Technical Appendix

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1 Notations

We use $\vec{C} = (C_1, ..., C_k)$ to denote a vector of k random variables representing costs of contractors submitting bids in a given First Price Sealed Bid auction. The support of the random vector $\vec{C}$ is given by $I_C = [\underline{c}, \overline{c}]^k$ with $\underline{c} > 0$. Functions $F_{\vec{C}} : I_C \to [0,1]$ and $f_{\vec{C}} : I_C \to [0, \infty)$ denote the cumulative distribution function and the probability density function of vector $\vec{C}$ correspondingly.

The support of the bid distribution that arises through the equilibrium bidding behavior is denoted by $I_B = [\underline{b}, \overline{b}]^k$ and $F_{\vec{B}}(.)$ and $f_{\vec{B}}(.)$ denote the corresponding cumulative distribution and probability density functions of the bid vector.

We use $F_{Z_{i_1,...i_l}}(.)$ and $f_{Z_{i_1,...i_l}}(.)$ to denote the cumulative distribution function and probability density function of the joint distribution of sub-vector $Z_{i_1,...i_l}$ of the random vector Z.

2 Sufficient Condition

We assume that data are generated by the First Price Sealed Bid Procurement auction. More specifically, we assume that assumptions $A_1 - A_5$ hold.

(A_1) Bidders are symmetric, i.e., $\vec{C}$ is a vector of exchangeable random variables:

$$f_{\vec{C}}(c) = f_{\pi(\vec{C})}(\pi(c))$$
$$F_{\vec{C}}(c) = F_{\pi(\vec{C})}(\pi(c))$$

for every permutation $\pi(.)$ of the vector $\vec{C}$ components.

(A_2) Function $f_{\vec{C}}(...)$ satisfies the strict affiliation property, i.e., $\frac{\partial^2 f_{\vec{C}}}{\partial C_i \partial C_j} > 0$.

(A_3) Function $f_{\vec{C}}(..)$ is continuously differentiable on the interim of I_C and continuous everywhere on the support, including the boundary.

(A_4) Function $f_{\vec{C}}(..) > 0$ everywhere on the support, including the boundary.

(A_5) For a sub set of four bidders the following inequality holds:

$$f_{C_{i_1,i_2,i_3,i_4}}(\underline{c}, \overline{c}, \underline{c}, \overline{c}) - f_{C_{i_1,i_2}}(\underline{c}, \overline{c}) f_{C_{i_3,i_4}}(\underline{c}, \overline{c}) \neq 0$$

Notice that if (A_5) holds for some sub set of four bidders, it also holds for all sub-sets due to the exchangeability of function $f_{\vec{C}}$.

McAdams (2006) proves that the model with affiliated private values has a unique equilibrium in monotone strategies if (A_1)-(A_4) are satisfied. The equilibrium bidding strategies are continuously differentiable. Moreover, in the procurement auction equilibrium bidding implies that the bid is at least as high as the underlying cost draw. Therefore, $\underline{b} > 0$. We next formulate a testable implication that holds in this environment.

Proposition 4 (as in the paper)

(A_1)-(A_5) *imply the following property:* (B_1) $\quad (\frac{B_{i_1}}{B_{i_2}}, \frac{B_{i_3}}{B_{i_4}})$ *are not independent for any* B_{i_1,i_2,i_3,i_4} *subset of* $\vec{B}$ *such that* $i_j \neq i_k$ *whenever* $j \neq k$.

Proof

Due to monotonicity and continuous differentiability of the bidding functions property (A_5) implies that

$$f_{B_{i_1,i_2,i_3,i_4}}(\underline{b}, \overline{b}, \underline{b}, \overline{b}) - f_{B_{i_1,i_2}}(\underline{b}, \overline{b}) f_{B_{i_3,i_4}}(\underline{b}, \overline{b}) \neq 0 \tag{1}$$

Next we show that under assumptions (A_1)-(A_5) the following equality breaks down:

$$f_{\frac{B_{i_1}}{B_{i_2}}, \frac{B_{i_3}}{B_{i_4}}}(z_1, z_2) = f_{\frac{B_{i_1}}{B_{i_2}}}(z_1) f_{\frac{B_{i_3}}{B_{i_4}}}(z_2)$$

for some z_1, z_2 on the support of $(\frac{B_{i_1}}{B_{i_2}}, \frac{B_{i_3}}{B_{i_4}})$. This implies that $(\frac{B_{i_1}}{B_{i_2}}, \frac{B_{i_3}}{B_{i_4}})$ cannot be independent, since z_1, z_2 are necessarily points of continuity for $f_{\frac{B_{i_1}}{B_{i_2}}, \frac{B_{i_3}}{B_{i_4}}}$ given (A_3) and $\underline{b} > 0$.

Indeed,

$$f_{\frac{B_{i_2}}{B_{i_1}}, \frac{B_{i_4}}{B_{i_3}}}(z_1, z_2) = \int_{\overline{b}}^{\underline{b}} \int_{\overline{b}}^{\underline{b}} f_{B_{i_1,i_2,i_3,i_4}}(b_1, z_1 * b_1, b_3, z_2 * b_3) db_3 db_1. \tag{2}$$

Similarly,

$$f_{\frac{B_{i_2}}{B_{i_1}}}(z_1) = \int_{\overline{b}}^{\underline{b}} f_{B_{i_1,i_2}}(b_1, z_1 * b_1) db_1. \tag{3}$$

and

$$f_{\frac{B_{i_4}}{B_{i_3}}}(z_2) = \int_{\overline{b}}^{\underline{b}} f_{B_{i_3,i_4}}(b_3, z_2 * b_3) db_3. \tag{4}$$

Therefore, we need to show that

$$\int_{\overline{b}}^{\underline{b}}\int_{\overline{b}}^{\underline{b}} f_{B_{i_1,i_2,i_3,i_4}}(b_1, z_1 * b_1, b_3, z_2 * b_3) - f_{B_{i_1,i_2}}(b_1, z_1 * b_1) * f_{B_{i_3,i_4}}(b_3, z_2 * b_3) db_3 db_1 \neq 0. \tag{5}$$

Through the choice of (z_1, z_2) we can ensure that only points in a small neighborhood of $(\underline{b}, \overline{b}, \underline{b}, \overline{b})$ appear in integration in (5).

Without loss of generality let's assume that

$$f_{B_1,B_2,B_3,B_4}(\underline{b}, \overline{b}, \underline{b}, \overline{b}) - f_{B_1,B_2}(\underline{b}, \overline{b}) f_{B_3,B_4}(\underline{b}, \overline{b}) > 0.$$

Then due to continuity of $f_{\vec{B}}$ this inequality holds in some neighborhood of $(\underline{b}, \overline{b}, \underline{b}, \overline{b})$. If we choose z_1 and z_2 in such a way that the values of $(b_1, z_1 * b_1, b_3, z_2 * b_3)$ that lie within the support are confined to the small neighborhood above then

$$f_{\frac{B_{i_2}}{B_{i_1}}, \frac{B_{i_4}}{B_{i_3}}}(z_1, z_2) > f_{\frac{B_{i_2}}{B_{i_1}}}(z_1) * f_{\frac{B_{i_4}}{B_{i_3}}}(z_2).$$

End of Proof

Remark 1 The proof easily extends to the case when the support of cost distribution, I_C, is a convex set in R^k, provided that the boundary has no more than a single point of tangency with any linear hyperspace of the type

$$\{(c_1, c_2, c_3, c_4) : c_i = zc_j \quad \text{for some} \quad z > 0 \quad \text{and} \quad i \neq j\}.$$

Remark 2 Property (B_1) forms the basis of the test for the model with affiliated values versus a model with unobserved heterogeneity. As will be shown below, property (A_5) holds for many affiliated distributions. Therefore, property (B_1) is consistent with affiliated values, while it is not supported by the model with unobserved auction heterogeneity. Hence, accepting (B_1) is indicative of affiliation and is equivalent to rejecting the model with unobserved heterogeneity. On the other hand, if we are able to reject property (B_1) in the data, then the results of the test are inconclusive.

Remark 3 Notice, however, that the conclusion of Proposition 4 obtains without imposing the requirement that the distribution of $\vec{B}$ needs to be affiliated. Therefore, the proposed test really separates the model with unobserved heterogeneity from the set of models with other sources of cost dependency. Further tests are needed to separate models with dependent costs from the model with strictly affiliated costs.

Remark 4 In Proposition 4 due to exchangeability if at least one pair of ratios of distinct bids is independent than all pairs of ratios of distinct bids are necessarily independent. In the non-exchangeable case property (B_1) should be modified to read:

(B_{1a}) There exists at least one subset of $\vec{B}$, B_{i_1,i_2,i_3,i_4} with $i_j \neq i_k$ whenever $j \neq k$, such that $(\frac{B_{i_1}}{B_{i_2}}, \frac{B_{i_3}}{B_{i_4}})$ are not independent.

3 Examples

In this section we study several examples that demonstrate the applicability of the condition in (A_5).

Example 1: Truncated Multivariate Normal Distribution.

Let us consider an auction environment where the joint distribution of bidders' costs is given by a truncated multivariate normal distribution.[1] More specifically, the joint probability density function of bidders' costs is given by

$$f_{\vec{C}}(c) = \frac{1}{(2\pi)^{\frac{k}{2}} * \Sigma^{\frac{1}{2}}} \exp(-\frac{(c-\vec{\mu})'\Sigma^{-1}(c-\vec{\mu})}{2}). \tag{6}$$

Here $\mu = (\mu, ..., \mu)$ denotes the vector of means and Σ represents the variance-covariance matrix with $\Sigma_{ii} = \sigma^2$ and $\Sigma_{ij} = \rho\sigma^2$ for $i \neq j$.

Figure 1 graphs the value of

$$w = f_{C_{i_1,i_2,i_3,i_4}}(\underline{c}, \overline{c}, \underline{c}, \overline{c}) - f_{C_{i_1,i_2}}(\underline{c}, \overline{c}) * f_{C_{i_3,i_4}}(\underline{c}, \overline{c})$$

as a function of $\underline{c}$ holding the difference between $\underline{c}$ and $\overline{c}$ fixed at 1.5 with $\vec{\mu} = (3, 3, 3, 3)$, $\sigma = 1.5$, and $\rho = 0.5$.

Similarly, figure 2 plots the value of w as a function of the value of σ as it changes from 0.4 to 2, holding $\underline{c}$, $\overline{c}$, $\vec{\mu}$ fixed and setting $\rho = 0.5$.

Even allowing the computational error these figures demonstrate that there exist

[1] We continue to assume that bidders are symmetric; i.e., their costs are exchangeable random variables.

many multivariate normal distributions that satisfy property (A_5).

Example 2: Frank's copula.

Next, we consider the set of cost distributions that can be represented by Frank's copula.[2] More specifically, we are interested in the set of multivariate distributions such that

$$F_{\vec{C}}(c) = -\frac{1}{\alpha}\log(1 + \frac{\prod_{i=1}^{i=k}(\exp(-\alpha * F_{C_i}(c_i)) - 1)}{(\exp(-\alpha) - 1)^(k-1)}). \tag{7}$$

Here $F_{C_i}(.)$ denotes marginal distribution of bidder $i's$ costs. $F_{C_i}(.)$ can be an arbitrary distribution of a real-valued variable with bounded support given by $[\underline{c}, \overline{c}]$. $F_{\vec{C}}(c)$ given by the expression in (7) represents a joint distribution of the affiliated random variables for any $\alpha > 0$ as shown in Genest (1987).

Figure 3 shows the value of $w(\alpha)$ as a function of α for the multivariate distribution given by (7) with an arbitrary $F_{C_i}(.)$, $\underline{c}$ and $\overline{c}$. The figure demonstrates that such distributions always satisfy condition (A_5) when $\alpha > 0$.

Example 3: Dependent Bids with Independent Ratios

Examples 1-2 consider distributions that satisfy property (B_1). Next, we describe how a distribution which violates property (B_1) can be constructed. We start by constructing a vector of the dependent random variables (bids) such that one pair of ratios of distinct bids is independent. This does not provide an example of the distribution for which (B_1) fails. However, the logic of this example can be extended to construct an example where more (or even all) pairs of ratios are independent.

A single independent pair of ratios

It is easy to construct an example of the vector of dependent random variables with independent ratios if pairs of bids involved in independent ratios are allowed to be independent. More specifically, consider a vector (B_1, B_2, B_3, B_4) of dependent random variables such that the sub-vectors (B_1, B_2) and (B_3, B_4) are independent. Then, the ratios $(\frac{B_1}{B_2}, \frac{B_3}{B_4})$ are necessarily independent. However, it is not possible to generalize this example to the case where random variables (B_1, B_2, B_3, B_4) are dependent while all pairs of ratios are independent. Below we construct an example of a random vector (B_1, B_2, B_3, B_4) such that the sub-vectors (B_1, B_2) and (B_3, B_4) are dependent and the ratios $(\frac{B_1}{B_2}, \frac{B_3}{B_4})$ are independent.

[2]For an extensive discussion of copulas and their application to the analysis of affiliated distribution, see Nelsen (2006).

More specifically, let us consider a vector $\vec{B} = (B_1, B_2, B_3, B_4)$ of bids with support in $I(B) = [\underline{b}, \overline{b}]^4$ such that the corresponding density function $f_{\vec{B}} : I_B \to [0, \infty)$ is given by

$$f_{\vec{B}}(b) = f_{B_1}(b_1) f_{B_2}(b_2) f_{B_3}(b_3) f_{B_4}(b_4) + h(b_1, b_2, b_3, b_4). \tag{8}$$

In this expression f_{B_i} are marginal distributions of $B_i's$. The function $h(b_1, b_2)$ is defined below. Let us set

$$\begin{aligned}
\Delta_0 &= \frac{\overline{b} - \underline{b}}{2} \\
\Delta_1 &= \frac{\Delta_0}{4} \\
\Delta_{b_3,b_4} &= \Delta_0 + \frac{(\Delta_1 - \Delta_0)(b_4 - \underline{b})(b_3 - \underline{b})}{(\overline{b} - \underline{b})^2}
\end{aligned}$$

Set $h(b_1, b_2, b_3, b_4) = 0$ if $\underline{b} \le b_2 < 0.5 * b_1$ or $2 * b_1 < b_2 \le \overline{b}$. Otherwise, define

$$\begin{aligned}
& h(b_1, b_2, b_3, b_4) = \\
& = 0 \quad \text{if } b_2 \le b_1 \text{ and } \frac{b_1 * \underline{b}}{b_2} \le b_1 \le \frac{\overline{b} + \frac{b_1 * \underline{b}}{b_2}}{2} - \Delta_{b_3,b_4} \\
& = c_1 \quad \text{if } b_2 \le b_1 \text{ and } \frac{\overline{b} + \frac{b_1 * \underline{b}}{b_2}}{2} - \Delta_{b_3,b_4} < b_1 \le \frac{\overline{b} + \frac{b_1 * \underline{b}}{b_2}}{2} \\
& = -c_1 \text{ if } b_2 \le b_1 \text{ and } \frac{\overline{b} + \frac{b_1 * \underline{b}}{b_2}}{2} < b_1 \le \frac{\overline{b} + \frac{b_1 * \underline{b}}{b_2}}{2} + \Delta_{b_3,b_4} \\
& = 0 \quad \text{if } b_2 \le b_1 \text{ and } \frac{\overline{b} + \frac{b_1 * \underline{b}}{b_2}}{2} + \Delta_{b_3,b_4} < b_1 \le \overline{b} \\
& = 0 \quad \text{if } b_1 \le b_2 \text{ and } \frac{b_2 * \underline{b}}{b_1} \le b_2 \le \frac{\overline{b} + \frac{b_2 * \underline{b}}{b_1}}{2} - \Delta_{b_3,b_4} \\
& = -c_2 \text{ if } b_1 \le b_2 \text{ and } \frac{\overline{b} + \frac{b_2 * \underline{b}}{b_1}}{2} - \Delta_{b_3,b_4} \le b_2 \le \frac{\overline{b} + \frac{b_2 * \underline{b}}{b_1}}{2} \\
& = c_2 \quad \text{if } b_1 \le b_2 \text{ and } \frac{\overline{b} + \frac{b_2 * \underline{b}}{b_1}}{2} < b_2 \le \frac{\overline{b} + \frac{b_2 * \underline{b}}{b_1}}{2} + \Delta_{b_3,b_4} \\
& = 0 \quad \text{if } b_1 \le b_2 \text{ and } \frac{\overline{b} + \frac{b_2 * \underline{b}}{b_1}}{2} + \Delta_{b_3,b_4} < b_2 \le \overline{b}
\end{aligned}$$

Figure 4 graphs $h(b_1, b_2, b_3, b_4)$ for $b_3 = \underline{b}$ and $b_4 = \underline{b}$. The function $h(b_1, b_2, b_3, b_4)$ is constructed in such a way that

(W_1) c_1 and c_2 are such that

$$\int_{\underline{b}}^{\overline{b}}\int_{\underline{b}}^{\overline{b}}\int_{\underline{b}}^{\overline{b}}\int_{\underline{b}}^{\overline{b}} h(b_1,b_2,b_3,b_4)db_1db_2db_3db_4 = 0.$$

(In this example, it is sufficient to choose $c_1 = c_2$.)

(W_2) $f_i(b_i|b_{-i})$, conditional distribution of b_i conditional on b_{-i}, implied by $f_{\vec{B}}(b)$ depends on b_{-i};

(W_3) $f_{12}(b_1,b_2|b_3,b_4)$ implied by $f_{\vec{B}}(b)$ depends on (b_3,b_4);

(W_4) $h(b_1,b_2,b_3,b_4)$ integrates to zero along any hyperplane

$$\{(b_1,b_2,b_3,b_4) : b_1 = z * b_2 \text{ for some } z \in [\frac{\underline{b}}{\overline{b}}, \frac{\overline{b}}{\underline{b}}]\}$$

over a segment where $\underline{b} \le b_i \le \overline{b}$ for all i.

(W_1) ensures that $f_{\vec{B}}(b)$ is a proper density. (W_2) implies that the random variables (B_1,B_2,B_3,B_4) are dependent, and (W_3) guarantees that the random sub-vector (B_1,B_2) is not independent of random sub-vector (B_3,B_4). Finally, (W_1) implies that $(\frac{B_1}{B_2},\frac{B_3}{B_4})$ are independent.

Remark 5 We have constructed an example of the vector of dependent random variables (bids) that has a pair of independent ratios of distinct bids. To provide an example of a model with affiliated private values that generates a distribution of dependent bids with a pair of independent ratios we need to modify an example above in such a way that (a) the density $f_{\vec{B}}$ is smooth; (b) f_{B_i}, c_1, c_2 and $\overline{b}-\underline{b}$ should be chosen so that $f_{\vec{B}}$ can be rationalized by a model with affiliated private values, i.e. (b_1) inverse bidding function implied by the first order condition is monotone and (b_2) $f_{\vec{B}}$ is affiliated. This is a much harder problem. While it is straightforward to adjust $h(.,.,.,.)$ in such a way as to achieve (a) and (b_1) it is not clear whether (b_2) can be accommodated.

Generalizing to Two Independent Pairs of Ratios

This example can be further generalized to construct a random vector with more pairs of independent ratios. The idea is to introduce a function $h(.,.,.)$ that would integrate to zero along several planes of type

$$\{(b_1,b_2,b_3,b_4) : b_i = z * b_j \text{ for } i \neq j \text{ and some } z \in [\frac{\underline{b}}{\overline{b}}, \frac{\overline{b}}{\underline{b}}]\}.$$

Define function $h(b_1,b_2,b_3)$ in the three-dimensional cube $\{(b_1,b_2,b_3) : \underline{b} \leq b_i \leq \overline{b}\}$ so that $h(b_1,b_2,b_3) = 0$ unless $\frac{1}{s} \leq \frac{b_1}{b_2} \leq s$ and $\frac{1}{s} \leq \frac{b_1}{b_3} \leq s$ where $s \in [\frac{\underline{b}}{\overline{b}}, \frac{\overline{b}}{\underline{b}}]$ is chosen to be close to one.

Let's denote by (b_i,b_j,z) a plane given by

$$\{(b_1,b_2,b_3) : \ b_i = z * b_j, \text{ for } i \neq j \text{ and } \frac{1}{s} \leq z \leq s\}.$$

We want to define $h(b_1,b_2,b_3)$ so that it integrates to zero over (b_1,b_j,z) for every z such that $\frac{1}{s} \leq z \leq s$ and $j = 2,3$. Also, we have to make sure that the definition of $h(b_1,b_2,b_3)$ is consistent for each intersection of (b_1,b_2,z_1) with (b_1,b_3,z_2).

To clarify the last idea let us consider an intersection of the plane (b_1,b_3,z_2) with the plane $(b_1,b_2,1)$. Let us denote this segment of the line by (n_1,n_2). This segment necessarily lies both in $(b_1,b_3,z_2)\bigcap[\underline{b},\overline{b}]^2$ and in $(b_1,b_2,1)\bigcap[\underline{b},\overline{b}]^2$. For illustration see figure 5. As z_2 changes from $\frac{1}{s}$ to s the intersection with $(b_1,b_2,1)\bigcap[\underline{b},\overline{b}]^2$ sweeps an area in $(b_1,b_2,1)\bigcap[\underline{b},\overline{b}]^2$. This is the only part of $(b_1,b_2,1)\bigcap[\underline{b},\overline{b}]^2$ where $h(b_1,b_2,b_3)$ can take non-zero values. Denote this area as Δ_1.

Similarly, we can consider an area, Δ_{z_1}, which is swept in $(b_1,b_2,z_1)\bigcap[\underline{b},\overline{b}]^2$ for some $\frac{1}{s} \leq z_1 \leq s$ by (b_1,b_3,z_2) as z_2 changes from $\frac{1}{s}$ to s. Again, this area outlines the part of $(b_1,b_2,z_1)\bigcap[\underline{b},\overline{b}]^2$ where $h(b_1,b_2,b_3) \neq 0$. The shape of this area changes in a "continuous way" with z_1. Symmetrically, such an area can be defined in $(b_1,b_3,z_2)\bigcap[\underline{b},\overline{b}]^2$ for some z_2, $\frac{1}{s} \leq z_2 \leq s$, as a collection of intersections of $(b_1,b_3,z_2)\bigcap[\underline{b},\overline{b}]^2$ with all (b_1,b_2,z_1) such that $\frac{1}{s} \leq z_1 \leq s$.

Denote by $g(z_1,z_2,b_1) = h(b_1, z_1 * b_1, z_2 * b_1)$. This function is defined on intersections of the two planes, therefore, it has to be constructed so that

$$g(z_1,z_2,b_1) = g(z_2,z_1,b_1). \tag{9}$$

This is quite feasible since the whole set-up is symmetric. Additionally, $g(z_1,z_2,b_1)$ has to be constructed in such a way that

$$\int_{\Delta_{z_1}} g(z_1, z_2, b_1) I_{(b_2,z_2)} db_1 dz_2 = 0 \text{ for every } z_1 \text{ and} \tag{10}$$
$$\int_{\Delta_{z_2}} g(z_1, z_2, b_1) I_{(b_3,z_1)} db_1 dz_1 = 0 \text{ for every } z_2.$$

Here $I_{(b_i,z_j)}$ denotes the Jacobian which arises during the change of variables from b_i to z_j.

Such a function can be constructed, for example, by initially specifying $g(1, z_2, b_1)$ as described in figure 5. More specifically, define $g(1, z_2, b_1)$ so that it is equal to C_1 if $z_2 < 1$ and C_2 if $z_2 \geq 1$. The constants C_1 and C_2 should be chosen in such a way that

$$\int_{\Delta_1} g(1, z_2, b_1) I_{(b_2,z_2)} db_1 dz_2 = 0.$$

Then, for every $\widetilde{z_2}$ the function $g(z_1, \widetilde{z_2}, b_1)$ should be constructed by adjusting $g(z_1, 1, b_1)$ to reflect how the length of the intersection of (b_1, b_2, z_1) with (b_1, b_3, z_2) changes as z_2 changes from 1 to $\widetilde{z_2}$. This construct should preserve (10) and will necessarily satisfy (9).

If $h(b_1, b_2, b_3)$ is constructed to meet (10) then the pairs of ratios given by $(\frac{b_1}{b_2}, \frac{b_3}{b_4})$ and $(\frac{b_1}{b_3}, \frac{b_2}{b_4})$ are independent.

To complete the example we also have to ensure that

(W_1) $h(b_1, b_2, b_3)$ is such that

$$\int_{\underline{b}}^{\overline{b}} \int_{\underline{b}}^{\overline{b}} \int_{\underline{b}}^{\overline{b}} h(b_1, b_2, b_3) db_1 db_2 db_3 = 0.$$

(W_2) $f_i(b_i|b_{-i})$, conditional distribution of b_i conditional on b_{-i}, implied by $f_{\vec{B}}(b)$ depends on b_{-i};

(W_3) $f_{ij}(b_{ij})|b_{-(ij)})$ implied by $f_{\vec{B}}(b)$ depends on $(b_{(ij)}$.

It should be possible since the definition of $g(z_1, z_2, b_1)$ has a lot of build-in flexibility, i.e. behavior of this function along "b_1" dimension is not restricted.

This approach when applied to the three types of planes: (b_1, b_2, z_1), (b_1, b_3, z_2), and (b_1, b_4, z_3) will produce an example of the random vector where all pairs of the ratios of distinct bids are independent.

4 Implications for Affiliated Distributions

The previous section established that there exist multiple affiliated distributions that satisfy condition (A_5). At the same time we pointed out that there exist affiliated distributions that violate this condition. This section undertakes to evaluate how large is the set of the affiliated distributions that violate condition (A_5) relative to the whole set of affiliated distributions and, therefore, how large is the set of alternatives against which our test potentially has no power.

Let us define for a given set $I_C \subset R^k$ a set of functions, Ω, such that:

(a) $f : I_C \to [0, \infty]$

(b) f is three times continuously differentiable in I_C, i.e. $f \in C^3(I_C)$;

(c) f is exchangeable and strictly affiliated, i.e., $\dfrac{\partial^2 \log(f_{\vec{C}})}{\partial C_i \partial C_j} > 0$;

(d) $f > 0$ everywhere in I_C;

(e) $\displaystyle\int_{I_C} f(c)\ dc = 1.$

The set Ω represents the set of all affiliated distributions with three times continuously differentiable density function with support in I_C.

Similarly, we can define a set of functions, Ω_0, such that:

(a) $f : I_C \to [0, \infty]$

(b) f is three times continuously differentiable in I_C, i.e. $f \in C^3(I_C)$;

(c) f is exchangeable and strictly affiliated, i.e., $\dfrac{\partial^2 \log(f_{\vec{C}})}{\partial C_i \partial C_j} > 0$;

(d) $f > 0$ everywhere in I_C;

(e) $\displaystyle\int_{I_C} f(c)\ dc = 1;$

(f) $f_{C_{i_1,i_2,i_3,i_4}}(c_l, c_u, c_l, c_u) - f_{C_{i_1,i_2}}(c_l, c_u) * f_{C_{i_3,i_4}}(c_l, c_u) = 0.$

The set Ω_0 is a subset of Ω, which consists of all affiliated distributions with three times continuously differentiable density function with support in I_C that violate condition in (A_5).

Finally, associated with the set Ω is a set Θ that consists of functions, $h(.)$, such that:

(a) $h : I_C \to [0, \infty]$;

(b) h is continuously differentiable in I_C, i.e. $h \in C^1(I_C)$;

(c) h is exchangeable;

(d) $h > 0$ everywhere in I_C;

(e) $\dfrac{\partial h}{\partial c_i} = \dfrac{\partial h}{\partial c_j}$ for every $i \neq j$;

The set, Θ, therefore, includes all pairs of continuously differentiable functions that could represent the second partial derivative of the log of exchangeable function, $f_{\vec{C}}$, that satisfies affiliation condition, $\frac{\partial^2 \log(f_{\vec{C}})}{\partial C_i \partial C_j} > 0$ and belongs to $C^3(I_C)$. Notice that condition (e) is superfluous because it is implied by (c).

Proposition 4a

(a) The set Ω is isomorphic to the set $\Theta \times R$.

(b) The set Ω_0 is isomorphic to the set Θ.

Proof

For every function, h, consider a system of partial differential equations:

$$\frac{\partial g(c)}{\partial c_j} = h(c) \text{ for every } j. \tag{11}$$

This is a a so-called 'full' system of partial differential equations that satisfies the regularity conditions, namely, conditions (b),(e) from the definition of Θ. Therefore, a solution of such a system exists and is unique up to a constant.[3] A generic solution of the system can be represented as

$$g(c) = g_0(c) + K_0, \text{ for some } K_0 \in R. \tag{12}$$

Notice that $g(.)$ by construction is exchangeable, continuously differentiable and satisfies conditions that $\frac{\partial g}{\partial c_i} = \frac{\partial g}{\partial c_j}$ for every $i \neq j$. Therefore, a solution of the system of partial

[3] See, for example, Valiron (1986).

differential equations given by

$$\frac{\partial t(c)}{\partial c_j} = g(c) \text{ for every } j. \tag{13}$$

exists and is unique up to a constant. That is, a generic solution of this system of equations is given by $t(c)$

$$t(c) = t_0(c) + K_0 \sum_{i=1}^{i=k} c_i + K_1, \text{ for some } K_1 \in R. \tag{14}$$

Thus, for every function $h(.) \in \Theta$ we have constructed a two-parameter family of functions $f(.|K_0, K_1) = exp(t(.|K_0, K_1)$ such that $\log(f)$ satisfies the affiliation condition. To obtain a conclusion of the lemma it is sufficient to notice that for every K_0 it is possible to find K_1 such that condition (e) in the definition of Ω is satisfied. Therefore, Ω is isomorphic to $\Theta \times R$. Further, constants K_0 and K_1 can always be chosen so that conditions (e) and (f) in the definition of the set Ω_0 is satisfied. Therefore, Ω_0 is isomorphic to Θ.

End of proof

Proposition 4a establishes that the set of the distributions of affiliated random variables that do not satisfy condition (A_5) is small relative to the set of all distributions of affiliated random variables with the same support. In fact, Ω_0 is a set of measure zero for an appropriately defined sigma additive measure on Ω.

5 References

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6 Figures

Figure 1: Multivariate normal: Changing the Lower End of Support

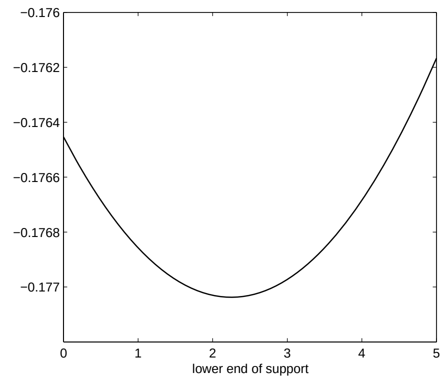
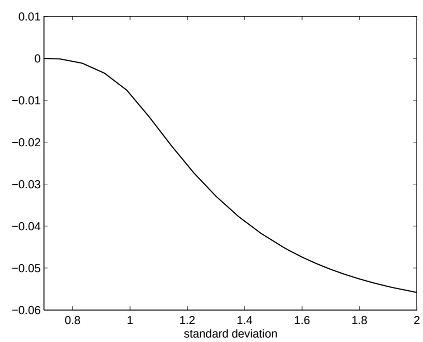


Figure 2: Multivariate Normal: Changing the Variance



The graph in the figure above approaches 0 as σ approaches 0. However, it is not equal to zero.

Figure 3: Frank's Copula: Changing α

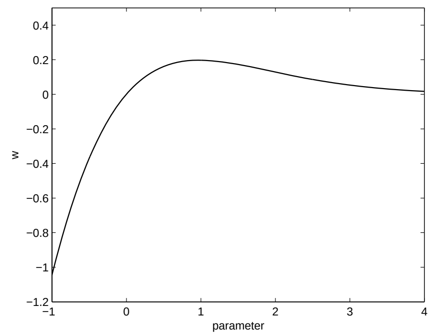


Figure 4: Example 3: Single Independent Pair of Ratios

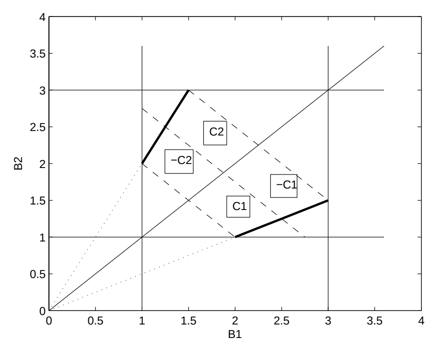


Figure 5

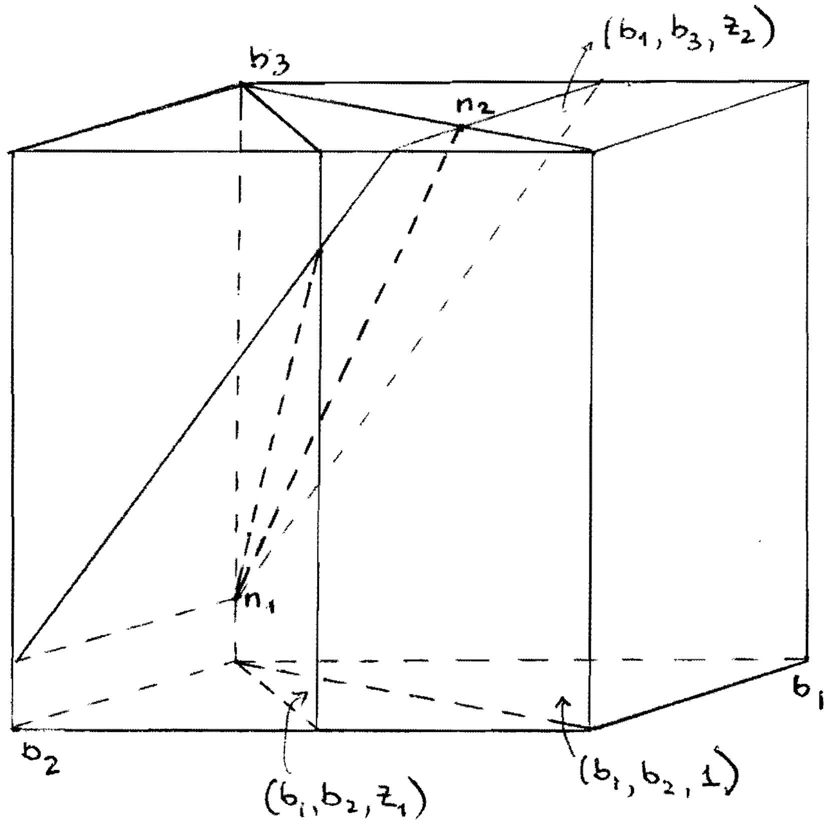


Figure 5: Example 3: Two Independent Pairs of Ratios