The Interest Rate, Learning and Inventory investment

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APPENDIX B

Derivations

Derivation of Equation (7). Using (3) in (5) the linearized Euler equation can be written as the following fourth- order difference equation in N_t:

$$E_t\Big\{\theta\big[N_t - N_{t-1} + X_t - \bar{\beta}\left(N_{t+1} - N_t + X_{t+1}\right)\big] + \gamma\big[N_t - N_{t-1} - \left(N_{t-1} - N_{t-2}\right) + X_t - X_{t-1}$$

$$-2\bar{\beta}\left(N_{t+1} - N_t - \left(N_t - N_{t-1}\right) + X_{t+1} - X_t\right) + \bar{\beta}^2\left(N_{t+2} - N_{t+1} - \left(N_{t+1} - N_t\right) + X_{t+2} - X_{t+1}\right)\big]$$

$$+\xi\left(W_t - \bar{\beta}W_{t+1}\right) + \delta\bar{\beta}\left(N_t - \alpha X_t\right) + \eta r_{t+1} + c\Big\} = 0\ .$$

Rearranging we have

$$E_t\left[f(L)N_{t+2}\right] = E_t\Psi_t \tag{A.1}$$

where

$$f(L) \equiv 1 - \frac{1}{\gamma\bar{\beta}}\left[\theta + 2\left(1+\bar{\beta}\right)\gamma\right]L + \frac{1}{\gamma\bar{\beta}^2}\left[\theta\left(1+\bar{\beta}\right) + \gamma\left(1 + 4\bar{\beta} + \bar{\beta}^2\right) + \delta\bar{\beta}\right]L^2$$

$$- \frac{1}{\gamma\bar{\beta}^2}\left[\theta + 2\gamma\left(1+\bar{\beta}\right)\right]L^3 + \frac{1}{\bar{\beta}^2}L^4$$

and

$$\Psi_t = -X_{t+2} + \frac{1}{\gamma\bar{\beta}}\left[\theta + \gamma\left(2+\bar{\beta}\right)\right]X_{t+1} - \frac{1}{\gamma\bar{\beta}^2}\left[\theta + \gamma\left(1+2\bar{\beta}\right) - \alpha\delta\bar{\beta}\right]X_t$$

$$+ \frac{1}{\bar{\beta}^2}X_{t-1} - \frac{\xi}{\gamma\bar{\beta}^2}(W_t - \bar{\beta}W_{t+1}) - \frac{\eta}{\gamma\bar{\beta}^2}r_{t+1} - \frac{c}{\gamma\bar{\beta}^2}\ . \tag{8}$$

Let λ_i, i = 1,2,3,4 , denote the roots of the fourth-order polynomial on the left-hand side of (A.1). Order these roots as $|\lambda_1| < |\lambda_2| < |\lambda_3| < |\lambda_4|$. It follows that

$\lambda_4 = \frac{1}{\bar{\beta}\lambda_1}$ and $\lambda_3 = \frac{1}{\bar{\beta}\lambda_2}$, with $|\lambda_1|, |\lambda_2| < \frac{1}{\bar{\beta}}$. Suppose further that $|\lambda_1|, |\lambda_2| < 1$.

Solve the unstable roots forward to obtain

$$E_t N_t = (\lambda_1 + \lambda_2) N_{t-1} - \lambda_1\lambda_2 N_{t-2} + \frac{\bar{\beta}\lambda_1\lambda_2}{(\lambda_1 - \lambda_2)} \sum_{j=0}^{\infty} \left[(\bar{\beta}\lambda_1)^{j+1} - (\bar{\beta}\lambda_2)^{j+1} \right] E_t \Psi_{t+j} \,. \qquad (7)$$

Note that λ_1 and λ_2 are either real or complex conjugates, so that $\lambda_1 + \lambda_2$ and $\lambda_1\lambda_2$ are real.

Derivation of (9). To resolve the forward sum on the right-hand side of (7) note that we assume that sales and input prices follow AR(1) processes:

$X_t = \mu_x + \rho_x X_{t-1} + \varepsilon_{xt}$, where $\varepsilon_{xt} \sim i.i.d.(0, \sigma_x^2)$ and

$W_t = \mu_w + \rho_w W_{t-1} + \varepsilon_{wt}$, where $\varepsilon_{wt} \sim i.i.d.(0, \sigma_w^2)$.

We allow for the special case of $\rho_x = \rho_w = 1$.

1.) The terms involving X on the right-hand side of (7) can be written as

$$\frac{\bar{\beta}\lambda_1\lambda_2}{(\lambda_1 - \lambda_2)} \left\{ \sum_{j=0}^{\infty} \left[(\bar{\beta}\lambda_1)^{j+1} - (\bar{\beta}\lambda_2)^{j+1} \right] E_t \left(-X_{t+2+j} + a_1 X_{t+1+j} - a_0 X_{t+j} + \frac{1}{\bar{\beta}^2} X_{t-1+j} \right) \right\} \qquad \text{(A.2)}$$

where $a_1 \equiv \frac{1}{\gamma\bar{\beta}} \left[\theta + \gamma (2 + \bar{\beta}) \right]$ and $a_0 \equiv \frac{1}{\gamma\bar{\beta}^2} \left[\theta + \gamma (1 + 2\bar{\beta}) - \alpha\delta\bar{\beta} \right]$.

Note that, for j = 0, 1, 2, … , $E_t X_{t+j} = \mu_x + \rho_x E_t X_{t-1+j}$

$E_t X_{t+j+1} = \mu_x (1 + \rho_x) + \rho_x^2 E_t X_{t-1+j}$, and $E_t X_{t+2+j} = \mu_x (1 + \rho_x + \rho_x^2) + \rho_x^3 E_t X_{t-1+j}$.

It therefore follows that

$$E_t\left(-X_{t+2+j}+a_1X_{t+1+j}-a_0X_{t+j}+\frac{1}{\bar{\beta}^2}X_{t-1+j}\right)=$$

$$\left[-(1+\rho_x+\rho_x^2)+a_1(1+\rho_x)-a_0\right]\mu_x+\left(-\rho_x^3+a_1\rho_x^2-a_0\rho_x+\frac{1}{\bar{\beta}^2}\right)E_tX_{t-1+j}$$

and thus,

$$\sum_{j=0}^{\infty}\left(\bar{\beta}\lambda_i\right)^{j+1}E_t\left(-X_{t+2+j}+a_1X_{t+1+j}-a_0X_{t+j}+\frac{1}{\bar{\beta}^2}X_{t-1+j}\right)=\frac{\bar{\beta}\lambda_i\left[-(1+\rho_x+\rho_x^2)+a_1(1+\rho_x)-a_0\right]}{1-\bar{\beta}\lambda_i}\mu_x$$

$$+\beta\lambda_i\left(-\rho_x^3+a_1\rho_x^2-a_0\rho_x+\frac{1}{\bar{\beta}^2}\right)\sum_{j=0}^{\infty}\left(\beta\lambda_i\right)^jE_tX_{t-1+j}. \qquad \text{(A.3)}$$

For the AR(1) process governing X_t ,the forward sum in (A.3) is

$$\sum_{j=0}^{\infty}\left(\bar{\beta}\lambda_i\right)^jE_tX_{t-1+j}=\frac{\bar{\beta}\lambda_i\mu_x}{(1-\bar{\beta}\lambda_i)(1-\bar{\beta}\lambda_i\rho_x)}+\frac{1}{(1-\bar{\beta}\lambda_i\rho_x)}X_{t-1}$$

This in (A.3) gives

$$\sum_{j=0}^{\infty}(\beta\lambda_i)^{j+1}E_t\left(-X_{t+2+j}+a_1X_{t+1+j}-a_0X_{t+j}+\frac{1}{\bar{\beta}^2}X_{t-1+j}\right)=$$

$$c(\rho_x,\lambda_i)\mu_x+\bar{\beta}\lambda_i\left(-\rho_x^3+a_1\rho_x^2-a_0\rho_x+\frac{1}{\bar{\beta}^2}\right)(1-\bar{\beta}\lambda_i\rho_x)^{-1}X_{t-1} \qquad \text{(A.4)}$$

where

$$c(\rho_x,\lambda_i)\equiv\frac{\beta\lambda_i\left[-(1+\rho_x+\rho_x^2)+a_1(1+\rho_x)-a_0\right]}{1-\beta\lambda_i}+\frac{(\bar{\beta}\lambda_i)^2\left(-\rho_x^3+a_1\rho_x^2-a_0\rho_x+\bar{\beta}^{-2}\right)}{(1-\bar{\beta}\lambda_i)(1-\bar{\beta}\lambda_i\rho_x)}.$$

Using (A.4) we can rewrite the term (A.2) as

$$\frac{\bar{\beta}\lambda_1\lambda_2}{\left(\lambda_1-\lambda_2\right)}\left\{\sum_{j=0}^{\infty}\left[\left(\beta\lambda_1\right)^{j+1}-\left(\beta\lambda_2\right)^{j+1}\right]E_t\left(-X_{t+2+j}+a_1X_{t+1+j}-a_0X_{t+j}+\frac{1}{\bar{\beta}^2}X_{t-1+j}\right)\right\}$$

$$=c_X+\Gamma_XX_{t-1} \qquad \text{(A.5)}$$

where $c_X \equiv \frac{\bar{\beta}\lambda_1\lambda_2}{(\lambda_1-\lambda_2)}\left[c(\rho_x,\lambda_1)-c(\rho_x,\lambda_2)\right]\mu_x$ and

$$\Gamma_X \equiv \bar{\beta}^2\lambda_1\lambda_2\left(-\rho_x^3+a_1\rho_x^2-a_0\rho_x+\bar{\beta}^{-2}\right)\left[\frac{1}{(1-\bar{\beta}\lambda_1\rho_x)(1-\bar{\beta}\lambda_2\rho_x)}\right].$$

2.) Proceeding as with the terms in X, the terms involving W on the right-hand side of (7) can be written as

$$\left(-\frac{\xi}{\gamma\bar{\beta}}\right)\frac{\lambda_1\lambda_2}{(\lambda_1-\lambda_2)}\left\{\sum_{j=0}^{\infty}\left[\left(\bar{\beta}\lambda_1\right)^{j+1}-\left(\bar{\beta}\lambda_2\right)^{j+1}\right]E_t(W_{t+j}-\bar{\beta}W_{t+1+j})\right\} = c_W+\Gamma_W W_t \qquad \text{(A.6)}$$

where $c_W \equiv \left(-\frac{\xi}{\gamma\bar{\beta}}\right)\frac{\lambda_1\lambda_2}{(\lambda_1-\lambda_2)}\left[c(\rho_w,\lambda_1)-c(\rho_w,\lambda_2)\right]\mu_w$,

$c(\rho_w,\lambda_i) \equiv \left[\frac{-\bar{\beta}^2\lambda_i}{1-\bar{\beta}\lambda_i}+\frac{(\bar{\beta}\lambda_i)^2(1-\bar{\beta}\rho_w)}{(1-\bar{\beta}\lambda_i)(1-\bar{\beta}\lambda_i\rho_w)}\right]\mu_w$, and

$\Gamma_W \equiv \left(-\frac{\xi}{\gamma}\right)\lambda_1\lambda_2\left[\frac{(1-\bar{\beta}\rho_w)}{(1-\bar{\beta}\lambda_1\rho_w)(1-\bar{\beta}\lambda_2\rho_w)}\right]$. Note that if $\lambda_1\lambda_2>0$ then $\Gamma_W<0$.

3.) Similarly, assuming that r_t follows $r_t=\mu_r+\rho_r r_{t-1}+\varepsilon_{rt}$, where $\varepsilon_{rt}\sim i.i.d.\left(0,\sigma_r^2\right)$, we obtain that

$$\left(-\frac{\eta}{\gamma\bar{\beta}}\right)\frac{\lambda_1\lambda_2}{(\lambda_1-\lambda_2)}\left\{\sum_{j=0}^{\infty}\left[\left(\bar{\beta}\lambda_1\right)^{j+1}-\left(\bar{\beta}\lambda_2\right)^{j+1}\right]E_t r_{t+1+j}\right\}=c_r+\Gamma_r r_t \qquad \text{(A.7)}$$

where

$$c_r \equiv \left(-\frac{\eta}{\gamma}\right)\lambda_1\lambda_2\left\{\left(\frac{1}{\left(1-\bar{\beta}\lambda_1\right)\left(1-\bar{\beta}\lambda_2\right)}\right)+\rho_r\left[\frac{\left(\bar{\beta}\lambda_1\right)^2}{\left(1-\bar{\beta}\lambda_1\right)\left(1-\bar{\beta}\lambda_1\rho_r\right)}-\frac{\left(\bar{\beta}\lambda_2\right)^2}{\left(1-\bar{\beta}\lambda_2\right)\left(1-\bar{\beta}\lambda_2\rho_r\right)}\right]\right\}\mu_r$$

and $\Gamma_r \equiv \left(-\frac{\eta}{\gamma}\right)(\lambda_1\lambda_2)\frac{\rho_r}{\left(1-\bar{\beta}\lambda_1\rho_r\right)\left(1-\bar{\beta}\lambda_2\rho_r\right)}$. Note that, if $\lambda_1\lambda_2 > 0$ then $\Gamma_r < 0$.

4.) Finally,

$$\left(\frac{-1}{\gamma\bar{\beta}^2}\right)\frac{\bar{\beta}\lambda_1\lambda_2}{(\lambda_1-\lambda_2)}\left\{\sum_{j=0}^{\infty}\left[\left(\bar{\beta}\lambda_1\right)^{j+1}-\left(\bar{\beta}\lambda_2\right)^{j+1}\right]c\right\}=\frac{-\lambda_1\lambda_2}{\gamma\left(1-\bar{\beta}\lambda_1\right)\left(1-\bar{\beta}\lambda_2\right)}c. \tag{A.8}$$

5.) Using the results from (A.5), (A.6), (A.7), and (A.8) in (7) we have

$$N_t = \Gamma_0 + \left(\lambda_1+\lambda_2\right)N_{t-1} - \lambda_1\lambda_2 N_{t-2} + \Gamma_X X_{t-1} + \Gamma_W W_t + \Gamma_r r_t + u_t^x, \tag{9}$$

where $\Gamma_0 = c_W + c_X + c_r + \frac{-\lambda_1\lambda_2}{\left(1-\bar{\beta}\lambda_1\right)\left(1-\bar{\beta}\lambda_2\right)}c$,

and where $\Gamma_X \gtrless 0 \qquad \Gamma_W < 0 \qquad \Gamma_r < 0$.

Equation (9) in the Model without Adjustment Costs.

In the model with $\gamma = 0$ equation (3) in (5) yields a second-order difference equation in N_t that has one stable and one unstable root. Denoting the stable root by λ_1, equation (9) becomes

$$N_t = \Gamma_0 + \lambda_1 N_{t-1} + \Gamma_X X_{t-1} + \Gamma_W W_t + \Gamma_r r_t + u_t^x, \tag{9'}$$

where $\Gamma_X \equiv \left(\frac{-\lambda_1}{\theta}\right)\left[\theta\rho_x - \left(\theta\bar{\beta}+\alpha\delta\bar{\beta}\right)\rho_x^2\right]\left(\frac{1}{1-\bar{\beta}\lambda_1\rho_x}\right)$,

$\Gamma_W \equiv \left(\frac{-\lambda_1}{\theta}\right)\xi\left(1-\bar{\beta}\rho_w\right)\left(\frac{1}{1-\bar{\beta}\lambda_1\rho_w}\right)$, and $\Gamma_r \equiv \left(\frac{-\lambda_1}{\theta}\right)\eta\rho_r\left(\frac{1}{1-\bar{\beta}\lambda_1\rho_r}\right)$.

Derivation of Equation (10).

Use equation (3) to substitute for Y_t and Y_{t+1} in equation (5) and rearrange to get

$$E_t\Big\{\gamma\left(\Delta Y_t - 2\bar{\beta}\Delta Y_{t+1} + \bar{\beta}^2\Delta Y_{t+2}\right) - \bar{\beta}\theta\left(\Delta N_{t+1} + \Delta X_{t+1}\right) + \theta\Delta N_t - \bar{\beta}\delta\alpha\Delta X_{t+1}$$

$$+\xi(W_t - \bar{\beta}W_{t+1}) + \delta\bar{\beta}\left[N_t - \left(\alpha - \frac{\theta\left(1-\bar{\beta}\right)}{\bar{\beta}\delta}\right)X_t\right] + \eta r_{t+1} + c\Big\} = 0\ . \qquad \text{(A.9)}$$

Use $\xi(W_t - \bar{\beta}W_{t+1}) = -\bar{\beta}\xi\Delta W_{t+1} + \left(1-\bar{\beta}\right)\xi W_t$ and $\eta r_{t+1} = \eta\Delta r_{t+1} + \eta r_t$ in (A.9) to get (10).

APPENDIX C

Description of Cost Simulations

In our simulations of the cost function in Section V.C, we hold W_t and X_t at their respective sample means in order to isolate the effect of variation in the interest rate. Inventories are determined from the decision rule, equations (7) and (8), using the Markov switching model to express expected future values of the real interest rate as functions of the current filter probabilities, π_{1t} and π_{3t}. Output is determined from sales and the change in N_t using the inventory identity, equation (3). Given output, sales, the cost shock, and the level of inventories, current costs are determined from equation (1). At each horizon we measure the cumulative present discounted value of costs, discounting with the ex post real interest rate. For each parameter setting we repeat the simulation 10,000 times and consider the average behavior of realized costs over the 10,000 repetitions.

We use our estimates of the cointegrating vector to set the values of $\theta, \alpha, \delta, \xi,$ and γ. For our baseline parameters we set $\alpha = 1$ implying that the target level of inventories equals one month's sales, and use the normalization $\delta = 1$. To determine θ and ξ, we equate the estimated coefficients on X_t and W_t from the cointegrating regression in levels (Table 7, row 1) to the corresponding composite parameters in the cointegrating vector (below equation 19). We then determine η using its definition, below equation (5). Since γ cannot be determined from the cointegrating vector, in the baseline simulations we use $\gamma = 0$.

In the first set of simulations we measure the cost of not adjusting to a switch from the low-interest-rate regime to the high-interest-rate regime. The economy begins in the low-interest-rate state and the vector of probabilities is initialized at π_t = [1 0 0] '. We initialize inventories at their long run mean conditional on π_t = [1 0 0] '. We run the model for 26 periods (two years plus two lags) holding π_t = [1 0 0] '. In the 27th period the economy switches to the high-interest-rate state. The firm that responds to this switch therefore begins to choose N_t using π_t = [0 0 1] '. The firm that does not respond continues to choose N_t using π_t = [1 0 0] '.

To measure the cost of not adjusting to a transitory shock we again begin in the low-interest-rate regime. However, in these simulations, π_t is allowed to vary in response to transitory movements in the interest rate. After the initial 26 periods we add to the realized interest rate, in the 27th period, a positive shock equal to one standard deviation of the transitory shock (i.e., equal to σ_1). The filter probabilities of the firm that responds to this shock, π_{1t} and π_{3t}, detect and adjust to this one-time transitory shock. The filter probabilities of the firm that does not respond do not detect or adjust to this shock.

For both simulations, we run the model forward from the 27th period (the period of the shock or regime change) and calculate the cumulative ex post present discounted value of costs at horizons of six, twelve, twenty-four, and forty-eight months. The cost of not responding is measured by subtracting the costs of the firm that responds from the costs of the firm that does not respond.