yt}\left(1+x_t\right)+\left(1+x_t\right)\right]\right\}=0 \qquad \text{(A-3)}$$

We assume that the ratios, R_{Nt}, R_{Yt}, J_t, and ψ_t, which are defined above, are stationary with finite expected values. The growth rate of sales, x_t, is also assumed to be stationary.

We assume that in steady state the expected values of the ratios, average production cost, average stockout avoidance cost, and the growth rates of variables are constants. The non-stochastic steady state is defined where shocks are zero, and the inventory sales ratio, the output-sales ratio, average production cost, average stockout avoidance cost, the growth rates, the real interest rate, and the multiplier are constant, so that $R_{Nt}=R_{Nt-1}=\overline{R}_N$, $R_{Yt}=\overline{R}_Y$, $J_t=\overline{J}$, $\psi_t=\overline{\psi}$, $x_{t+1}=x_t=\overline{x}$, $\beta_{t+1}=\beta_t=\overline{\beta}$, and $\xi_{t+1}^1=\xi_t^1=\overline{\xi}^1$. The steady state implied by (A-1)-(A-1) is then

$$\theta_1\overline{J}=\overline{\xi}^1 \qquad \text{(A-4)}$$

$$\overline{\beta}\left(\delta_2\overline{\psi}+\delta_3\right)+\left(1-\overline{\beta}\right)\overline{\xi}^1=0 \qquad \text{(A-5)}$$

$$\overline{x}\overline{R}_N+1=\overline{R}_Y \qquad \text{(A-6)}$$

where a "bar" above a variable denotes a constant expected steady state value and where to derive (A-6) we divide the steady state expression for (A-3) by $1+\overline{x}$ and use the approximation $\overline{R}_N\left(\frac{1}{1+\overline{x}}\right)\approx\overline{R}_N\left(1-\overline{x}\right)$.

On notation, a “^” above an upper case letter denotes a log-deviation from the steady state, while a “^” above a lower case letter denotes a deviation from the steady state growth rate. So, for example, the log-deviation of R_{Nt} from its steady state value is $\hat{R}_{Nt}=\ln R_{Nt}-\ln\overline{R}_N$, while the deviation of the growth rate of sales is $\hat{x}_t=x_t-\overline{x}$. Similar notation applies to other variables. The log-linearized optimality conditions are then

$$\theta_1 \bar{J} E_{t-1} \hat{J}_t - \bar{\xi}^1 E_{t-1} \hat{\xi}_t^1 = 0 \tag{A-7}$$

$$\bar{\xi}^1 E_{t-1} \hat{\xi}_t^1 - \bar{\beta} \bar{\xi}^1 E_{t-1} \hat{\xi}_{t+1}^1 + \bar{\xi}^1 E_{t-1} \hat{r}_{t+1} + \bar{\beta}\left(\delta_2 - 1\right)\delta_2 \overline{\psi} E_{t-1}\left[\hat{R}_{Nt} - \hat{x}_{t+1}\right] = 0 \tag{A-8}$$

$$\bar{R}_N E_{t-1} \hat{R}_{Nt} - \bar{R}_N \hat{R}_{Nt-1} - \bar{R}_Y E_{t-1} \hat{R}_{Yt} + \bar{R}_N E_{t-1} \hat{x}_t = 0 \tag{A-9}$$

where $\bar{\beta} = \dfrac{1}{1+\bar{r}} \approx 1 - \bar{r}$ and where in (A-9) we have assumed that $\bar{x}\hat{R}_{Nt} = \bar{x}\hat{R}_{Yt} \approx 0$.

Now, use (A-7) to eliminate $\bar{\xi}^1 E_{t-1} \hat{\xi}_t^1$ and $\bar{\xi}^1 E_{t-1} \hat{\xi}_{t+1}^1$ from (A-8), and use (A-4) to get

$$\theta_1 \bar{J} E_{t-1} \hat{J}_t - \bar{\beta}\theta_1 \bar{J} E_{t-1} \hat{J}_{t+1} + \bar{\beta}\left(\delta_2 - 1\right)\delta_2 \overline{\psi} E_{t-1}\left[\hat{R}_{Nt} - \hat{x}_{t+1}\right] + \theta_1 \bar{J} E_{t-1} \hat{r}_{t+1} = 0 \tag{A-10}$$

Now, recognize that

$$\hat{R}_{Nt} = \ln R_{Nt} - \ln \bar{R}_N = \ln N_t - \ln X_t - \ln \bar{R}_N \tag{A-11-a}$$

$$\hat{R}_{Yt} = \ln R_{Yt} - \ln \bar{R}_Y = \ln Y_t - \ln X_t - \ln \bar{R}_Y \tag{A-11-b}$$

$$\hat{J}_t = \ln J_t - \ln \bar{J} = \ln A_t + \left(\theta_1 - 1\right)\ln Y_t + \theta_2 \ln W_t - \ln \bar{J} \tag{A-11-c}$$

$$\hat{x}_t = x_t - \bar{x} = \Delta \ln X_t - \bar{x} \tag{A-11-d}$$

$$\hat{r}_t = r_t - \bar{r} \tag{A-11-e}$$

Substituting (A-11-a)-(A11-e) into (A-10) and collecting terms yields

$$\theta_1 \bar{J}\left[E_{t-1} \ln A_t - \bar{\beta} E_{t-1} \ln A_{t+1}\right] +$$

$$\left(\theta_1 - 1\right)\theta_1 \bar{J}\left[E_{t-1} \ln Y_t - \bar{\beta} E_{t-1} \ln Y_{t+1}\right] + \theta_2 \theta_1 \bar{J}\left[E_{t-1} \ln W_t - \bar{\beta} E_{t-1} \ln W_{t+1}\right] \tag{A-12}$$

$$+ \bar{\beta}\left(\delta_2 - 1\right)\delta_2 \overline{\psi}\left[E_{t-1} \ln N_t - E_{t-1} \ln X_{t+1}\right] + \theta_1 \bar{J} E_{t-1} r_{t+1} + c = 0$$

where c is a constant that depends on steady state values. Let $\ln A_t$ be stationary. Define $\tilde{u}_{t+1}^{A} \equiv \ln A_t - \overline{\beta} \ln A_{t+1}$ and note that $\tilde{u}_{t+1}^{A}$ will also be stationary.[1] Equation (A-12) then yields

$$E_{t-1}\Big\{(\theta_1 - 1)\theta_1 \overline{J}\left[\ln Y_t - \overline{\beta}\ln Y_{t+1}\right] + \theta_2\theta_1\overline{J}\left[\ln W_t - \overline{\beta}\ln W_{t+1}\right] + \overline{\beta}(\delta_2 - 1)\delta_2\overline{\psi}\left[\ln N_t - \ln X_{t+1}\right] + \theta_1\overline{J} r_{t+1} + \theta_1\overline{J}\tilde{u}_{t+1}^{A} + c\Big\} = 0 \qquad (10)$$

QED, Proposition 1.

II. Decision Rule.

Proposition 2*. The model implies that the firm's decision rule is*

$$\ln N_t = \Gamma_0 + \lambda_1 \ln N_{t-1} + \Gamma_X \ln X_{t-1} + \Gamma_W \ln W_{t-1} + \Gamma_{\pi 1}\pi_{1t-1} + \Gamma_{\pi 3}\pi_{3t-1} + u_t \qquad (17)$$

where

$$\lambda_1 = 1 + \frac{\overline{r} + \zeta}{2} - \frac{1}{2}\left[\left(\overline{r} + \zeta\right)^2 + 4\zeta\right]^{\frac{1}{2}}, \qquad (21\text{-a})$$

$$\zeta = \frac{(\delta_2 - 1)\delta_2\overline{\psi}}{(\theta_1 - 1)\theta_1\overline{J}}\frac{\overline{R}_Y}{\overline{R}_N}, \qquad (21\text{-b})$$

where $0 < \lambda_1 < 1, \zeta > 0,$ Γ_0 *is a constant, and* u_t *is a stationary shock.*

Proof: Substitute (A-11-a,b,d) into (A-9) to get

$$\overline{R}_Y E_{t-1} \ln Y_t = \overline{R}_N E_{t-1}\Delta \ln N_t + \overline{R}_Y E_{t-1} \ln X_t + c_1 \qquad (A\text{-}13)$$

[1] In our empirical work we allow for $\ln A_t$ to contain a deterministic trend. As we discuss in the proof of Proposition 6 below, our empirical results confirm the assumption that $\tilde{u}_{t+1}^{A}$ is stationary.

where c_1 is a constant. Then, using (A-13) to eliminate $E_{t-1}\ln Y_t$ and $E_{t-1}\ln Y_{t+1}$ from (10) yields

$$(\theta_1-1)\theta_1\bar{J}\left[\frac{\bar{R}_N}{\bar{R}_Y}E_{t-1}\Delta\ln N_t+E_{t-1}\ln X_t\right]-\bar{\beta}(\theta_1-1)\theta_1\bar{J}\left[\frac{\bar{R}_N}{\bar{R}_Y}E_{t-1}\Delta\ln N_{t+1}+E_{t-1}\ln X_{t+1}\right]$$

$$+\theta_2\theta_1\bar{J}\left[E_{t-1}\ln W_t-\bar{\beta}E_{t-1}\ln W_{t+1}\right]+\bar{\beta}(\delta_2-1)\delta_2\bar{\psi}\left[E_{t-1}\ln N_t-E_{t-1}\ln X_{t+1}\right] \quad \text{(A-14)}$$

$$+\theta_1\bar{J}\left[(\gamma_1-\gamma_2)\pi_{1t-1}+(\gamma_3-\gamma_2)\pi_{3t-1}+\gamma_2\right]+\theta_1\bar{J}E_{t-1}\tilde{u}^A_{t+1}+c_2=0$$

where c_2 is a constant. Combining terms gives

$$(\theta_1-1)\theta_1\bar{J}\frac{\bar{R}_N}{\bar{R}_Y}\left[E_{t-1}\Delta\ln N_t-\bar{\beta}E_{t-1}\Delta\ln N_{t+1}\right]+(\theta_1-1)\theta_1\bar{J}\left[E_{t-1}\ln X_t-\bar{\beta}E_{t-1}\ln X_{t+1}\right]$$

$$+\theta_2\theta_1\bar{J}\left[E_{t-1}\ln W_t-\bar{\beta}E_{t-1}\ln W_{t+1}\right]+\bar{\beta}(\delta_2-1)\delta_2\bar{\psi}E_{t-1}\ln N_t \quad \text{(A-15)}$$

$$-\bar{\beta}(\delta_2-1)\delta_2\bar{\psi}E_{t-1}\ln X_{t+1}+\theta_1\bar{J}\left[(\gamma_1-\gamma_2)\pi_{1t-1}+(\gamma_3-\gamma_2)\pi_{3t-1}+\gamma_2\right]+\theta_1\bar{J}E_{t-1}\tilde{u}^A_{t+1}+c_2=0$$

Collecting and using (15), equation (A-15) can be written as

$$E_{t-1}\left[f(L)\frac{\bar{R}_N}{\bar{R}_Y}\ln N_{t+1}\right]=E_{t-1}\Xi_t \quad \text{(A-16)}$$

where

$$f(L)\equiv 1-\left[1+\frac{1}{\bar{\beta}}+\frac{(\delta_2-1)\delta_2\bar{\psi}}{(\theta_1-1)\theta_1\bar{J}}\frac{\bar{R}_Y}{\bar{R}_N}\right]L+\frac{1}{\bar{\beta}}L^2 \quad \text{(A-17)}$$

and

$$\Xi_t=-\left[1+\frac{(\delta_2-1)\delta_2\bar{\psi}}{(\theta_1-1)\theta_1\bar{J}}\right]\ln X_{t+1}+\frac{1}{\bar{\beta}}\ln X_t$$
$$-\frac{\theta_2}{(\theta_1-1)}\ln W_{t+1}+\frac{\theta_2}{\bar{\beta}(\theta_1-1)}\ln W_t+\frac{1}{\bar{\beta}(\theta_1-1)}r_{t+1}+\frac{1}{\bar{\beta}(\theta_1-1)}\tilde{u}^A_{t+1}-c_4 \quad \text{(A-18)}$$

where c_4 is a constant.

Let λ_i, i = 1,2 denote the roots of the second-order polynomial in (A-17). The roots must satisfy the quadratic equation:

$$\lambda^2 - \left[1 + \frac{1}{\overline{\beta}} + \zeta\right]\lambda + \frac{1}{\overline{\beta}} = 0 \tag{A-19}$$

where

$$\zeta = \frac{(\delta_2 - 1)\delta_2 \overline{\psi}}{(\theta_1 - 1)\theta_1 \overline{J}} \frac{\overline{R}_Y}{\overline{R}_N}. \tag{21-b}$$

Note that $\zeta > 0$ follows from $\theta_1 > 1, \delta_2 < 0, \overline{\psi} > 0, \overline{J} > 0, \overline{R}_Y > 0,$ and $\overline{R}_N > 0$. From (A-19), using $\overline{\beta} = \frac{1}{1+\overline{r}}$ we have

$$\lambda_1, \lambda_2 = \frac{1}{2}\left(1 + \frac{1}{\overline{\beta}} + \zeta\right) \pm \frac{1}{2}\left[\left(1 + \frac{1}{\overline{\beta}} + \zeta\right)^2 - 4\frac{1}{\overline{\beta}}\right]^{\frac{1}{2}} = \frac{2 + \overline{r} + \zeta}{2} \pm \frac{1}{2}\left[\left(\overline{r} + \zeta\right)^2 + 4\zeta\right]^{\frac{1}{2}}$$

or

$$\lambda_1 = 1 + \frac{\overline{r} + \zeta}{2} - \frac{1}{2}\left[\left(\overline{r} + \zeta\right)^2 + 4\zeta\right]^{\frac{1}{2}} \tag{21-a}$$

and

$$\lambda_2 = 1 + \frac{\overline{r} + \zeta}{2} + \frac{1}{2}\left[\left(\overline{r} + \zeta\right)^2 + 4\zeta\right]^{\frac{1}{2}}. \tag{A-20}$$

Since $\zeta > 0$ it is clear that $\lambda_2 > 1$. Also, from (A-20) it is clear that $\lambda_1 < 1$. Observe from (A-17) that $\lambda_1 \lambda_2 = \frac{1}{\overline{\beta}}$ > 0. It follows that $\lambda_1 > 0$. Collecting, we have $0 < \lambda_1 < 1$.

Since λ_1 is the stable root and λ_2 is the unstable root, solve λ_2 forward in (A-17) and use $\lambda_2 = \frac{1}{\overline{\beta}\lambda_1}$ to obtain

$$\frac{\overline{R}_N}{\overline{R}_Y} E_{t-1} \ln N_t = \lambda_1 \frac{\overline{R}_N}{\overline{R}_Y} \ln N_{t-1} - \sum_{j=0}^{\infty} \left[\left(\frac{1}{\lambda_2} \right)^{j+1} E_{t-1} \Xi_{t+j} \right]$$

$$= \lambda_1 \frac{\overline{R}_N}{\overline{R}_Y} \ln N_{t-1} - \overline{\beta} \lambda_1 \sum_{j=0}^{\infty} \left[\left(\overline{\beta} \lambda_1 \right)^j E_{t-1} \Xi_{t+j} \right] \quad . \qquad \text{(A-21)}$$

To resolve the forward sum on the right-hand side of (A-21), we assume that sales and input prices are I(1) processes of the form:

$$\ln X_t = \mu_x + \ln X_{t-1} + u_t^X \qquad \text{(A-22-a)}$$

$$\ln W_t = \mu_w + \ln W_{t-1} + u_t^W \qquad \text{(A-22-b)}$$

where $u_t^X \sim i.i.d.(0, \sigma_X^2)$ and $u_t^W \sim i.i.d.(0, \sigma_W^2)$. For the theoretical derivations, we impose no distributional restriction.[2]

Using the definition of Ξ_t in (A-18), the terms involving sales on the right-hand side of (A-21) can be written as

$$-\overline{\beta} \lambda_1 \sum_{j=0}^{\infty} \left[\left(\overline{\beta} \lambda_1 \right)^j E_{t-1} \left(-\overline{a} \ln X_{t+1+j} + \frac{1}{\overline{\beta}} \ln X_{t+j} \right) \right] \qquad \text{(A-23)}$$

where $\overline{a} = 1 + \dfrac{(\delta_2 - 1)\delta_2 \overline{\psi}}{(\theta_1 - 1)\theta_1 \overline{J}}$.

Note from (A-22-a) that $E_{t-1} \ln X_{t+j}$ is a linear function of X_{t-1} for j = 0, 1, 2, … .

It therefore follows that

$$-\overline{\beta} \lambda_1 \sum_{j=0}^{\infty} \left[\left(\overline{\beta} \lambda_1 \right)^j E_{t-1} \left(-\overline{a} \ln X_{t+1+j} + \frac{1}{\overline{\beta}} \ln X_{t+j} \right) \right] = c_X + \tilde{\Gamma}_X \ln X_{t-1} . \qquad \text{(A-24)}$$

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[2] For the simulations that explore DOLS bias, we assume that u_t^X and u_t^W have Gaussian distributions with σ_X^2 and σ_W^2 set equal to their sample values (e.g., variance of the change in log sales) and μ_X and μ_W set equal to 0.

The terms involving real input prices on the right-hand side of (A-21) can, using the same argument applied to (A-23), be written as

$$-\bar{\beta}\lambda_1 \sum_{j=0}^{\infty}\left[\left(\bar{\beta}\lambda_1\right)^j E_{t-1}\left(-\frac{\theta_2}{(\theta_1-1)}\ln W_{t+1}+\frac{\theta_2}{\bar{\beta}(\theta_1-1)}\ln W_t\right)\right]= c_W+\tilde{\Gamma}_W W_{t-1}\,. \qquad \text{(A-25)}$$

Consider next the terms involving the real interest rate on the right-hand side of (A-21),

$$-\bar{\beta}\lambda_1 \sum_{j=0}^{\infty}\left[\left(\bar{\beta}\lambda_1\right)^j \frac{1}{\bar{\beta}(\theta_1-1)}E_{t-1}r_{t+1+j}\right]. \qquad \text{(A-26)}$$

Assuming that the real interest rate follows a three-state Markov-switching process and using the learning process developed above in the text, we have that

$$E_{t-1}r_{t+1+j}=\mathrm{r_v}'\left[P^{j+2}\pi_{t-1}\right] \qquad \text{(A-27)}$$

where

$$\mathrm{r_v}'=[r_1,r_2,r_3],\quad \pi_{t-1}=\begin{bmatrix}\pi_{1t-1}\\ \pi_{2t-1}\\ \pi_{3t-1}\end{bmatrix}, \text{ and } P=\begin{bmatrix}p_{11} & p_{21} & p_{31}\\ p_{12} & p_{22} & p_{32}\\ p_{13} & p_{23} & p_{33}\end{bmatrix}.$$

Since $\pi_{2t-1}=1-\left(\pi_{1t-1}+\pi_{3t-1}\right)$, we have from (A-27) that $E_{t-1}r_{t+1+j}$ is a linear function of π_{1t-1} and π_{3t-1} for j = 0,1,2, … . It follows that

$$-\bar{\beta}\lambda_1 \sum_{j=0}^{\infty}\left[\left(\bar{\beta}\lambda_1\right)^j \frac{1}{\bar{\beta}(\theta_1-1)}E_{t-1}r_{t+1+j}\right]=c_\pi+\tilde{\Gamma}_{\pi_1}\pi_{1t-1}+\tilde{\Gamma}_{\pi_3}\pi_{3t-1} \qquad \text{(A-28)}$$

The terms involving $\tilde{u}^A_{t+1+j}$ on the right-hand side of (A-21) can be written as

$$-\bar{\beta}\lambda_1 \sum_{j=0}^{\infty}\left[\left(\bar{\beta}\lambda_1\right)^j \frac{1}{\bar{\beta}(\theta_1-1)}E_{t-1}\tilde{u}^A_{t+1+j}\right]\equiv u^A_{t-1}\,. \qquad \text{(A-29)}$$

Since the $\tilde{u}^A_{t+1+j}$ are stationary, u^A_{t-1} will be stationary.

Finally,

$$\sum_{j=0}^{\infty}\left[\left(\bar{\beta}\lambda_1\right)^{j+1}\right]E_{t-1}c_4=\bar{\beta}\lambda_1\sum_{j=0}^{\infty}\left[\left(\bar{\beta}\lambda_1\right)^{j}\right]c_4=\frac{\bar{\beta}\lambda_1}{\left(1-\bar{\beta}\lambda_1\right)}c_4=c_{con} \qquad \text{(A-30)}$$

Thus, using the definition of Ξ_t, (A-18), in (A-21) and then substituting from (A-24), (A-25), (A-28), (A-29), and (A-30) we have

$$\begin{aligned}\frac{\bar{R}_N}{\bar{R}_Y}E_{t-1}\ln N_t=\lambda_1\frac{\bar{R}_N}{\bar{R}_Y}\ln N_{t-1}+\tilde{\Gamma}_X\ln X_{t-1}+\tilde{\Gamma}_W\ln W_{t-1}\\+\tilde{\Gamma}_{\pi1}\pi_{1t-1}+\tilde{\Gamma}_{\pi3}\pi_{3t-1}+\tilde{\Gamma}_o+u_{t-1}^A\end{aligned} \qquad \text{(A-31)}$$

where $\tilde{\Gamma}_0=c_X+c_W+c_\pi+c_{con}$.

Now, log-linearizing the accumulation equation, (4), yields

$$\bar{R}_N\hat{R}_{Nt}-\bar{R}_N\hat{R}_{Nt-1}-\bar{R}_Y\hat{R}_{Yt}+\bar{R}_N\hat{x}_t=0 \qquad \text{(A-32)}$$

Substitute (A-11-a), (A-11-b), and (A-11-d) into (A-32) to get

$$\begin{aligned}\bar{R}_N\left[\ln N_t-\ln X_t-\ln\bar{R}_N\right]-\bar{R}_N\left[\ln N_{t-1}-\ln X_{t-1}-\ln\bar{R}_N\right]\\-\bar{R}_Y\left[\ln Y_t-\ln X_t-\ln\bar{R}_Y\right]+\bar{R}_N\left[\ln X_t-\ln X_{t-1}-\bar{x}\right]=0\end{aligned}$$

or

$$\frac{\bar{R}_N}{\bar{R}_Y}\ln N_t-\frac{\bar{R}_N}{\bar{R}_Y}\ln N_{t-1}-\ln Y_t+\ln X_t+c_5=0 \qquad \text{(A-33)}$$

Taking expectations of (A-33) gives

$$\frac{\bar{R}_N}{\bar{R}_Y}E_{t-1}\ln N_t-\frac{\bar{R}_N}{\bar{R}_Y}\ln N_{t-1}-E_{t-1}\ln Y_t+E_{t-1}\ln X_t+c_5=0 \qquad \text{(A-34)}$$

Subtract (A-34) from (A-33) to get

$$\frac{\bar{R}_N}{\bar{R}_Y}\left[\ln N_t - E_{t-1}\ln N_t\right] - \left[\ln Y_t - E_{t-1}\ln Y\right] + \left[\ln X_t - E_{t-1}\ln X_t\right] = 0 \qquad \text{(A-35)}$$

Define $u_t^Y = \ln Y_t - E_{t-1}\ln Y_t$ as the production error and $u_t^X = \ln X_t - E_{t-1}\ln X_t$ as the sales error, then (A-35) becomes

$$\frac{\bar{R}_N}{\bar{R}_Y}\ln N_t = E_{t-1}\frac{\bar{R}_N}{\bar{R}_Y}\ln N_t + u_t^Y - u_t^X \qquad \text{(A-36)}$$

Substituting (A-31) into (A-36) gives

$$\begin{aligned}\frac{\bar{R}_N}{\bar{R}_Y}\ln N_t = \lambda_1\frac{\bar{R}_N}{\bar{R}_Y}\ln N_{t-1} + \tilde{\Gamma}_X \ln X_{t-1} + \tilde{\Gamma}_W \ln W_{t-1} \\ + \tilde{\Gamma}_{\pi 1}\pi_{1t-1} + \tilde{\Gamma}_{\pi 3}\pi_{3t-1} + \tilde{\Gamma}_o + \tilde{u}_t\end{aligned} \qquad \text{(A-37)}$$

where $\tilde{u}_t = u_{t-1}^A + u_t^Y - u_t^X$.

Finally, multiply (A-37) by $\frac{\bar{R}_Y}{\bar{R}_N}$ to get the decision rule, equation (17) in the text, where $\Gamma_X = \frac{\bar{R}_Y}{\bar{R}_N}\tilde{\Gamma}_X$, $\Gamma_W = \frac{\bar{R}_Y}{\bar{R}_N}\tilde{\Gamma}_W$, $\Gamma_{\pi_1} = \frac{\bar{R}_Y}{\bar{R}_N}\tilde{\Gamma}_{\pi_1}$, $\Gamma_{\pi_3} = \frac{\bar{R}_Y}{\bar{R}_N}\tilde{\Gamma}_{\pi_3}$, $u_t = \frac{\bar{R}_Y}{\bar{R}_N}\tilde{u}_t$ and $\Gamma_o = \frac{\bar{R}_Y}{\bar{R}_N}\tilde{\Gamma}_o$.

From the definitions of $\tilde{u}_t$ and u_t it follows that

$$u_t = \left(\bar{R}_Y / \bar{R}_N\right)\left(u_{t-1}^A + u_t^Y - u_t^X\right). \qquad \text{(A-38)}$$

Note that u_t acts as a buffer, absorbing unanticipated shocks to production and sales. Rational expectations implies that $u_t^Y = \ln Y_t - E_{t-1}\ln Y_t$ and $u_t^X = \ln X_t - E_{t-1}\ln X_t$ are both i.i.d. mean zero and, therefore, stationary. Since, in (A-29), u_{t-1}^A is the conditional expectation of a weighted sum of stationary shocks it is itself stationary. Since u_{t-1}^A, u_t^Y and u_t^X are stationary it follows from (A-38) that u_t is stationary. **QED, Proposition 2.**

Proposition 3. *Decision Rule Coefficient on Sales: The coefficient on sales in the decision rule,* Γ_X , *is*

$$\Gamma_X = \left[\frac{(\delta_2 - 1)\delta_2 \overline{\psi}}{(\theta_1 - 1)\theta_1 \overline{J}} - \overline{r} \right] \frac{\overline{R}_Y}{\overline{R}_N} \left(\frac{\lambda_1}{1 + \overline{r} - \lambda_1} \right) \tag{18}$$

Further, $\Gamma_X \underset{<}{\overset{>}{-}} 0$ *as* $\frac{(\delta_2 - 1)\delta_2 \overline{\psi}}{\overline{r}} \underset{<}{\overset{>}{-}} (\theta_1 - 1)\theta_1 \overline{J}$.

Proof: From (A-22-a) it follows that

$$E_{t-1} \ln X_{t+j+1} = \mu_x + E_{t-1} \ln X_{t+j}$$

and therefore

$$E_{t-1}\left(-\overline{a} \ln X_{t+1+j} + \frac{1}{\overline{\overline{\beta}}} \ln X_{t+j} \right) = -\overline{a}\mu_x + \left(\frac{1}{\overline{\overline{\beta}}} - \overline{a} \right) E_{t-1} \ln X_{t+j} .$$

Also,

$$E_{t-1} \ln X_{t+j} = \mu_x + E_{t-1} \ln X_{t-1+j} .$$

Hence,

$$E_{t-1}\left(-\overline{a} \ln X_{t+1+j} + \frac{1}{\overline{\overline{\beta}}} \ln X_{t+j} \right) = \left(\frac{1}{\overline{\overline{\beta}}} - 2\overline{a} \right) \mu_x + \left(\frac{1}{\overline{\overline{\beta}}} - \overline{a} \right) E_{t-1} \ln X_{t-1+j} .$$

This in (A-23) gives

$$\begin{aligned} -\overline{\beta}\lambda_1 \sum_{j=0}^{\infty} \left(\overline{\beta}\lambda_1 \right)^j E_{t-1}\left(-\overline{a} \ln X_{t+1+j} + \frac{1}{\overline{\overline{\beta}}} \ln X_{t+j} \right) &= \frac{-\overline{\beta}\lambda_1}{1 - \overline{\beta}\lambda_1} \left(\frac{1}{\overline{\overline{\beta}}} - 2\overline{a} \right) \mu_x \\ &\quad - \overline{\beta}\lambda_1 \left(\frac{1}{\overline{\overline{\beta}}} - \overline{a} \right) \sum_{j=0}^{\infty} \left(\overline{\beta}\lambda_1 \right)^j E_{t-1} \ln X_{t-1+j} \end{aligned} \tag{A-39}$$

For the stochastic process governing $\ln X_t$, (A-22-a), we can use the formulas for geometric and arithmetic-geometric series to re-write the forward sum in (A-39) as

$$\sum_{j=0}^{\infty} \left(\overline{\beta}\lambda_1 \right)^j E_{t-1} \ln X_{t-1+j} = \frac{\overline{\beta}\lambda_1 \mu_x}{(1 - \overline{\beta}\lambda_1)^2} + \frac{1}{(1 - \overline{\beta}\lambda_1)} \ln X_{t-1} \tag{A-40}$$

Using (A-40) we can rewrite (A-39) as

$$-\bar{\beta}\lambda_1 \sum_{j=0}^{\infty}\left[\left(\bar{\beta}\lambda_1\right)^j E_{t-1}\left(-\bar{a}\ln X_{t+1+j} + \frac{1}{\bar{\beta}}\ln X_{t+j}\right)\right] = c_X + \tilde{\Gamma}_X \ln X_{t-1} \qquad \text{(A-41)}$$

where

$$c_X = -\frac{\beta\lambda_1}{\left(1-\beta\lambda_1\right)^2}\left[\frac{1}{\bar{\beta}} + \bar{a}\left(\beta\lambda_1 - 2\right)\right]\mu_x$$

and

$$\tilde{\Gamma}_X = -\left(\frac{1}{\bar{\beta}} - \bar{a}\right)\frac{\bar{\beta}\lambda_1}{1-\bar{\beta}\lambda_1}$$

or, using the definition of $\bar{a}$ and $\bar{\beta} = \frac{1}{1+\bar{r}}$,

$$\tilde{\Gamma}_X = \left[\frac{\left(\delta_2 - 1\right)\delta_2\overline{\psi}}{\left(\theta_1 - 1\right)\theta_1\bar{J}} - \bar{r}\right]\frac{\lambda_1}{1+\bar{r}-\lambda_1}. \qquad \text{(A-42)}$$

We then have (18) from (A-42) and $\Gamma_X = \frac{\bar{R}_Y}{\bar{R}_N}\tilde{\Gamma}_X$.

To show that $\Gamma_X \underset{<}{\overset{>}{}} 0$ as $\frac{\left(\delta_2-1\right)\delta_2\overline{\psi}}{\bar{r}} \underset{<}{\overset{>}{}} \left(\theta_1-1\right)\theta_1\bar{J}$, note that $\bar{R}_Y$ and $\bar{R}_N$ are both positive, and $0<\lambda_1<1$ implies that $\lambda_1 \big/ \left(1+\bar{r}-\lambda_1\right) > 0$, so the sign of Γ_X depends on the term in square brackets, which will be positive if and only if

$$\frac{\left(\delta_2-1\right)\delta_2\overline{\psi}}{\bar{r}} > \left(\theta_1-1\right)\theta_1\bar{J}$$

And vice-versa. **QED, Proposition 3.**

Proposition 4. *Decision Rule Coefficient on Input Costs: The coefficient on input costs in the decision rule* $\boldsymbol{\Gamma_W}$ *is*

$$\Gamma_W = -\frac{\bar{r}\theta_2}{(\theta_1 - 1)} \frac{\bar{R}_Y}{\bar{R}_N} \left(\frac{\lambda_1}{1 + \bar{r} - \lambda_1} \right) < 0 \qquad (19)$$

Proof: Proceeding as with the terms in sales and using (A-22-b), the terms involving real input prices on the right-hand side of (A-21) can be written as (A-25), where

$$c_W = \frac{\theta_2}{(\theta_1 - 1)} \left(\frac{\bar{\beta}\lambda_1}{1 - \bar{\beta}\lambda_1} \right) \left[1 + \left(1 - \frac{1}{\bar{\beta}} \right) \left(\frac{1}{1 - \bar{\beta}\lambda_1} \right) \right] \mu_w$$

$$\tilde{\Gamma}_W \equiv \frac{\theta_2}{(\theta_1 - 1)} \left(1 - \frac{1}{\bar{\beta}} \right) \left(\frac{\bar{\beta}\lambda_1}{1 - \bar{\beta}\lambda_1} \right) = -\frac{\bar{r}\theta_2}{(\theta_1 - 1)} \left(\frac{\lambda_1}{1 + \bar{r} - \lambda_1} \right) \qquad \text{(A-43)}$$

where the last equality follows from $\frac{1}{\bar{\beta}} = 1 + \bar{r}$. We then obtain (19) from (A-43) and

$$\Gamma_W = \frac{\bar{R}_Y}{\bar{R}_N} \tilde{\Gamma}_W .$$

To show that $\Gamma_W < 0$ note that under the assumptions of the model $\theta_1 > 1$ and $\theta_2 > 0$. We assume that $\bar{r}$, the unconditional mean real interest rate, is positive. Thus $\bar{r}\theta_2 / (\theta_1 - 1) > 0$. We have that $0 < \lambda_1 < 1$ and so $\lambda_1 / (1 + \bar{r} - \lambda_1) > 0$. Since $\bar{R}_Y > 0$ and $\bar{R}_N > 0$ it then follows from (19) that $\Gamma_W < 0$. **QED, Proposition 4.**

Proposition 5. *Decision Rule Coefficients on the Interest-Rate-Regime Probabilities: The model implies that the decision rule coefficients on the Interest-Rate-Regime Probabilities are*

$$\Gamma_{\pi_1} \equiv \frac{-\lambda_1}{(\theta_1 - 1)} \frac{\bar{R}_Y}{\bar{R}_N} \gamma' \left[I - \frac{\lambda_1}{1 + \bar{r}} P \right]^{-1} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \qquad \text{(20-a)}$$

$$\Gamma_{\pi_3} \equiv \frac{-\lambda_1}{(\theta_1 - 1)} \frac{\bar{R}_Y}{\bar{R}_N} \gamma' \left[I - \frac{\lambda_1}{1 + \bar{r}} P \right]^{-1} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \qquad \text{(20-b)}$$

where $\gamma' \equiv [\gamma_1 \quad \gamma_2 \quad \gamma_3]$. *Furthermore, if*

$$(p_{11} + p_{22})/2 > 0.5 \tag{A-44-a}$$

$$(p_{22} + p_{33})/2 > 0.5 \tag{A-44-b}$$

$$p_{13} = p_{31} = 0, \tag{A-44-c}$$

then, $\Gamma_{\pi_1} > 0$ *and* $\Gamma_{\pi_3} < 0$.

Proof: Since the eigenvalues of $\bar{\beta}\lambda_1 P$ lie inside the unit circle[3], we can use (A-27) to write

$$\sum_{j=0}^{\infty} (\bar{\beta}\lambda_1)^j E_{t-1} r_{t+1+j} = \sum_{j=0}^{\infty} (\bar{\beta}\lambda_1)^j \mathrm{r_v}' \left[P^{j+2} \pi_{t-1} \right] = \mathrm{r_v}' P^2 \sum_{j=0}^{\infty} (\bar{\beta}\lambda_1 P)^j \pi_{t-1}$$

$$= \mathrm{r_v}' P^2 \left[I - \bar{\beta}\lambda_1 P \right]^{-1} \pi_{t-1} \tag{A-45}$$

Using (A-45) and noting that $\mathrm{r_v}' P^2 \equiv [\gamma_1 \quad \gamma_2 \quad \gamma_3]$, (A-27) can be written as

$$-\bar{\beta}\lambda_1 \sum_{j=0}^{\infty} \left[(\bar{\beta}\lambda_1)^j \frac{1}{\bar{\beta}(\theta_1 - 1)} E_{t-1} r_{t+1+j} \right] = \frac{-\lambda_1}{(\theta_1 - 1)} [\gamma_1 \quad \gamma_2 \quad \gamma_3] \left[I - \bar{\beta}\lambda_1 P \right]^{-1} \pi_{t-1} \tag{A-46}$$

Since $\frac{-\lambda_1}{(\theta_1 - 1)} [\gamma_1 \quad \gamma_2 \quad \gamma_3] \left[I - \bar{\beta}\lambda_1 P \right]^{-1}$ is (1x3) and $\pi_{2t-1} = 1 - (\pi_{1t-1} + \pi_{3t-1})$ we can rewrite (A-46) as (A-28) where, using $\bar{\beta} = \frac{1}{1+\bar{r}}$,

$$\tilde{\Gamma}_{\pi_1} = \frac{-\lambda_1}{(\theta_1 - 1)} [\gamma_1 \quad \gamma_2 \quad \gamma_3] \left[I - \frac{\lambda_1}{1+\bar{r}} P \right]^{-1} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \tag{A-47}$$

$$\tilde{\Gamma}_{\pi_3} = \frac{-\lambda_1}{(\theta_1 - 1)} [\gamma_1 \quad \gamma_2 \quad \gamma_3] \left[I - \frac{\lambda_1}{1+\bar{r}} P \right]^{-1} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \tag{A-48}$$

and

[3] Hamilton (1994) pages 681 and 732.

$$c_{\pi} \equiv \frac{-\lambda_1}{(\theta_1 - 1)} [\gamma_1 \quad \gamma_2 \quad \gamma_3] \left[I - \bar{\beta}\lambda_1 P \right]^{-1} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}. \qquad \text{(A-49)}$$

We then have (20-a) from (A-47) and $\Gamma_{\pi_1} = \frac{\bar{R}_Y}{\bar{R}_N} \tilde{\Gamma}_{\pi_1}$, and we have (20-b) from (A-48) and $\Gamma_{\pi_3} = \frac{\bar{R}_Y}{\bar{R}_N} \tilde{\Gamma}_{\pi_3}$.

To establish $\tilde{\Gamma}_{\pi 1} > 0$ and $\tilde{\Gamma}_{\pi 3} < 0$, recall that $[\gamma_1 \quad \gamma_2 \quad \gamma_3] \equiv \mathrm{r_v}' P^2$ and let $[s_1, s_2, s_3]_{(1\times 3)} \equiv \mathrm{r_v}' P^2 \left[I - \bar{\beta}\lambda_1 P \right]^{-1}$. Since $\left(\frac{-\lambda_1}{\theta_1 - 1} \right) < 0$, it then follows from (A-47) and (A-48) that $\tilde{\Gamma}_{\pi 1} > 0$ and $\tilde{\Gamma}_{\pi 3} < 0$ if and only if $s_1 < s_2 < s_3$.

We can expand the definition of $[s_1, s_2, s_3]$ to get

$$[s_1, s_2, s_3] = \mathrm{r_v}' P^2 \left[I + \bar{\beta}\lambda_1 P + \left(\bar{\beta}\lambda_1\right)^2 P^2 + \left(\bar{\beta}\lambda_1\right)^3 P^3 + \ldots \right]$$

$$= \left[\mathrm{r_v}' P^2 + \bar{\beta}\lambda_1 \mathrm{r_v}' P^3 + \left(\bar{\beta}\lambda_1\right)^2 \mathrm{r_v}' P^4 + \left(\bar{\beta}\lambda_1\right)^3 \mathrm{r_v}' P^5 + \ldots \right].$$

Define

$$\left[g_1(0), \quad g_2(0), \quad g_3(0) \right] = \mathrm{r_v}' = [r_1, r_2, r_3] \qquad \text{(A-50)}$$

and let $\left[g_1(j), \quad g_2(j), \quad g_3(j) \right]$ be defined by

$$\left[g_1(j+1), \quad g_2(j+1), \quad g_3(j+1) \right] = \left[g_1(j), \quad g_2(j), \quad g_3(j) \right] P \qquad \text{(A-51)}$$

for j = 0, 1, 2, 3, … .

Then from (A-51) $\left[g_1(2), \quad g_2(2), \quad g_3(2) \right] = \mathrm{r_v}' P^2$ and

$$[s_1, s_2, s_3] = \sum_{j=2}^{\infty} \left(\bar{\beta}\lambda_1\right)^j [g_1(j) \quad g_2(j) \quad g_3(j)], \text{ or}$$

$$s_1 = g_1(2) + \bar{\beta}\lambda_1 g_1(3) + \left(\bar{\beta}\lambda_1\right)^2 g_1(4) + \left(\bar{\beta}\lambda_1\right)^3 g_1(5) + \ldots$$

$$s_2 = g_2(2) + \bar{\beta}\lambda_1 g_2(3) + \left(\bar{\beta}\lambda_1\right)^2 g_2(4) + \left(\bar{\beta}\lambda_1\right)^3 g_2(5) + \ldots$$

$$s_3 = g_3(2) + \bar{\beta}\lambda_1 g_3(3) + \left(\bar{\beta}\lambda_1\right)^2 g_3(4) + \left(\bar{\beta}\lambda_1\right)^3 g_3(5) + ...$$

We establish an intermediate proposition to show that (A-44-a)-(A-44-c) are sufficient for $g_1(j) < g_2(j) < g_3(j), \quad j = 1, 2, 3, \ldots$.

Lemma: Let $\left[g_1(j), \quad g_2(j), \quad g_3(j)\right]$ be defined by (A-51). If $g_1(j) < g_2(j) < g_3(j)$ and conditions (A-44-a)-(A-44-c) are true, then $g_1(j+1) < g_2(j+1) < g_3(j+1)$.

Proof of Lemma: Note from the definition of P that the elements of each of its columns must sum to one. Using (A-44-c) we can therefore write that

$$P = \begin{bmatrix} p_{11} & p_{21} & 0 \\ (1-p_{11}) & p_{22} & (1-p_{33}) \\ 0 & (1-p_{22}-p_{21}) & p_{33} \end{bmatrix}. \qquad \text{(A-52)}$$

From (A-52) and (A-51) we then have

$$g_1(j+1) = g_1(j)\,p_{11} + g_2(j)(1-p_{11}), \qquad \text{(A-53)}$$

$$g_2(j+1) = g_1(j)\,p_{21} + g_2(j)\,p_{22} + g_3(j)(1-p_{22}-p_{21}), \qquad \text{(A-54)}$$

$$g_3(j+1) = g_2(j)(1-p_{33}) + g_3(j)\,p_{33}. \qquad \text{(A-55)}$$

Subtracting (A-53) from (A-54) gives

$$g_2(j+1) - g_1(j+1) = g_1(j)(p_{21}-p_{11}) + g_2(j)(p_{11}+p_{22}-1) + g_3(j)(1-p_{22}-p_{21}). \qquad \text{(A-56)}$$

From (A-56) it follows that

$$\frac{\partial\left[g_2(j+1) - g_1(j+1)\right]}{\partial p_{21}} = g_1(j) - g_3(j) < 0. \qquad \text{(A-57)}$$

Evaluating (A-56) at the maximum possible value of p_{21}, that is at $p_{21} = 1 - p_{22}$, and using (A-44-a) gives $g_2(j+1) - g_1(j+1) = \left[g_2(j) - g_1(j)\right](p_{11} + p_{22} - 1) > 0.$ It follows that $g_1(j+1) < g_2(j+1)$ for all values of p_{21}.

Next, subtracting (A-54) from (A-55) we have that

$$g_3(j+1) - g_2(j+1) = g_1(j)(-p_{21}) + g_2(j)(1-p_{22}-p_{33}) + g_3(j)(p_{22}+p_{33}+p_{21}-1). \qquad \text{(A-58)}$$

From (A-58) it follows that

$$\frac{\partial\left[g_3(j+1)-g_2(j+1)\right]}{\partial p_{21}}=g_3(j)-g_1(j)>0. \qquad \text{(A-59)}$$

Evaluating (A-58) at the minimum value of p_{21}, that is at $p_{21}=0$, and using (A-44-b) gives $g_3(j+1)-g_2(j+1)=\left[g_3(j)-g_2(j)\right](p_{22}+p_{33}-1)>0.$ It follows that $g_2(j+1)<g_3(j+1)$ for all values of p_{21}. Collecting, we have

$$g_1(j+1)<g_2(j+1)<g_3(j+1).$$

QED, Lemma.

Note that since $r_1<r_2<r_3$, (A-50) gives $g_1(0)<g_2(0)<g_3(0)$. The lemma then gives that $g_1(j)<g_2(j)<g_3(j)$, for $j=1,2,3,\ \ldots$. This in turn implies that $s_1<s_2<s_3$ and therefore that $\tilde{\Gamma}_{\pi 1}>0$ and $\tilde{\Gamma}_{\pi 3}<0$. **QED, Proposition 5.**

III. Cointegrating Regression.

Proposition 6. *The model in Section II implies that inventories, sales, input costs, and the interest-rate-regime probabilities are cointegrated, with cointegrating regression*

$$\ln N_t=b_0+b_X\ln X_t+b_W\ln W_t+b_{\pi_1}\pi_{1,t-1}+b_{\pi_3}\pi_{3,t-1}+v_t, \qquad (22)$$

where

$$b_X=1-\frac{\bar{r}(\theta_1-1)\theta_1\bar{J}}{(\delta_2-1)\delta_2\bar{\psi}} \qquad \text{(23-a)} \qquad\qquad b_W=-\frac{\bar{r}\theta_2\theta_1\bar{J}}{(\delta_2-1)\delta_2\bar{\psi}} \qquad \text{(23-b)}$$

$$b_{\pi_1}=-(\gamma_1-\gamma_2)\frac{(1+\bar{r})\theta_1\bar{J}}{(\delta_2-1)\delta_2\bar{\psi}} \qquad \text{(23-c)} \qquad\qquad b_{\pi_3}=-(\gamma_3-\gamma_2)\frac{(1+\bar{r})\theta_1\bar{J}}{(\delta_2-1)\delta_2\bar{\psi}} \qquad \text{(23-d)}$$

b_0 *is a constant, and* v_t *is a stationary error term.*

Proof: Begin from (A-15). Add and subtract $\bar{\beta}E_{t-1}\ln W_t$ and $\bar{\beta}E_{t-1}\ln X_t$ where appropriate, recognize that $\ln X_{t+1}=\ln X_t+\Delta\ln X_{t+1}$, and re-arrange terms to get

$$\theta_2\theta_1\bar{J}\left[\left(1-\bar{\beta}\right)E_{t-1}\ln W_t-\bar{\beta}E_{t-1}\Delta\ln W_{t+1}\right]+\left(\theta_1-1\right)\theta_1\bar{J}\,\frac{\bar{R}_N}{\bar{R}_Y}\left[E_{t-1}\Delta\ln N_t-\bar{\beta}E_{t-1}\Delta\ln N_{t+1}\right]$$

$$+\left(\theta_1-1\right)\theta_1\bar{J}\left[\left(1-\bar{\beta}\right)E_{t-1}\ln X_t-\bar{\beta}E_{t-1}\Delta\ln X_{t+1}\right]+\bar{\beta}\left(\delta_2-1\right)\delta_2\overline{\psi}E_{t-1}\ln N_t \qquad \text{(A-60)}$$

$$-\bar{\beta}\left(\delta_2-1\right)\delta_2\overline{\psi}E_{t-1}\left[\ln X_t+\Delta\ln X_{t+1}\right]+\theta_1\bar{J}\left[\left(\gamma_1-\gamma_2\right)\pi_{1t-1}+\left(\gamma_3-\gamma_2\right)\pi_{3t-1}+\gamma_2\right]$$

$$+\theta_1\bar{J}E_{t-1}\tilde{u}^A_{t+1}+c_2=0$$

Then, combining terms appropriately yields

$$-\bar{\beta}\theta_2\theta_1\bar{J}E_{t-1}\Delta\ln W_{t+1}+\left(\theta_1-1\right)\theta_1\bar{J}\,\frac{\bar{R}_N}{\bar{R}_Y}\left[E_{t-1}\Delta\ln N_t-\bar{\beta}E_{t-1}\Delta\ln N_{t+1}\right]$$

$$-\bar{\beta}\left[\left(\theta_1-1\right)\theta_1\bar{J}+\left(\delta_2-1\right)\delta_2\overline{\psi}\right]E_{t-1}\Delta\ln X_{t+1}$$

$$+\bar{\beta}\left(\delta_2-1\right)\delta_2\overline{\psi}\left\{E_{t-1}\ln N_t+\frac{\left(1-\bar{\beta}\right)\left(\theta_1-1\right)\theta_1\bar{J}-\bar{\beta}\left(\delta_2-1\right)\delta_2\overline{\psi}}{\bar{\beta}\left(\delta_2-1\right)\delta_2\overline{\psi}}E_{t-1}\ln X_t\right. \qquad \text{(A-61)}$$

$$\left.+\frac{\left(1-\bar{\beta}\right)\theta_2\theta_1\bar{J}}{\bar{\beta}\left(\delta_2-1\right)\delta_2\overline{\psi}}E_{t-1}\ln W_t+\frac{\theta_1\bar{J}}{\bar{\beta}\left(\delta_2-1\right)\delta_2\overline{\psi}}\left[\left(\gamma_1-\gamma_2\right)\pi_{1t-1}+\left(\gamma_3-\gamma_2\right)\pi_{3t-1}+\gamma_2\right]\right\}$$

$$+\theta_1\bar{J}E_{t-1}\tilde{u}^A_{t+1}+c_2=0$$

We have assumed that $\ln A_t$ is stationary.[4] If $\ln A_t$ were nonstationary, then we would not obtain a cointegrating vector. In the data, the Johansen-Juselius test rejects the null hypothesis of no cointegrating vector (as reported in the paper). The stochastic process for $\ln A_t$ implies that $\ln A_t-\bar{\beta}\ln A_{t+1}\equiv\tilde{u}^A_{t+1}$ is stationary.

Re-write (A-61) to get

[4] If $\ln A_t$ contains a deterministic trend, the cointegrating relationship will contain a trend, which we allow for in the empirical work.

$$E_{t-1}\left\{\chi_t + \overline{\beta}\left(\delta_2 - 1\right)\delta_2\overline{\psi}\left[\ln N_t - b_X \ln X_t - b_W \ln W_t - b_{\pi_1}\pi_{1t-1} - b_{\pi_3}\pi_{3t-1}\right]\right\} = 0 \qquad \text{(A-62)}$$

where

$$\chi_t = -\overline{\beta}\theta_2\theta_1\overline{J}\Delta\ln W_{t+1} + \left(\theta_1 - 1\right)\theta_1\overline{J}\,\frac{\overline{R}_N}{\overline{R}_Y}\left[\Delta\ln N_t - \overline{\beta}\Delta\ln N_{t+1}\right]$$
$$-\overline{\beta}\left[\left(\theta_1 - 1\right)\theta_1\overline{J} + \left(\delta_2 - 1\right)\delta_2\overline{\psi}\right]\Delta\ln X_{t+1} + \theta_1\overline{J}\tilde{u}^A_{t+1} + c_3 \qquad \text{(A-63)}$$

where b_X, b_W, b_{π_1}, and b_{π_3} are given by equations (23-a)-(23-d) in the text and c_3 is a constant. Observe that χ_t is stationary since $\Delta\ln N_{t+i}$, $\Delta\ln W_{t+i}$, $\Delta\ln X_{t+i}$, and $\tilde{u}^A_{t+1}$ are all I(0).

Let $\chi_{2t} \equiv \chi_t + \overline{\beta}\left(\delta_2 - 1\right)\delta_2\overline{\psi}\left[\ln N_t - b_X \ln X_t - b_W \ln W_t - b_{\pi_1}\pi_{1t-1} - b_{\pi_3}\pi_{3t-1}\right]$. Rational expectations and then implies that $\chi_{2t} - E_{t-1}\chi_{2t}$ is i.i.d mean zero and, hence, stationary. As (A-62) gives that $\chi_{2t} - E_{t-1}\chi_{2t} = \chi_{2t}$, it follows that χ_{2t} is stationary, and since χ_t is stationary, it follows that

$$\left[\ln N_t - b_X \ln X_t - b_W \ln W_t - b_{\pi_1}\pi_{1t-1} - b_{\pi_3}\pi_{3t-1}\right] \sim I(0). \qquad \text{(A-64)}$$

Writing the cointegrating relationship implied by (A-64) as a cointegrating regression we have equation (22) where b_0 is a constant, and v_t is a stationary error term. **QED, Proposition 6.**

Proposition 7*. Signs of the Coefficients in the Cointegrating Regression:*

A. $b_X \underset{<}{\overset{>}{}} 0$ *as* $\frac{\left(\delta_2 - 1\right)\delta_2\overline{\psi}}{\overline{r}} \underset{<}{\overset{>}{}} \left(\theta_1 - 1\right)\theta_1\overline{J}$

B. $b_W < 0$

C. If (A-44-a), (A-44-b), and (A-44-c) hold, then $b_{\pi_1} > 0$*, and* $b_{\pi_3} < 0$.

Proof of A. From the definition of b_X in (23-a) it follows that

$b_X > 0$ if and only if $\frac{\left(\delta_2 - 1\right)\delta_2\overline{\psi}}{\overline{r}} > \left(\theta_1 - 1\right)\theta_1\overline{J}$,

$b_X < 0$ if and only if $\frac{\left(\delta_2 - 1\right)\delta_2\overline{\psi}}{\overline{r}} < \left(\theta_1 - 1\right)\theta_1\overline{J}$, and

$b_X = 0$ if and only if $\frac{(\delta_2 - 1)\delta_2 \overline{\psi}}{\bar{r}} = (\theta_1 - 1)\theta_1 \bar{J}$. **QED, Proposition 7A.**

Proof of B: Under the assumptions of the model $\theta_1 > 1$, $\theta_2 > 0$, and $\delta_2 < 0$. Since $\bar{J}$ denotes the steady-state value of average production costs, it follows that $\bar{J} >$ 0. Note that $\overline{\psi} = \delta_1 \left[\bar{R}_N (1 - \bar{x}) \right]^{\delta_2 - 1}$. $\bar{R}_N$ denotes the steady-state inventory/sales ratio, so $\bar{R}_N >$ 0, and $\bar{x}$ denotes the steady-state growth rate of sales which is assumed to be weakly positive and less than one. It follows that $\overline{\psi} > 0$. We have assumed that $\bar{r}$, the unconditional mean real interest rate, is positive. Collecting we have $\bar{r} > 0$, $\theta_1 > 1$, $\theta_2 > 0$, $\bar{J} >$ 0, $\delta_2 < 0$, and $\overline{\psi} > 0$. Thus, from (23-b), $b_W < 0$. **QED, Proposition 7 B.**

Proof of C: Under the assumptions of the model $\theta_1 > 1$ and $\delta_2 < 0$. Also, from the proof of B, above, we have $\bar{r} > 0$, $\bar{J} > 0$, and $\overline{\psi} > 0$. It follows from (23-c) and (23-d) that $b_{\pi 1} > 0$ if and only if $\gamma_1 - \gamma_2 < 0$ and that $b_{\pi 3} < 0$ if and only if $\gamma_3 - \gamma_2 > 0$ or, equivalently, that $b_{\pi 1} > 0$ and $b_{\pi 3} < 0$ if and only if $\gamma_1 < \gamma_2 < \gamma_3$.

Recall that $[\gamma_1 \quad \gamma_2 \quad \gamma_3] \equiv \mathrm{r}_{\mathrm{v}}{}' P^2 = [r_1 \quad r_2 \quad r_3] P^2$. Since $r_1 < r_2 < r_3$, it follows from Lemma 1 that if (A-44-a)-(A-44-c) hold then $\gamma_1 < \gamma_2 < \gamma_3$. **QED, Proposition 7 C.**

IV. Analytical Conditional Variance Ratio.

Proposition 8.

$$\frac{\mathrm{Var}[\ln Y_t | \ln Y_{t-n}]}{\mathrm{Var}\left[\ln X_t \mid \ln X_{t-(n+1)}\right]} = \frac{1}{1 + \frac{1}{n}} + \frac{\left(1 - \lambda_1 + \tilde{\Gamma}_X\right)}{(1+n)} \left[\left(1 - \lambda_1 + \tilde{\Gamma}_X\right) \left(\frac{1 - \lambda_1^{2n}}{1 - \lambda_1^2} \right) + 2 \left(\frac{1 - \lambda_1^n}{1 - \lambda_1} \right) \right] + \left[\frac{\tilde{\Gamma}_W^2}{1+n} \right] \left(\frac{1 - \lambda_1^{2n}}{1 - \lambda_1^2} \right) \frac{\sigma_W^2}{\sigma_X^2} \tag{33}$$

where $\mathrm{Var}\left(\ln Y_t \mid \ln Y_{t-n}\right)$ *is the variance of* $\ln Y_t$ *conditional on* $\ln Y_{t-n}$, $\mathrm{Var}\left(\ln X_t \mid \ln X_{t-(n+1)}\right)$ *is the variance of* $\ln X_t$ *conditional on* $\ln X_{t-(n+1)}$, σ_X^2 *is the variance of the sales shock,* σ_W^2 *is the variance of the cost shock, and*

$$\tilde{\Gamma}_X = \frac{\bar{R}_N}{\bar{R}_Y}\Gamma_X = \left[\frac{\left(\delta_2 - 1\right)\delta_2\overline{\psi}}{\left(\theta_1 - 1\right)\theta_1\bar{J}} - \bar{r}\right]\frac{\lambda_1}{(1+\bar{r}-\lambda_1)} \tag{34-a}$$

$$\tilde{\Gamma}_W = \frac{\bar{R}_N}{\bar{R}_Y}\Gamma_W = -\frac{\bar{r}\theta_2}{\theta_1 - 1}\left(\frac{\lambda_1}{1+\bar{r}-\lambda_1}\right) \tag{34-b}$$

where $\tilde{\Gamma}_X$ *and* $\tilde{\Gamma}_W$ *are the elasticities of output with respect to sales and input costs, respectively.*

Proof: To derive the variance of output, re-write the production error so that

$$\ln Y_t = E_{t-1}\ln Y_t + u_t^Y \tag{A-65}$$

Then, solve (A-34) for $E_{t-1}\ln Y_t$ and substitute into (A-65) to get

$$\ln Y_t = \frac{\bar{R}_N}{\bar{R}_Y}E_{t-1}\ln N_t - \frac{\bar{R}_N}{\bar{R}_Y}E_{t-1}\ln N_{t-1} + E_{t-1}\ln X_t + c_5 + u_t^Y \tag{A-66}$$

Then, substitute (A-31) for $\frac{\bar{R}_N}{\bar{R}_Y}E_{t-1}\ln N_t$ into (A-66) and combine terms to get

$$\begin{aligned}\ln Y_t = {} & -\left(1-\lambda_1\right)\frac{\bar{R}_N}{\bar{R}_Y}\ln N_{t-1} + \tilde{\Gamma}_X \ln X_{t-1} + \tilde{\Gamma}_W \ln W_{t-1} + \tilde{\Gamma}_{\pi_1}\pi_{1t-1} + \tilde{\Gamma}_{\pi_3}\pi_{3t-1} + u_{t-1}^A \\ & + E_{t-1}\ln X_t + c_6 + u_t^Y\end{aligned} \tag{A-67}$$

where $\tilde{\Gamma}_X$, $\tilde{\Gamma}_W$, $\tilde{\Gamma}_{\pi_1}$, and $\tilde{\Gamma}_{\pi_3}$ are defined above by (A-42), (A-43), (A-47) and (A-48), respectively, and where $c_6 = \Gamma_o + c_5$. Finally, use the assumption that sales is an I(1) process of the form given by (A-22-a) and combine terms to get

$$\ln Y_t = -(1-\lambda_1)\frac{\overline{R}_N}{\overline{R}_Y}\ln N_{t-1} + \left(1+\widetilde{\Gamma}_X\right)\ln X_{t-1} + \widetilde{\Gamma}_W \ln W_{t-1} + \widetilde{\Gamma}_{\pi_1}\pi_{1t-1} + \widetilde{\Gamma}_{\pi_3}\pi_{3t-1}$$
$$+ u_{t-1}^A + c_7 + u_t^Y \tag{A-68}$$

where $c_7 = c_6 + \mu_X$.

In order to simplify the analysis and focus on the traditional explanations for the variance ratio puzzle, we assume that the interest rate is constant (which implies that the probabilities are fixed over time) and that $u_t^Y = 0$ and $u_{t-1}^A = 0$ in the remainder of this section. Then, first-differencing (A-68) and using these assumptions yields

$$\Delta \ln Y_t = -(1-\lambda_1)\frac{\overline{R}_N}{\overline{R}_Y}\Delta \ln N_{t-1} + \left(1+\widetilde{\Gamma}_X\right)\Delta \ln X_{t-1} + \widetilde{\Gamma}_W \Delta \ln W_{t-1} \tag{A-69}$$

Use (A-33) to eliminate $\frac{\overline{R}_N}{\overline{R}_Y}\Delta \ln N_{t-1}$ from (A-69) to get

$$\Delta \ln Y_t = -(1-\lambda_1)\left(\ln Y_{t-1} - \ln X_{t-1} + c_5\right) + \left(1+\widetilde{\Gamma}_X\right)\Delta \ln X_{t-1} + \widetilde{\Gamma}_W \Delta \ln W_{t-1} \tag{A-70}$$

Or,

$$\ln Y_t = \lambda_1 \ln Y_{t-1} + (1-\lambda_1)\ln X_{t-1} + \left(1+\widetilde{\Gamma}_X\right)\Delta \ln X_{t-1} + \widetilde{\Gamma}_W \Delta \ln W_{t-1} + c_8 \tag{A-71}$$

Then, using the processes for sales and input prices, (A-22-a) and (A-22-b), to eliminate $\Delta \ln X_{t-1}$ and $\Delta \ln W_{t-1}$ from (A-71) gives

$$\begin{aligned} \ln Y_t &= \lambda_1 \ln Y_{t-1} + (1-\lambda_1)\ln X_{t-1} + \left(1+\widetilde{\Gamma}_X\right)\left(u_{t-1}^X + \mu_X\right) + \widetilde{\Gamma}_W\left(u_{t-1}^W + \mu_W\right) + c_9 \\ &= \lambda_1 \ln Y_{t-1} + (1-\lambda_1)\ln X_{t-1} + \left(1+\widetilde{\Gamma}_X\right)u_{t-1}^X + \widetilde{\Gamma}_W u_{t-1}^W + c_Y. \end{aligned} \tag{A-72}$$

Next, using backward substitution, (A-72) can be written as

$$\ln Y_t = \lambda_1^n \ln Y_{t-n} + \frac{(1-\lambda_1)}{\lambda_1}\sum_{i=1}^{n}\lambda_1^i \ln X_{t-i} + \frac{\left(1+\widetilde{\Gamma}_X\right)}{\lambda_1}\sum_{i=1}^{n}\lambda_1^i u_{t-i}^X + \frac{\widetilde{\Gamma}_W}{\lambda_1}\sum_{i=1}^{n}\lambda_1^i u_{t-i}^W + c_9 \tag{A-73}$$

Through backward substitution, the stochastic process for sales may be written as

$$\ln X_{t-k} = \ln X_{t-(n+1)} + \sum_{j=k}^{n} u_{t-j}^{X} \tag{A-74}$$

for $k = 0,1,...n$ and where we have assumed $\mu_X = 0$. Substituting (A-74) for the terms involving $\sum_{i=1}^{n} \lambda_1^i \ln X_{t-i}$ in (A-73), we have that

$$\begin{aligned}\ln Y_t &= \lambda_1^n \ln Y_{t-n} + (1-\lambda_1)\left(\sum_{i=0}^{n-1} \lambda_1^i\right) \ln X_{t-(n+1)} \\ &+ (1-\lambda_1)\left[\sum_{j=1}^{n} u_{t-j}^{X} + \lambda_1 \sum_{j=2}^{n} u_{t-j}^{X} + \lambda_1^2 \sum_{j=3}^{n} u_{t-j}^{X} + ... + \lambda_1^{n-1} \sum_{j=n}^{n} u_{t-j}^{X}\right] \\ &+ \frac{(1+\tilde{\Gamma}_X)}{\lambda_1} \sum_{i=1}^{n} \lambda_1^i u_{t-i}^{X} + \frac{\tilde{\Gamma}_W}{\lambda_1} \sum_{i=1}^{n} \lambda_1^i u_{t-i}^{W} + c_{10}\end{aligned} \tag{A-75}$$

Combining terms appropriately, (A-75) may be re-written as

$$\begin{aligned}\ln Y_t &= \lambda_1^n \ln Y_{t-n} + (1-\lambda_1)\left(\sum_{i=0}^{n-1} \lambda_1^i\right) \ln X_{t-(n+1)} \\ &+ \left[1-\lambda_1 + 1 + \tilde{\Gamma}_X\right] u_{t-1}^{X} + \left[(1-\lambda_1)\sum_{i=0}^{1} \lambda_1^i + (1+\tilde{\Gamma}_X)\lambda_1\right] u_{t-2}^{X} \\ &+ \left[(1-\lambda_1)\sum_{i=0}^{2} \lambda_1^i + (1+\tilde{\Gamma}_X)\lambda_1^2\right] u_{t-3}^{X} \cdots + \left[(1-\lambda_1)\sum_{i=0}^{n-1} \lambda_1^i + (1+\tilde{\Gamma}_X)\lambda_1^{n-1}\right] u_{t-n}^{X} \\ &+ \tilde{\Gamma}_W \sum_{i=1}^{n} \lambda_1^{i-1} u_{t-i}^{W} + \tilde{c}_Y\end{aligned} \tag{A-76}$$

Or, more concisely,

$$\begin{aligned}\ln Y_t &= \lambda_1^n \ln Y_{t-n} + (1-\lambda_1)\left(\sum_{i=0}^{n-1} \lambda_1^i\right) \ln X_{t-(n+1)} + \sum_{k=0}^{n-1}\left[(1-\lambda_1)\sum_{i=0}^{k} \lambda_1^i + (1+\tilde{\Gamma}_X)\lambda_1^k\right] u_{t-k-1}^{X} \\ &+ \tilde{\Gamma}_W \sum_{i=1}^{n} \lambda_1^{i-1} u_{t-i}^{W} + \tilde{c}_Y .\end{aligned} \tag{A-77}$$

Our objective is to calculate the variance of $\ln Y_t$ at horizon n. We therefore treat the initial values of sales and output as known (non-stochastic) quantities, implying that

$\text{Var}\left[\ln X_{t-(n+1)}\right]=\text{Var}\left[\ln Y_{t-n}\right]=0$. Further, we assume that the sales shock and the cost shock are uncorrelated at all leads and lags, specifically that $\text{cov}(u_\tau^X, u_s^W)=0 \quad \forall \tau, s$. Then, taking the variance of (A-76), or equivalently, (A-77), yields

$$\text{Var}\left[\ln Y_t\right]=\left[1-\lambda_1+1+\tilde{\Gamma}_X\right]^2\sigma_X^2+\left[\left(1-\lambda_1\right)\sum_{i=0}^{1}\lambda_1^i+\left(1+\tilde{\Gamma}_X\right)\lambda_1\right]^2\sigma_X^2$$

$$+\left[\left(1-\lambda_1\right)\sum_{i=0}^{2}\lambda_1^i+\left(1+\tilde{\Gamma}_X\right)\lambda_1^2\right]^2\sigma_X^2 \ldots.+\left[\left(1-\lambda_1\right)\sum_{i=0}^{n-1}\lambda_1^i+\left(1+\tilde{\Gamma}_X\right)\lambda_1^{n-1}\right]^2\sigma_X^2$$

$$+\left[\tilde{\Gamma}_W^2\sum_{i=1}^{n}\lambda_1^{2(i-1)}\right]\sigma_W^2$$

$$=\sum_{k=0}^{n-1}\left[\left(1-\lambda_1\right)\sum_{i=0}^{k}\lambda_1^i+\left(1+\tilde{\Gamma}_X\right)\lambda_1^k\right]^2\sigma_X^2+\left[\tilde{\Gamma}_W^2\sum_{i=1}^{n}\lambda_1^{2(i-1)}\right]\sigma_W^2 \qquad \text{(A-78)}$$

where σ_X^2 is the variance of u_{t-i}^X and σ_W^2 is the variance of u_{t-i}^W.
Computing geometric sums and combining terms appropriately gives

$$\text{Var}\left[\ln Y_t\right]=\left[n+\left(1-\lambda_1+\tilde{\Gamma}_X\right)^2\left(\frac{1-\lambda_1^{2n}}{1-\lambda_1^2}\right)+2\left(1-\lambda_1+\tilde{\Gamma}_X\right)\left(\frac{1-\lambda_1^n}{1-\lambda_1}\right)\right]\sigma_X^2$$
$$+\tilde{\Gamma}_W^2\left(\frac{1-\lambda_1^{2n}}{1-\lambda_1^2}\right)\sigma_W^2 \qquad \text{(A-79)}$$

Recognize from (A-74) and the assumption that $\text{Var}\left[\ln X_{t-(n+1)}\right]=0$ that

$$\text{Var}\left[\ln X_t\right]=\left(n+1\right)\sigma_X^2. \qquad \text{(A-80)}$$

Then, dividing (A-79) by (A-80), we have equation (33) as stated in the text.

QED, Proposition 8.

APPENDIX C: Vector Autoregression

Following Bernnke and Mihov (1998) we estimate the following VAR:

$$Z_t = \sum_{i=0}^{n} B_i Z_{t-i} + \sum_{i=0}^{n} C_i P_{t-i} + A^Z \mathrm{v}_t^Z \qquad \text{(C-1)}$$

$$P_t = \sum_{i=0}^{n} D_i Z_{t-i} + \sum_{i=0}^{n} G_i P_{t-i} + A^P \mathrm{v}_t \qquad \text{(C-2)}$$

where Z denotes a vector of macroeconomic variables and P denotes a vector of policy variables. In our model the macroeconomic variables are the natural logarithms of real sales ($\ln X_t$), the GDP deflator, real input prices ($\ln W_t$), and real inventories ($\ln N_t$). Our policy block is the same as Bernakne and Mihov's, so that the elements of P are total reserves, non-borrowed reserves, and the Fed funds rate. $B_i, C_i, A^Z, D_i, G_i,$ and A^P are matrices, v_t^Z and v_t are vectors of structural shocks whose elements are mutually uncorrelated by assumption. Policy variables, by assumption, have no contemporaneous affect on macroeconomic variables so that $C_0 = 0$.

Re-write equation (C-2) as

$$P_t = (I - G_0)^{-1} \sum_{i=0}^{n} D_i Z_{t-i} + (I - G_0)^{-1} \sum_{i=1}^{n} G_i P_{t-i} + \mathrm{e}_t \qquad \text{(C-3)}$$

where,

$$\mathrm{e}_t = (I - G_0)^{-1} A^P \mathrm{v}_t \,. \qquad \text{(C-4)}$$

Note that e_t, the vector of residuals from the policy-block VAR, is orthogonal to the residuals from the macroeconomic block. If the elements of $(I - G_0)^{-1} A^P$ are known (C-4) can be used to recover the unobservable v_t, which includes the monetary policy shock, from the observable e_t.

To identify the elements of $(I - G_0)^{-1} A^P$ Bernanke and Mihov (1998) characterize the Fed funds market. Omitting time subscripts let $\mathrm{e}_{\mathrm{FFR}}$ denote innovations in the Fed funds rate, and let v^d denote exogenous shocks to the demand for total reserves. Innovations in total reserve demand, e_{TR}, are then given by

$$\mathrm{e}_{\mathrm{TR}} = -\alpha \mathrm{e}_{\mathrm{FFR}} + v^d \qquad \text{(C-5)}$$

where $\alpha \geq 0$. Also, if $\mathrm{e}_{\mathrm{DISC}}$ denotes innovations in the discount rate, then e_{BR}, which denotes innovations in the demand for borrowed reserves, is given by

$$\mathrm{e}_{\mathrm{BR}} = -\omega(\mathrm{e}_{\mathrm{FFR}} - \mathrm{e}_{\mathrm{DISC}}) + v^b \qquad \text{(C-6)}$$

where v^b denotes exogenous shocks to the demand for borrowed reserves and where $\omega \geq 0$. Innovations in the demand for non-borrowed reserves, $\mathrm{e}^{\mathrm{D}}_{\mathrm{NBR}}$, are by definition

$$\mathrm{e}^{\mathrm{D}}_{\mathrm{NBR}} = \mathrm{e}_{\mathrm{TR}} - \mathrm{e}_{\mathrm{BR}}. \qquad \text{(C-7)}$$

Innovations in the supply of non-borrowed reserves, $\mathrm{e}^{\mathrm{S}}_{\mathrm{NBR}}$, are governed by Federal Reserve policy. Let

$$\mathrm{e}^{\mathrm{S}}_{\mathrm{NBR}} = \phi^d v^d + \phi^b v^b + v^s. \qquad \text{(C-8)}$$

Here v^s, the monetary policy shock, is an exogenous shock to the supply of non-borrowed reserves. The policy parameters, ϕ^d and ϕ^b, describe how the Fed will react to the shocks v^d and v^b. Bernannke and Mihov show that (C-5) – (C-8) can be used to express each of the elements of $(I - G_0)^{-1} A^P$ as a function of the parameters α, ω, ϕ^d, and ϕ^b. Order variables so that $\mathrm{e}_t = [\mathrm{e}_{\mathrm{TR}}\ \mathrm{e}_{\mathrm{NBR}}\ \mathrm{e}_{\mathrm{FFR}}]'$ and $\mathrm{v}_t = [v^d\ v^s\ v^b]'$ and let $V\hat{a}r_T[\mathrm{e}_t] = \left(\frac{1}{T-k}\right)\sum_{t=1}^{T} \hat{\mathrm{e}}_t \hat{\mathrm{e}}_t'$. (We use a constant term plus six lags of the seven variables, so that k = 43.) Since $V\hat{a}r_T[\mathrm{e}_t]$ is a (3x3) symmetric matrix it has six unique elements. Note from (C-4) that

$$E(\mathrm{e}_t \mathrm{e}'_t) = \left[\left(I - G_0\right)^{-1} A^P\right] E(\mathrm{v}_t \mathrm{v}'_t) \left[\left(I - G_0\right)^{-1} A^P\right]'.$$

Since the elements of v_t are i.i.d. by assumption, we can write

$$E(\mathrm{v}_t \mathrm{v}'_t) = \begin{bmatrix} \sigma_d^2 & 0 & 0 \\ 0 & \sigma_s^2 & 0 \\ 0 & 0 & \sigma_b^2 \end{bmatrix}.$$

The matrix $E(\mathrm{e}_t \mathrm{e}'_t)$ is also (3x3) and symmetric. Equating $E(\mathrm{e}_t \mathrm{e}'_t)$ to $V\hat{a}r_T[\mathrm{e}_t]$ therefore places six restrictions on the seven unknown structural parameters: $\alpha, \omega, \phi^d, \phi^b, \sigma_s^2, \sigma_d^2$, and σ_b^2. At least one more restriction is needed to identify these parameters and, hence, the elements of $\left(I - G_0\right)^{-1} A^P$.

Bernanke and Mihov (1998) examine five alternative sets of identifying restrictions. Four of these sets impose two additional restrictions so that the model is over identified. Bernanke and Mihov call their fifth set the "just identified" model as it imposes the single additional restriction that $\alpha = 0$. This restriction is motivated by Strongin's (1995) argument that the demand for total reserves is inelastic in the short run. Impulse-response functions show that a monetary policy shock has qualitatively similar effects under all five sets of restrictions. We therefore take the simplest approach, set $\alpha = 0$, and solve $E(\mathrm{e}_t \mathrm{e}_t') = V\hat{a}r_T[\mathrm{e}_t]$ for the remaining six structural parameters.

We estimate the VAR we using monthly data from December 1961 through August 2004. We obtain monthly observations of the GDP deflator using the state-space procedure of Bernanke, Gertler, and Watson (1997). This procedure uses several monthly series on prices to infer the unobserved monthly value of the GDP Deflator. In the policy block, following Bernanke and Mihov (1998) we render total reserves and non-borrowed reserves stationary by measuring each as a ratio to a 36-month moving average of total reserves. [5] Not surprisingly, our parameter estimates are quite similar to Bernanke and Mihov's.

Having identified the parameters that characterize the money market it is then possible to identify the monetary policy shocks by inverting equation (C-4) to obtain

$$\begin{bmatrix} v^d \\ v^s \\ v^b \end{bmatrix} = \left[\left(I - G_0 \right)^{-1} A^z \right]^{-1} \begin{bmatrix} \mathrm{e}_{\mathrm{TR}} \\ \mathrm{e}_{\mathrm{NBR}} \\ \mathrm{e}_{\mathrm{FFR}} \end{bmatrix}$$

The middle row of this equation is

$$v^s = -\left(\phi^d + \phi^b\right)\mathrm{e}_{\mathrm{TR}} + \left(1 + \phi^b\right)\mathrm{e}_{\mathrm{NBR}} - \left(\alpha\phi^d - \omega\phi^b\right)\mathrm{e}_{\mathrm{FFR}} \qquad \text{(C-9)}$$

Inserting the policy-block residuals for e_{TR}, $\mathrm{e}_{\mathrm{NBR}}$, and $\mathrm{e}_{\mathrm{FFR}}$ on the right-hand side of (C-9) gives the time series of monetary policy shocks, $\left\{v_t^s\right\}_{t=1}^{T}$.

[5] There is a dramatic spike in the reserves data in the months of September and October 2001, following the September 11th attacks. We eliminate this spike by interpolating the series from August 2001 to November 2001.

Additional References for Appendix C.

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