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Money-metrics in local welfare analysis: Pareto improvements and equity considerations ☆

Edward E. Schlee [a,*], M. Ali Khan [b]

[a] Department of Economics, Arizona State University, United States of America

[b] Department of Economics, The Johns Hopkins University, United States of America

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Abstract

We identify local Pareto improvements from a valuation equilibrium, and extend the results of Hirshleifer, Arrow-Lind, Milleron, and Radner on the evaluation of small projects to behavioral or nonstandard choice models. We use the sign of directional derivative of the sum of McKenzie-Samuelson money metrics to evaluate small projects, but, rather than assume its differentiability, furnish preference conditions that guarantee it. Our methods yield, as an unintended consequence, (i) a refutation of Samuelson's (1974) conjecture that the money metric is locally concave in a neighborhood of a demand point, thereby settling an issue open for five decades; and (ii) a substantive extension of the 1988 Blackorby-Donaldson theorem that the money metric is concave in consumption only if preferences are quasihomothetic. We explain some equity implications of our local-welfare result, and as part of the rehabilitation of money metrics, suggest a case for using a second-order approximation to a money metric for local welfare. We illustrate when our results hold and don't hold with several non-standard choice models.

JEL classification: D110; C61; D610

Keywords: Money-metric; Local cost-benefit analysis; Saddlepoint inequalities; Non-ordered preferences; Equity; Samuelson's conjecture

☆ In an earlier version of the paper the authors had penned the following dedication "It is with great pleasure, and a greater sense of gratitude, that the authors dedicate this paper to Roy Radner as a token appreciation of the inspiration that they have received from his work and from the intellectual standards that have guided it." It with sadness that the authors learnt of Radner's passing on 6 October 2022: they now dedicate this work to his hallowed memory.

* Corresponding author.

E-mail addresses: edward.schlee@asu.edu (E.E. Schlee), akhan@jhu.edu (M. Ali Khan).

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Since money can be added across people, those obsessed by Pareto-optimality in welfare economics as against interpersonal equity may feel tempted to add money-metric utilities across people and think that there is ethical warrant for maximizing the resulting sum. That would be an illogical perversion, and any such temptation should be resisted.

[Samuelson (1974, p. 1266)]

1. Introduction

The modern second-best theory of optimum commodity and income taxation, as pioneered by Ramsey (1927) and Mirrlees (1971) respectively, has been read as an attempt to identify taxes on commodities and/or income to raise a fixed level of government revenue in the presence of a given price equilibrium when lump-sum taxes are not available.[1] Another approach, associated with Guesnerie (1977), does not seek a second-best optimum, but rather simply *local directions of improvement* in taxes starting from a given system of specific taxes. Our work is in the spirit of this directions-of-improvement tax literature, except that, inspired by a long tradition of cost-benefit for project evaluation stretching from Hirshleifer (1966), Arrow and Lind

[1] In addition to the follow-up treatments of Hicks, Samuelson, Boiteax and Diamond-Mirrleess, see Samuelson (1986 [1951]) and Samuelson (1982) for the classical references, and Stern (1987), Hammond (2000), and Bierbrauer et al. (2023) for current work. The interested reader may also wish to connect to Hotelling (1938), Chapter 3 in Debreu (1983), and to Hammond (1990).

(1970), Milleron (1970), and Radner (1993), we remain silent about what policy changes lead to consumption changes.[2]

But unlike the tax reform and local cost-benefit literature, we do not assume that consumer preferences are either complete or transitive. Such nonstandard preferences could arise simply because the consumer is really a household, and collective household choices cannot be rationalized by ordered preferences[3]; or because individual choices cannot be so rationalized.[4] We thereby bring together the already-mentioned tradition on project evaluation and cost-benefit analysis in the small with the behavioral underpinnings of welfare economics as envisioned, for example, by Bernheim and Rangel (2009).[5] We do this through the use of the McKenzie-Samuelson money-metric function. To the point, we show that if, starting from a valuation equilibrium, the directional derivative of the sum of individual money metrics is positive, then a small change in the aggregate production plan in that direction can be allocated to give a strict Pareto improvement (Proposition 3). This result gives an optimality foundation for those who, like Deaton (2003), recommend a first-order approximation to the money metric for welfare analysis. This is a primary contribution of the paper, but one which necessitates for its execution additional lines of investigation.

These additional investigations concern properties of the *money metric* of McKenzie (1957) and Samuelson (1974). The money metric is the smallest income or wealth at fixed prices that leaves a consumer at least as well off as with a given consumption bundle. Continuity of the money metric in consumption has been extensively investigated for the classical case of ordered preferences,[6] but it is of some surprise that differentiability in consumption has not been touched in the antecedent literature even for ordered preferences. In particular, we give sufficient conditions for its differentiability for nonordered preferences (Proposition 1).

Why is our local welfare result surprising or even interesting for applied welfare economics? By way of background, a startling theorem in an influential paper of Blackorby and Donaldson (1988) asserts that a money metric cannot be a concave function of consumption for every

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[2] Hirshleifer and Arrow-Lind emphasize uncertainty with subjective-expected-utility (SEU) preferences. Radner and Milleron can accommodate uncertainty with non-SEU preferences. Milleron (1970) is an English-language version of the earlier-published Milleron (1969). Varian (1978, Section 7.5) summarizes the content of Radner's note, and cites it as a 1974 working paper. Radner (1993, p. 141) cites Milleron (1970) "for a related discussion," and Luenberger (1996, p. 454) summarizes the Milleron-Radner results.

[3] Chiappori and Mazzocco (2017) survey models of collective household choice. The unitary model assumes that household choice can be rationalized by ordered preferences. The main interest for us are non-unitary models. As a simple, if implausible, example, a household could rank alternatives by majority rule.

[4] For example, Echenique et al. (2011) find that 80% of households—single- and multi-person—in their empirical application using grocery store scanner data violate the generalized axiom of revealed preference (GARP). Even so, they fail to reject GARP, using their "money-pump" statistic to test for severity of GARP departures. Statistical tests of GARP notoriously suffer from weak power, however; they do not report any measure of the power and indeed point to difficulties in developing such a measure. In their comment, Smeulders et al. (2013) emphasize the *heterogeneity* of conformity with GARP in their data, an observation which suggests that welfare analysis must take seriously departures from GARP. Regarding experimental data, Cason and Plott (2014) point to the need to distinguish between what they call misperceptions of the game form, and recovering preferences, including their consistency with GARP.

[5] Green and Hojman (2007) is complementary. See also Rubinstein and Salant (2012), Fleurbaey and Schokkaert (2013), and Mandler (2014).

[6] Shafer (1980), Weymark (1985), and Honkapohja (1987). Martínez-Legaz and Santos (1996) consider an infinite-commodity version.

price except for the knife-edge case of quasihomothetic preferences.[7] The point is that since the money-metric is not generally concave in consumption, then the sum of money metrics is not generally even quasiconcave, and the implied social preferences over allocations are not convex. This follows as a consequence of a theorem of Debreu and Koopmans (1982) that the quasi-concavity of a sum of functions implies that all but one of the individual functions are concave. Blackorby and Donaldson (1988) draw out what they see to be the negative implications of their result coupled with that of Debreu-Koopmans this way:

> Quasiconcavity [of a social evaluation function on allocations] *ensures* that social judgements provide goods for everyone rather than giving them exclusively to the few. Samuelson called this requirement "the foundation for the economics of the good society." Use of money metrics ... may result in a social-evaluation function ... that is not quasiconcave (p. 121). [S]ince many plausible preference orderings yield nonconcave money metrics for all reference prices, we must conclude that social welfare analysis based on money metrics is flawed, despite the fact that money metrics provide exact indices of individual households' well-being (p. 129, emphasis added).

Schlee and Khan (2022) document both the widespread estimation and use of money metrics in applied welfare economics in the last century, and how the theorems and arguments of Blackorby-Donaldson helped lead to its withering away.[8] But, at the risk of pedantry, we remind the reader that quasiconcavity of an objective does not imply that its maximizers are interior; nor does non-quasiconcavity imply corner solutions. These facts were already pointed out clearly by Gorman (1959, pp. 490-1, Figs. 1-2). Gorman's point is that quasiconcavity of a representation of social preferences is neither a necessary nor sufficient condition for an allocation to be equitable.[9]

Indeed, Schlee and Khan (2022, Theorem 2) prove that any competitive equilibrium allocation maximizes the money-metric sum, provided that each consumer's money metric is evaluated at the associated competitive equilibrium price. Obviously, competitive allocations can be wildly inegalitarian, giving everything to one person, for example; or *fair*, in the sense of being Pareto optimal and envy-free: no consumer prefers another's consumption to its own Kolm (1997 [1972]). The result suggests that, at least in some applications, maximizers of the money-metric sum on feasible allocations can be largely neutral with respect to equity; this very neutrality lies at the heart of Samuelson's disavowal of the money-metric sum as a welfare function that appears as the epigraph of this paper.[10]

[7] Special cases include *homothetic*, *quasi-linear*, and *Stone-Geary* preferences; see Deaton and Muellbauer (1980, chapter 6), or Mas-Colell et al. (1995, chapter 4) for textbook treatments. Such preferences generate straight-line Engel curves when demand is single-valued, an implausible implication.

[8] Decancq et al. (2015, Chapter 2) summarize the history of money metrics thus: "[M]oney-metric utility ... has a somewhat surprising history. [It] had some impact on the applied welfare economic literature during the eighties [but] lost popularity ... as authors argued that it relied on an arbitrary choice of a reference values and could have nonegalitarian implications." A notable exception is Angus Deaton, who recommends using a first-order approximation to the money metric for welfare; see Footnote 14 for a summary of his argument.

[9] Bosmans et al. (2018, p. 467-9) present a related critique of Blackorby-Donaldson, expanded to include Fleurbaey and Maniquet (2011). Gorman (1959) had also pointed out that non-convexity of social preferences over allocations could lead to discontinuities in optimal allocations as a function of the economy's parameters, another point of complaint in Blackorby and Donaldson (1988, p. 129). An emphasis on discontinuities constitutes Guesnerie's second corner of second-best theory; see Chiappori (2010).

[10] As we wrote in our 2022 *IER* paper, we stress that Samuelson's criticism is based on a lack of virtue (ethical warrant) and not a positive vice (that it is apt to select inegalitarian allocations); to our knowledge, Samuelson was not aware

The nonconcavity of the money metric in consumption *does* sometimes imply that the money-metric sum increases with an increase in inequality. This can be vividly illustrated by a two-consumer exchange economy with identical monotone and convex ordered preferences. (See also Fleurbaey and Maniquet (2011, pp. 20-1).)

Example 1 *(Two-person symmetric and convex exchange economy with nonconcave money metrics).* Consider a two-person exchange economy with identical preferences on the consumption set $X = \mathbb{R}^2_+$ that are complete, transitive, continuous, monotone, and strictly convex, but not quasihomothetic.[11] By the Blackbory-Donaldson theorem, the associated money metric is not concave for *some* price vector $p' \in \mathbb{R}^2_{++}$. Denote that money metric by $f : X \to \mathbb{R}_+$. (In words, $f(x)$ gives the smallest wealth at prices p' needed to be at least as well off as consuming the bundle x.) Since f is in addition continuous under these preference assumptions (Shafer, 1980, Lemma), we know that it is not *mid-point concave*, which is to say, for some points x and y in X,

$$2f(\tfrac{1}{2}x + \tfrac{1}{2}y) < f(x) + f(y). \tag{1}$$

The sum of these money-metrics is higher at the unequal allocation which gives one consumer x and the other y than if they both get $(x + y)/2$. Now suppose that the aggregate endowment in this exchange economy is $x + y$. Then the equal-allocation Pareto optimum in which each consumer gets $(x + y)/2$ can be supported as a competitive equilibrium by some price p^*. Letting each consumer's money metric now be evaluated at $p = p^*$, the already-mentioned Schlee-Khan Theorem 2 implies that *this* money-metric sum is maximized at this equal-consumption allocation of the aggregate endowment at $p = p^*$, so (1) would fail for those money metrics. □

The example shows two things: that the nonconcavity of the money metric at some prices implies that the money-metric sum sometimes violates a principle of equity that, if two people have the same preferences, then each should be indifferent between its own and the other's consumption—a simple application of the idea of *envy-free* allocations, that no consumer should prefer another's consumption to its own.[12] But that, when evaluated at a different price vector—one that supports equal consumption as an equilibrium – that implication disappears. Still, whatever its merits, the Schlee-Khan theorem is silent about this question:

> Suppose we start from an equal-consumption competitive equilibrium in Example 1 and evaluate each consumer's money metric at the supporting price p^*; if we perturb the economy's aggregate endowment, then what allocation of the perturbed endowment maximizes the money-metric sum? in particular, does an equal-consumption allocation maximize it?

of the money-metric nonconcavity. For what it is worth, we *emphatically* agree with Samuelson that there is no *ethical* warrant for maximizing the sum, just as is there is no ethical warrant for mere Pareto optimality. At one level, our goal is to restore the money-metric sum to the contempt that Samuelson held it in, as ethically neutral, that is, just concerned with optimality. Here we do address equity in Section 5.

[11] In the context of this example, preferences are quasihomothetic if there is a representation U of preferences such that the expenditure function is of the affine-in-utility form $e(p, \bar{U}) = a(p) + b(p)\bar{U}$, with $\bar{U}$ in the range of U; or equivalently, there is a representation of the indirect utility that takes the affine-in-wealth form $V(p, w) = \alpha(p) + \beta(p)w$. Application of Roy's Identity reveals that demand is affine in wealth for each price.

[12] Kolm (1997 [1972], p. 79) writes the simple implication thus: "A state is equitable with identical preferences if and only if the x_i all belong to the same indifference class." It is also related to the *transfer among equals* axiom in social choice, as in Fleurbaey and Maniquet (2011, p. 31).

This question is closely related to our main result, Proposition 3, on small changes in the aggregate production plan and local Pareto improvements.

Samuelson (1974, p. 1276) makes an intriguing conjecture relevant to this question. He asserts that a consumer's money metric is concave in a *neighborhood* of a demand point, provided that the money metric is evaluated at a price that supports that point as a demand.[13] If true, Samuelson's local-concavity conjecture would preclude unequal treatment of equals for a small-enough perturbation of the economy's endowment in Example 1, starting from an equal-allocation competitive equilibrium at price p^*: the money-metric sum would not only identify Pareto improvements, but would avoid at least this glaring inequity in its distribution. Unfortunately for this hope, we prove in Section 5 (Proposition 5) that Samuelson's conjecture is false: if the money metric is concave in consumption on some convex set containing a demand point, then demand on that set is severely restricted: if demand is single valued, then the wealth expansion path must be a straight line on that set, an implication of ordered quasihomothetic preferences. An unintended consequence of this result is a new proof of and extension of the Blackorby-Donaldson theorem to non-ordered preferences: the convex set in our Proposition 5 can be taken to be the entire consumption set.

Now what Samuelson (1974) does correctly assert without proof is this: if the money-metric is C^2, then its Hessian is negative semidefinite when evaluated at an interior demand point. It follows that a *second-order approximation* of a money metric in consumption around a demand point *is* concave. As already mentioned, Angus Deaton advocates using a first-order approximation to the money metric for welfare analysis. But he reluctantly comes to this conclusion in spite of the Blackorby-Donaldson criticism.[14] Our Proposition 3 asserts that the sum of such first-order approximations can identify strict Pareto improvements. And Samuelson's observation about the money-metric Hessian implies that a second-order approximation of the money metric sum is concave, and so avoids any inegalitarian implications implied by its nonquasiconcavity. Our combined results suggest an argument for using a second-order approximation to the money-metric sum for local welfare analysis, at least as a back-of-the-envelope guide.[15]

The fact that we use to prove our results is surprising in view of the Blackorby-Donaldson proof that money metrics are not generally concave in consumption: even though money metrics are not concave, and even though they do not in general represent consumer preferences, any demand point must solve the celebrated saddlepoint inequalities from *concave programming* for the problem of choosing a consumption plan to maximize a money metric on a budget set; see Schlee and Khan (2022). Moreover the multiplier always equals unity, giving a concrete meaning to Marshall's elusive "constancy of the marginal utility of money." We reproduce this money-metric saddlepoint theorem in Section 2.3.

Before proceeding, we warn the reader about terminology: sometimes "money metric" refers to any welfare measure expressed in money. In this usage, the textbook compensating variation is a money metric, as is the welfare measure that Blackorby and Donaldson (1987) propose as

[13] We reproduce the exact text in Section 5.

[14] Deaton and Zaidi (2002) compare the merits of the money-metric sum and an alternative to money metrics developed in Blackorby and Donaldson (1987) called *welfare ratios* and conclude, that, "though while money metric utility is more problematic for distributional calculations, ... [o]ur own choice is to stick with money metric utility" (p. 11). Their reasoning is that welfare ratios represent continuous, ordered preferences on $\mathbb{R}^L_+$ *only if* preferences are homothetic. See also Deaton (2003) and Deaton (1985).

[15] Bosmans et al. (2018) present a complementary rationale for money metrics in welfare analysis. We discuss their contribution in more detail in Schlee and Khan (2022, Section 5), and in Section 4.3.

an *alternative* to the McKenzie-Samuelson money metric—at least after it is converted, as is customary, to a monetary amount as in Deaton (2003, p. 140). Here we specifically refer to the McKenzie-Samuelson money metric function, formally defined in Section 2.2.[16]

At the risk of some repetition, in terms of a broad overview of this paper, we see the contribution of this paper along five directions: (i) local-cost benefit analysis executed through directional derivatives of the money metric sum in a setting of non-ordered preferences; (ii) justification of this differentiability through preference assumptions, a result not available in the antecedent literature, and relate (i) and (ii) to a notion of smoothness analyzed by Neilson (1991) and Rubinstein (2012); (iii) a refutation of a conjecture of Samuelson's that has remained open since 1974; (iv) a substantive extension of the Blackorby-Donaldson theorem that the money metric is concave in consumption if and only if demand is quasihomothetic; and finally, (v) an illustration of when our results hold or not for non-ordered preferences using three nonstandard or behavioral choice models: *regret theory, multiple-selves models,* and *price-dependent preferences,* the last a special *case of menu-dependent preferences.*[17]

2. Preliminaries

This section consists of three parts: first, we remind the reader of a weakening of transitivity that we use for one of our results, and highlight an axiom due to Danan (2008); second, we briefly remind the reader of some definitions and properties of money metrics; and finally, we reproduce our money-metric saddle-point theorem that we use throughout.[18]

2.1. Weakenings of the transitivity postulate

There are $L > 1$ commodities, and we take the consumption set X to be $\mathbb{R}^L_+$. Let $\succsim \subset X \times X$ be a reflexive binary relation, interpreted as weak preference, with asymmetric part $\succ$ and symmetric part $\sim$. The relation $\succsim$ is *reflexive* if $x \succsim x$ for every $x \in X$; it is *complete* if, for every x and y in X, either $x \succsim y$ or $y \succsim x$; it is transitive if, for every x, y, and z in X with $x \succsim y$ and $y \succsim z$, $x \succsim z$. It is closed if the set $\{(x, y) \in X^2 \mid x \succsim y\}$ is closed; it is locally nonsatiated (LNS) if, for every $x \in X$ and open neighborhood N of x in $\mathbb{R}^L$, there is a $y \in N \cap X$ such that $y \succ x$. The relation $\succ$ is *open* if $\{(x, y) \in X \times X \mid x \succ y\}$ is an open relative to $X \times X$. For some results we replace transitivity with one of these two postulates.

[16] As a prominent example, Feldman and Serrano (2006, Chapter 10.8) consistently use the phrase "money metric" in this broad sense. The money metric *is* closely related to the textbook equivalent variation: For preferences represented by a function u with indirect utility function V and expenditure function e, the *indirect money metric function* for price-wealth pairs (p, w) is defined by $e(p^0, V(p, w))$, which after subtracting w, equals the equivalent variation; see Mas-Colell et al. (1995, p. 81) and set their reference price $\bar{p}$ equal to their p^0. But since the compensating variation does not in general equal the equivalent variation, the indirect money-metric function does not equal the compensating variation, even up to an additive constant. All the more, the money metric does not equal Marshallian consumer's surplus, even for quasilinear preferences, as we point out in Schlee and Khan (2022, Section 2.2).

[17] We remind readers of these models as they appear in the text, but here we just write that we picked these three examples out of the ocean of possibilities simply to illustrate the results, not to be exhaustive.

[18] Schlee and Khan (2022) give more detail on money metrics and Khan and Schlee (2016) trace the influence of McKenzie's minimum income function on 20^{th}-century demand theory.

Definition 1. A binary relation $\succsim \subset X \times X$ is **semitransitive** if $x \succ y \succ z \succsim t$ or $x \succsim y \succ z \succ t$ implies that $x \succ t$. It is **strongly upper nonsatiated** (SUN) if, for every $x \succsim y$ and every neighborhood N of x, there is a $z \in N \cap X$ with $z \neq x$ and $z \succ y$.

Strong upper nonsatiation rules out 'thick' indifference sets (Danan, 2008, pp. 28-9). If $\succsim$ is reflexive, then SUN implies LNS. We use the next fact.

Lemma 1. *If $\succsim \subset X \times X$ is locally non-satiated (LNS) and semitransitive, then it is strongly upper nonsatiated (SUN).*

Proof. Suppose that $\succsim$ is LNS and semitransitive. Let $x \succsim y$ and let N be any open neighborhood of x. By LNS there is a $z \in N \cap X$ with $z \succ x$. Now let N' be any open neighborhood of z that is contained in N. By LNS, there is a $z' \in N' \cap X$ with $z' \succ z$. So $z' \succ z \succ x \succsim y$. By semitransitivity, $z' \succ y$. □

2.2. *Money metrics for nonordered preferences*

For a price-wealth pair $(p, w) \gg 0$, the budget set and the demand correspondence are

$$B(p, w) = \{x \in X \mid p \cdot x \leq w\} \text{ and } d(p, w) = \{y \in B(p, w) \mid x \succ y \Rightarrow x \notin B(p, w)\}. \quad (2)$$

We define the money metric to be

$$M(x, p) = \inf_{x' \succ x} p \cdot x'. \quad (3)$$

If $\succsim$ is strongly upper nonsatiated, then this definition agrees with McKenzie's (1958) definition, $\inf_{x' \succsim x} p \cdot x'$; and if $\succsim$ is both closed and strongly upper nonsatiated, it agrees with McKenzie's (1957) definition, $\min_{x' \succsim x} p \cdot x'$; see Schlee and Khan (2022, Proposition).[19]

If $\succsim \subset \mathbb{R}^L_+ \times \mathbb{R}^L_+$ is complete, transitive, LNS, and closed, with $x \succsim 0$ for every $x \in X$, then $M(\cdot, p)$ represents $\succsim$ Shafer (1980); Weymark (1985) is a standard reference for counterexamples when the consumption set does not equal $\mathbb{R}^L_+$. None of our results assume transitivity, so the money metric obviously does not represent preferences here.

2.3. *A money-metric saddlepoint theorem*

Schlee and Khan (2022) prove this saddlepoint theorem for a money metric.[20]

[19] In Schlee and Khan (2022), we used a weaker version of SUN that requires $x \sim y$ for the conclusion $z \succ y$ to hold. And the Proposition showing the equivalence between the money-metric definitions imposes our maintained assumptions in that paper, but they are not required for the equivalence.

[20] Our summary combines Theorem 1 and Corollary 1 from Schlee and Khan (2022). We stress that here we assume that $X = \mathbb{R}^L_+$. This assumption allows us to write the saddlepoint theorem in a streamlined form. The theorem is closely related to Propositions 4.8 and 4.9 in Martínez-Legaz and Santos (1996). They however assume transitivity, whereas our novelty is to drop ordered preferences altogether. We achieve this by using what Schlee and Khan (2022) call the *upper* money metric, equation (3), whereas they use what we call the lower money metric, McKenzie's (1958) definition, replacing strict preference in (3) with weak preference. They also assume an infinite-dimensional commodity space; in ongoing work we show that our money-metric SP theorem generalizes to infinite dimensional spaces.

Theorem *(Money-Metric Saddlepoint). Suppose that $X = \mathbb{R}^L_+$, $\succsim \subset X \times X$ is reflexive and locally nonsatiated, and $\succ$ is open. Fix $p \gg 0$. Define $\mathcal{L}(x, \lambda) = M(x, p) + \lambda[w - p \cdot x]$, and consider the saddlepoint inequalities for $(x^*, 1)$ with $x^* \in X$:*

$$\mathcal{L}(x, 1) \leq \mathcal{L}(x^*, 1) \leq \mathcal{L}(x^*, \lambda) \qquad \text{for every } x \in X \text{ and } \lambda \geq 0. \tag{4}$$

The saddlepoint inequalities (4) hold (a) if and (b) only if $x^* \in d(p, p \cdot x^*)$.

Since the multiplier in the saddlepoint inequalities is constant and equal to 1, an immediate implication is that $x^* \in d(p, p \cdot x^*)$ if and only if x^* maximizes $M(x, p) - p \cdot x$ on $X = \mathbb{R}^L_+$. The theorem allows us to convert the standard consumer's constrained preference-maximization problem into an *unconstrained* optimization problem. This is so despite the money-metric non-concavity—precluding application of Uzawa's (1958) classic *concave*-programming saddlepoint Theorem 2—and despite that the money metric generally does not represent preferences, and indeed when there is no preference representation at all. Another way to phrase the saddlepoint theorem is this: p^* is a supergradient of $M(\cdot, p^*)$ at $x = x^*$ if and only if $x^* \in d(p^*, p^* \cdot x^*)$.[21]

3. Money-metric differentiability

We now turn to differentiability of the money metric in consumption at a demand point, on which our results on first-order welfare properties of the money-metric sum turn. Define $S = \{p \in \mathbb{R}^L_{++} \mid \sum p_\ell = 1\}$ and a correspondence $g : X \rightrightarrows S$ by

$$g(x) = \{p \in S \,|\, y \succ x \Rightarrow p \cdot y > p \cdot x\}. \tag{5}$$

It is sometimes called an inverse demand correspondence since $x \in d(p, p \cdot x)$ if and only if $p \in g(x)$.[22]

Define a correspondence $R : X \rightrightarrows \mathbb{R}^L_+$ by $R(x) = cl\{y \in X \mid y \succ x\}$, where "*cl*" denotes the closure of a set. And define the compensated demand correspondence to be

$$h(p, x) = \left\{y \in X \mid p \cdot y = M(x, p) \text{ and } y \in R(x)\right\}. \tag{6}$$

Proposition 1 *(Money metric differentiability). Suppose that $\succ$ is open and LNS and that R has a closed graph. Let $(x^0, p^0) \gg 0$ satisfy $x^0 \in d(p^0, p^0 \cdot x^0)$ and $h(p^0, x^0) = \{x^0\}$. Then $M(\cdot, p^0)$ is differentiable at $x = x^0$ if in addition the correspondence g*

(a) *is nonempty on some open neighborhood $N_0 \subset X$ of x^0 and upper-hemicontinuous at $x = x^0$; and*
(b) *satisfies a Lipschitz-like condition that, for some open neighborhood $N \subseteq N_0$ of x^0, there is a real number $K > 0$ such that, for every $x \in N$, there exists $\gamma_x \in g(x)$ with*

$$\|p^0 - \gamma_x\| \leq K\|x^0 - x\|.$$

[21] Khan and Piazza (2011) and Khan and Piazza (2012) use this supportability condition at a point in place of concavity of the felicity functions.

[22] Mas-Colell (1974) uses (the weak inequality version of) this correspondence to prove existence of equilibrium in an economy whose consumers have nonordered preferences. We use the notation of Chapter 15 of Debreu (1983) as does he. Georgescu-Roegen (1936) uses substantively the same construct for consumers with intransitive preferences. When it is nonempty and single-valued, it is sometimes called a *vector-space representation* of preferences; Al-Najjar (1993) uses it to analyze smooth non-transitive preferences. For a comprehensive treatment of inverse demand functions in the context of duality theory, see Weymark (1980).

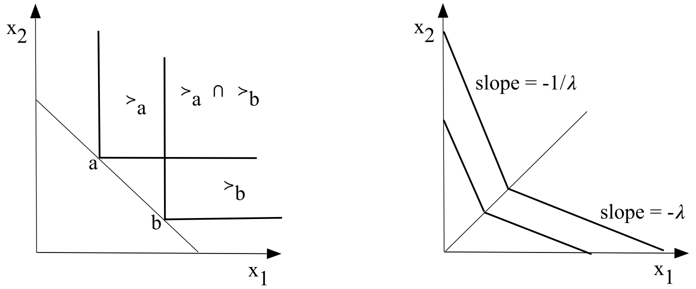

Fig. 1. Illustrations for Examples 2 and 4. The left diagram depicts the better-than-sets for two points, a and b, $\succ_a$ and $\succ_b$, for preferences that generate the money metric $M(x, p) = p \cdot x$. The demand is the entire budget line, but the compensated demand is always a singleton. The right diagram depicts an indifference curve for the utility function $u(x) = \min\{\lambda x_1 + x_2, x_1 + \lambda x_2\}$ for $\lambda \in (0, 1)$, whose money metric is not differentiable at any point along the line $x_1 = x_2$. Every assumption of Proposition 1 holds except either (b) or the single-valuedness of h.

We prove this result in Section 7. Here we make a few remarks. First, we do not assume that preferences are complete, transitive, or convex. We do assume that the compensated demand h is single-valued at (p^0, x^0). We present three examples to illustrate Proposition 1, and refer to them later to illustrate possible failures of our local welfare results.

Example 2 *(Linear money metric).* Consider the money metric $M(x, p) = p \cdot x$. It is generated by any relation $\succsim$ with asymmetric part $\succ$ satisfying $x \succ y$ if and only if $x \gg y$; the left diagram of Fig. 1 depicts a couple of better-than-sets for this example. The relation $\succsim$ is not ordered; $R(x) = \{y \in X \mid y \geq x\}$, and $g(x) = S$ for any $x \gg 0$; and, for any $(x, p) \gg 0$, $h(p, x) = \{x\}$, $x \in d(p, p \cdot x) = \{y \in X \mid p \cdot y = p \cdot x\}$. It is easy to verify that all the conditions of Proposition 1 are satisfied; in particular, g is constant and so uhc at any $x \gg 0$; and we can take $\gamma_x = p^0$ in part (b) for every $x \in N$, so that the left side of the inequality is 0. □

Example 3 *(Bewley "multiple selves" example).* In Bewley's (2002) model of incomplete preferences, there is a single physical good and a finite number of states of the world. There is a fixed utility u on $\mathbb{R}_+$, but a *set* of probability distributions on the states. Let there be just two states, and two probability distributions, π^1 and π^2, with $1 > \pi_1^1 > \pi_1^2 > 0$. In his model $(y_1, y_2) \succ (x_1, x_2)$ if and only if $\sum u(y_s)\pi_s > \sum u(x_s)\pi_s$ for *both* $\pi = \pi^1$ and $\pi = \pi^2$. Suppose in addition that u is strictly increasing, strictly concave, and C^1; see the right diagram in Fig. 2 below for an illustration of a typical kinked indifference curve (kinked because of multiple beliefs). Here it is easy to verify that

$$g(x) = \left\{ p \in S \mid p_1 \in \left[\frac{u'(x_1)\pi_1^2}{\sum u'(x_s)\pi_s^2}, \frac{u'(x_1)\pi_1^1}{\sum u'(x_s)\pi_s^1} \right] \right\}.$$

Since the endpoints of the interval defining g are continuous, g is uhc at any $x \gg 0$. Fix any $x^0 \gg 0$ and any p^0 in the interior of $g(x^0)$. As with Example 2, condition (b) is satisfied because we can take $\gamma_x = p^0$ provided that x is close enough to x^0. And $h(p^0, x^0) = \{x^0\}$. The money metric is $M(x, p) = p \cdot x$ on some neighborhood of (x^0, p^0) and so differentiable in x at (x^0, p^0). □

A *sufficient* condition for (b) is that $g(\cdot)$ is single-valued on a neighborhood of x^0 and differentiable at $x = x^0$: If g is differentiable at x^0, then

$$\frac{|g(x) - g(x^0) - \nabla g(x^0) \cdot (x - x^0)|}{\|x - x^0\|}$$

is bounded by some number, call it B; by the triangle inequality,

$$\frac{|g(x) - g(x^0)|}{\|x - x^0\|} \leq B + \frac{|\nabla g(x^0) \cdot (x - x^0)|}{\|x - x^0\|} \leq B + \sqrt{L}\|\nabla g(x^0)\|_\infty,$$

so that (b) holds.

To illustrate the role of (b), suppose that preferences are represented by a continuous, concave, strictly increasing, and homogeneous-of-degree-1 function u, so that $\succsim$ is closed, convex, strongly monotone, and *homothetic*. Then the money metric takes the form $M(x, p) = u(x)b(p)$ for some function b of the prices. Here R is certainly upper hemicontinuous. If u is continuously differentiable, then the inverse demand g is single-valued and continuous at any $x \gg 0$. And if u is twice differentiable, then g certainly satisfies (b), by the argument in the last paragraph. But if u is not differentiable at a demand point, then obviously the money metric is not differentiable there (at any price that supports the demand point). What fails in Proposition 1 is either condition (b), or that the compensated demand is single-valued. The next example illustrates this point.

Example 4 *(Ordered preferences and kinks: a nondifferentible money metric).* Let $L = 2$ and $u(x_1, x_2) = \min\{\lambda x_1 + x_2, x_1 + \lambda x_2\}$ for $\lambda \in (0, 1)$, which is continuous, strictly increasing, concave, and homogenous of degree 1. We show typical indifference curves in the right diagram of Fig. 1. In the limit as $\lambda \to 0$, preferences converge to "Leontief" perfect complements. Set $x^0 = (r, r)$ for any positive real number r. The money metric is not differentiable in x at (x^0, p) at any $p \in g(x^0)$. If either $p = (\lambda, 1 - \lambda)$ or $p = (1 - \lambda, \lambda)$, then $p \in g(x^0)$, but $h(p, x^0)$ is not single valued. For any other $p^0 \in g(x^0)$, condition (b) fails (and in particular g is not lower-hemicontinuous at x^0): $g(x) \in \{(\lambda, 1 - \lambda), (1 - \lambda, \lambda)\}$ for any $x \gg 0$ with $x_1 \neq x_2$, and the left side of the inequality in (b) is bounded away from 0 for any such sequence of x's with limit x^0. □

Example 4 has a negligible set of kinks but a non-differentiable money metric. Examples 2 and 3 have *everywhere*-kinked preferences but a differentiable money metric. Interestingly, everywhere-kinked preferences with convex better-than sets naturally have differentiable money metrics.

4. Local Pareto improvements

In this section, we turn to the central results of the paper concerning project evaluation and local welfare analysis. We show how the money-metric sum can either rule out or identify Pareto improvements.

Consider an economy with I consumers. Consumer i's reflexive preference relation is $\succsim_i \subset X \times X$, with $X = \mathbb{R}^L_+$, antisymmetric part $\succ_i$ and symmetric part $\sim_i$. An *allocation* is a point $\mathbf{x} = (x_1, ..., x_I) \in X^I$, and *aggregate consumption* is $x = \sum_i x_i$. We say that an allocation $\mathbf{x}'$ is a *Pareto improvement* over allocation $\mathbf{x}$ if $x'_i \succsim x_i$ for every consumer $i = 1, ..., I$, with $x'_i \succ x_j$ for at least one consumer $1 \leq j \leq I$. It is a *strict* Pareto improvement if the preference can be taken to be strict for *each* consumer $i = 1, ..., I$.

4.1. Why Pareto improvements?

Workaday economists often use the criterion of *potential* Pareto improvements for policy recommendations, requiring merely that there is some reallocation of the new supply that would make everyone better off than the starting allocation. As is well-understood, one objection to the criterion is it can give contradictory rankings if compensation is not carried out to make it an actual Pareto improvement: both a change in total supply and its undoing can be potential Pareto improvements.[23] Partly for this reason, we follow the spirit of the tax reform literature pioneered by Guesnerie (1977) in seeking actual Pareto improvements.

A potential objection to an actual Pareto improvement is that it can reinforce inequities in the starting allocation or introduce new ones. Of course it can also reinforce equitable properties of the starting allocation. We find Pareto improvements most defensible as a policy criterion when they reinforce desirable equity properties of the starting allocation (or undo inequities). For example, if the starting allocation is fair—Pareto optimal and envy-free—then the Pareto improvement should remain envy-free or at least preserve equal treatment of equals. The rest of this section we show how the money-metric sum can be used either to rule out or to find Pareto improvements; we address some potential equity concerns related to money-metric nonconcavity in Section 5.

4.2. A necessary condition on the money-metric sum

The next result gives a necessary condition for one allocation to be a Pareto improvement over another, and so *rules out* Pareto improvements, local or not. We re-emphasize that we do not assume that any consumer's preferences are complete or transitive, or even LNS. Although simple, the result is also useful, and we think worth reporting, given the hard earth of welfare theory with non-ordered preferences that we are plowing here.

Proposition 2. *Suppose that for each $i = 1, ..., I$, $\succ_i$ is open, and $x \succsim_i 0$ for every $x \in X$. Consider an allocation $\mathbf{x} = (x_1, .., x_I)$. The allocation $\mathbf{x}' = (x'_1, .., x'_I)$ with aggregate consumption x' is a strict Pareto improvement over $\mathbf{x}$ only if $p \cdot x' > \sum_i M_i(x_i, p)$ for every $p \gg 0$.*

Proof. Consider an allocation $\mathbf{x}$. Let $\mathbf{x}' = (x'_1, ..., x'_I)$ be an allocation with total consumption x', and suppose that $p \cdot x' \leq \sum_i M_i(x_i, p)$ for some $p \gg 0$, and that $x'_i \succ x_i$ for $i = 1, ..., I - 1$. We will show that $x'_I \not\succ x_I$. Since for $i = 1, ..., I - 1$, each $\succ_i$ is open, and $x'_i \neq 0$ (from $z \succsim_i 0$ for every $z \in X$), there exists a $\lambda_i \in (0, 1)$ such that $\lambda_i x'_i \succ_i x_i$, implying that $p \cdot x'_i > p \cdot \lambda_i x'_i \geq M_i(x_i, p)$. Sum to find $p \cdot \sum_1^{I-1} x'_i > \sum_1^{I-1} M_i(x_i, p)$. Since $p \cdot x' \leq \sum_i M_i(x_i, p)$, it follows that $p \cdot x'_I < M_I(x_I, p)$, so $x'_I \not\succ x_I$. □

Proposition 2 requires the strict inequality $p \cdot x' > \sum_i M_i(x_i, p)$ to hold for *every* $p \gg 0$, so if it fails for a single price, then no feasible allocation of the supply x' can be a strict Pareto improvement over $\mathbf{x}$.

[23] In addition to the classic reference of Scitovsky (1941), Feldman and Serrano (2006, Chapter 10) has an elegant overview of compensation criteria, and Fleurbaey (2022) has a philosophical overview.

4.3. A sufficient condition on the money-metric sum

The necessary condition of the last section is not sufficient, a fact evident in a single-consumer economy (Fig. 2 below). Here we prove a partial converse to Proposition 2. It is partial in four senses. First, it is only a local result (Fig. 2, left diagram). Second, we impose the condition that the starting consumption allocation is a *valuation equilibrium with respect to some price* $p \gg 0$.[24] Third, we require that each consumer's money metric be differentiable at the starting valuation equilibrium. Fourth, we impose the condition that the *completion* of each $\succsim_i$ is *strongly upper nonsatiated* (Definition 1). (As we explain later, the right diagram of Fig. 2 is relevant for the third and fourth senses.)

Let $\tilde{y} : [0, 1] \to Y$, where $Y \subseteq \mathbb{R}^L$ is a connected aggregate production set with nonempty interior. Letting $\omega \in \mathbb{R}^l_+$ be the aggregate endowment of the economy, the total supply of goods is $\omega + \tilde{y}(\alpha)$ for $\alpha \in [0, 1]$. This framework we take from Radner (1993); as mentioned, it is similar to that used in the *tax reform* literature that looks for local improvements from some equilibrium. But we emphasize that we are silent about how the aggregate production plan is chosen; in particular it need not be part of a competitive equilibrium allocation. It could, for example, be determined in part by regulation, or could be part of an imperfectly competitive equilibrium. If the aggregate production plan is partly determined by distortionary taxes, then we take the price in a valuation equilibrium to be consumer prices, not producer prices. More generally, we imagine that the production plan changes because of different *policies*, which we abstractly index by the real number α.

A *feasible* consumption allocation of the total supply $\omega + \tilde{y}(\alpha) \in \mathbb{R}^L_+$ is a point $\mathbf{x} \in X^I = \mathbb{R}^{I\times L}_+$ such that total consumption $x = \sum_{i=1}^I x_i$ equals $\omega + \tilde{y}(\alpha)$, where x_i is consumer i's bundle. The valuation equilibrium assumption at $\alpha = 0$ assures that the consumption allocation of the supply $\tilde{y}(0) + \omega$ is Pareto optimal: any inefficiency is in the choice of an aggregate production plan in $\{y \in Y \mid y = \tilde{y}(\alpha) \text{ for some } \alpha \in [0, 1]\}$. In what follows $\tilde{\mathbf{x}}(\alpha) = (\tilde{x}_1(\alpha), ..., \tilde{x}_I(\alpha))$ will be a particular feasible consumption allocation of the supply $\omega + \tilde{y}(\alpha)$. Define $\mathcal{M}(\alpha) = \sum_i M_i(\tilde{x}_i(\alpha), p^0)$, the sum of money-metrics. Our main result of this subsection uses the next two easy lemmata, which we write separately for ease of reference.

Lemma 2. *Suppose that, for* $i = 1, ..., I$

(a) $\succsim_i$ *is reflexive and LNS, and* $\succ_i$ *is open;*
(b) $\tilde{\mathbf{x}}(0) \gg 0$ *is a valuation equilibrium relative to some* $p^0 \in \mathbb{R}^L_{++}$*;*
(c) $\tilde{x}_i(\alpha) = \tilde{x}_i(0) + s_i\big(\tilde{y}(\alpha) - \tilde{y}(0)\big)$, *where* $s_i > 0$, $\sum_i s_i = 1$, *and* $\tilde{y}(\cdot)$ *is differentiable at* $\alpha = 0$*; and*
(d) $M_i(\cdot, p^0)$ *is differentiable at* $x_i = \tilde{x}_i(0)$.

Then

$$\mathcal{M}'(0) = p^0 \cdot \tilde{y}'(0). \tag{7}$$

[24] A consumption allocation $\mathbf{x} = (x_1, ..., x_I)$ is a *valuation equilibrium* relative to a price $p \in \mathbb{R}^L_{++}$ if $x_i \in d_i(p, p \cdot x_i)$ for each $i = 1, ..., I$.

Lemma 3. *Suppose that (a), (b) and (d) from Lemma 2 hold. Then*

$$\nabla_{x_i} M_i(\tilde{x}_i(0), p^0) = p^0. \tag{8}$$

Proof of Lemma 3. Since $\tilde{x}_i(0) \in d_i(p^0, p^0 \cdot \tilde{x}_i(0))$, $\tilde{x}_i(0)$ maximizes $M_i(x, p^0) - p^0 \cdot x$ on $\mathbb{R}^L_+$ by the money-metric SP Theorem(a) in Section 2.3. And since $M_i(\cdot, p^0)$ is differentiable at $\tilde{x}_i(0)$, the necessary Kuhn-Tucker conditions must hold at $x_i = \tilde{x}_i(0)$. Since $\tilde{x}_i(0) \gg 0$, the Kuhn-Tucker conditions hold as equalities, and (8) holds. □

Proof of Lemma 2. By Lemma 3, and (a)-(d), each $\alpha \mapsto M_i(\tilde{x}_i(\alpha), p^0)$ is differentiable at $\alpha = 0$ and the derivative equals $s_i\, p^0 \cdot y'(0)$. Sum over all consumers to find $\mathcal{M}'(0) = p^0 \cdot \tilde{y}'(0)$. □

Let $\succsim_i^c$ be the completion of $\succsim_i$: $x \succsim_i^c y$ if and only if $y \not\succ_i x$. The dot product on the right side of (7) is a local measure of welfare,[25] as the next result, our main one, confirms, even for non-ordered preferences.

Proposition 3 *(Local Strict Pareto Improvement). Suppose that (a)-(d) of Lemma 2 hold and that $\succsim_i^c$ is strongly upper nonsatiated for $i = 1, .., I$. If $\mathcal{M}'(0) > 0$, then there is an $\bar{\alpha} > 0$ such that, for every $\alpha' \in (0, \bar{\alpha})$, the allocation $\tilde{\mathbf{x}}(\alpha) = (\tilde{x}_1(\alpha), ..., \tilde{x}_I(\alpha))$ strictly Pareto dominates $(\tilde{x}_1(0), ..., \tilde{x}_I(0))$, for small-enough $\alpha > 0$.*

Proof of Proposition 3. Fix a consumer i. By Lemmata 2 and 3, $\mathcal{M}_i'(0) = s_i\, p^0 \cdot \tilde{y}'(0) > 0$, which implies $\mathcal{M}_i(\alpha) > \mathcal{M}_i(0)$ for every α in some interval $(0, \alpha_i)$. Fix $\alpha \in (0, \alpha_i)$. By the money-metric SP theorem(a), $M_i(\tilde{x}_i(0), p^0) = p^0 \cdot \tilde{x}_i(0)$, and so

$$M_i(\tilde{x}_i(\alpha), p^0) > p^0 \cdot \tilde{x}_i(0). \tag{9}$$

It follows from the definition of the money metric (3) that $\tilde{x}_i(0) \not\succ_i \tilde{x}_i(\alpha)$, and so, $\tilde{x}_i(\alpha) \succsim_i^c \tilde{x}_i(0)$. Strong upper nonsatiation (SUN) of $\succsim_i^c$ implies that, for any $x' \in X$, if $x' \succsim_i^c \tilde{x}_i(\alpha)$, then $p^0 \cdot x' \geq M_i(\tilde{x}_i(\alpha), p^0)$, since x' is the limit of a sequence $x^n \succ_i \tilde{x}(\alpha)$. From this implication and (9), it follows that $\tilde{x}_i(\alpha) \succ_i \tilde{x}_i(0)$. To finish the argument, set $\bar{\alpha} = \min\{\alpha_1, ..., \alpha_I\}$. □

Proposition 3—together with Proposition 1—gives a foundation for using the derivative of the money-metric sum for local welfare analysis, even absent transitivity.[26]

Note that Lemma 2 implies that the derivative of the money-metric sum does not depend on how the extra output is allocated to the consumers (the s_i's). This fact gives a sense in which the money-metric sum is ethically neutral, at least in this application. We will take up higher-order properties of the money-metric sum—which do have implications for equity—in the next section.

Next we point out implications of Proposition 3 for a familiar weakening of transitivity, *semi-transitivity*. It follows immediately from Lemma 1 and Proposition 3.

[25] See Hirshleifer (1966, p. 271), building on Hirshleifer (1965), Arrow and Lind (1970, p. 368-73), Milleron (1969, p. 94), and Radner (1993).

[26] Fountain (1981) points out difficulties in measuring welfare changes "in the large" when preferences are intransitive, but does not consider small changes starting from a valuation equilibrium, the subject of Proposition 3. Compare his Fig. 1 to the left diagram in our Fig. 2, where $x^0 \in h(p^0, x^0)$ (by the SP theorem).

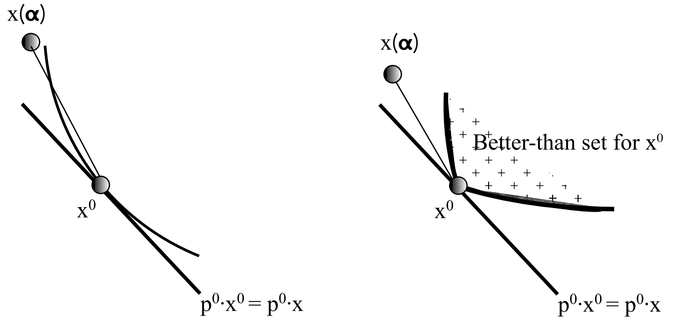

Fig. 2. Illustrations of Proposition 3 for a single-consumer economy, where $x^0 \in d(p^0, p^0 \cdot x^0)$ and $x^0 = \tilde{x}(0)$. The left diagram shows that if the policy change is too large, then the conclusion can fail, even if the indifference set is smooth. The right diagram shows that the conclusion can fail if the indifference set is kinked at x^0: $p^0 \cdot \tilde{x}(\alpha) > p^0 \cdot x^0$ for every $\alpha \in (0, 1]$, but $\tilde{x}(\alpha) \not\succ x^0$ for every $\alpha \in (0, 1]$, where $\tilde{x}(\alpha) = \alpha\tilde{x}(1) + (1 - \alpha)x^0$.

Corollary 1 *(Semitransitivity). If semitransitivity replaces SUN in Proposition 3, then the conclusion of Proposition 3 holds.*

Assumption (d) on the differentiability of the money metric in Lemma 2 and Proposition 3 can of course be replaced by the assumptions of Proposition 1 at $(p, x) = (p^0, \tilde{x}_i(0))$ for $i = 1, ..., I$. Radner (1993, p. 136) goes beyond us—and Milleron (1969) and Arrow and Lind (1970)—in assuming that the parameter α indexes a family of valuation equilibria, and that equilibrium prices and quantities vary smoothly with it, closer to the spirit of the directions-of-improvement-tax-reform literature Guesnerie (1977). Radner's (1993) stronger condition can be justified from primitives, and indeed extended to smooth *non-transitive* preferences, a subset of nontransitive preferences considered by Shafer (1974). The example also illustrates one way to implement a strict Pareto improvement: simply transfer a fraction of change in output to consumers and let them trade in competitive markets.

Example 5 *(Smooth economies).* Let $\tilde{y} : (-1, 1) \to Y$ be differentiable. Consider a family of economies with fixed preferences but the entire supply $\omega + \tilde{y}(\alpha)$ is given to consumers as an endowment indexed by $\alpha \in (-1, 1)$, and consumers trade in the resulting exchange economy. In particular set $\tilde{\omega}_i(\alpha) = \omega_i + s_i(\tilde{y}(\alpha) - \tilde{y}(0))$ as consumer i's endowment, $s_i > 0$, $\sum_i s_i = 1$. The Remark in Debreu (1970, p. 390) implies that, if each consumer has a C^1 demand (satisfying a boundary condition), then, unless the point $\tilde{\boldsymbol{\omega}}(0) = (\tilde{\omega}_1(0), ..., \tilde{\omega}_I(0))$ happens to lie in a particular closed subset of $\mathbb{R}^{LI}_+$ of Lebesgue-measure zero, an exchange economy with endowment in some neighborhood V of $\tilde{\boldsymbol{\omega}}(0)$ has a finite and constant number of equilibria, each given by a C^1 function on V. Take $\tilde{p}(\alpha)$ to be one of these and restrict α to an open interval J containing 0 with $\tilde{\boldsymbol{\omega}}(\alpha) \in V$ on J. It follows that each consumer's equilibrium consumption for this equilibrium selection, $\tilde{x}_i(\alpha) = d_i(\tilde{p}(\alpha), \tilde{p}(\alpha) \cdot \tilde{\omega}_i(\alpha))$, is C^1 on J. Since Debreu (1970) works directly with demand, rather than preferences, the conclusion is unaffected by preference intransitivity. In particular, Al-Najjar (1993) identifies a class of smooth intransitive preferences that generate C^1 demands and so can satisfy the demand conditions in Debreu (1970). This justifies the smoothness condition in Radner (1993) for nontransitive consumers. □

Proposition 3 uses the money-metric sum to identify Pareto improvements, but the conclusion holds for other money-metric aggregates. Let $W : \mathbb{R}^I_+ \to \mathbb{R}$ be differentiable with positive

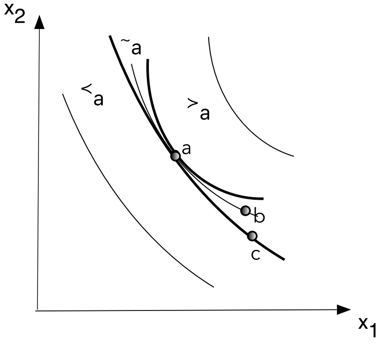

Fig. 3. An example that violates SUN, but is Rubinstein-smooth at $x = a$. Here $b \succ c$ and $b \sim a \sim c$, indeed, a is indifferent to any point between the two thicker curves: an open neighborhood of b is indifferent to a. For a single consumer economy, if p^0 supports $x = a$ as a demand, and $p^0 \cdot \tilde{y}'(0) > 0$, then the conclusion of Proposition 3 holds. The relation R is not upper hemicontinuous at $x = a$; and, for $p = p^0$, the money metric is not continuous at $x = a$.

first derivatives. If we replace the derivative of the money-metric sum with the derivative of $W(M_1, ..., M_I)$, then the conclusion of Propostion 3 holds, since the derivatives of both money-metric aggregates are proportional to $p^0 \cdot \tilde{y}'(0)$. Because of their concavity and symmetry, Schur-concave functions are a natural candidate for W. The already-mentioned Bosmans et al. (2018) axiomatize such a money-metric aggregate, and so the conclusion of Proposition 3 certainly extends to their class of welfare functions. They assume that consumer preferences are ordered, monotone, and convex. Two natural questions arise for nonordered preferences: what welfare properties does their functional form retain; and is there an axiomatic foundation for it?

4.3.1. Relationship to Neilson-Rubinstein smoothness

The conclusion of Proposition 3 is related to Rubinstein's (2012, Lecture 4) notion of smoothness. Unlike Rubinstein, we do not impose monotonicity or convexity or transitivity in our definition.

Formally,

Definition 2 *(Rubinstein smoothness).* Let $\succsim \subset X \times X$ be complete and LNS, and $\succ$ open. It is **Rubinstein-smooth** at $x^* \gg 0$ if there is a point $q \in \mathbb{R}^L_+$ such that these properties are equivalent for any $z \in \mathbb{R}^L$

(a) $q \cdot z > 0$;
(b) There is a $\lambda^* > 0$ such that $x^* + \lambda^* z \gg 0$ and, for every $\lambda \in (0, \lambda^*)$, $x^* + \lambda z \succ x^*$.

Diasakos and Gerasimou (2022) is a recent application of this notion of smoothness. They prove that, under some conditions, Rubinstein-smoothness is equivalent to a smoothness notion analyzed by Neilson (1991). Such smoothness directly rules out preference kinks in the sense that the inverse demand g defined in (5) is sometimes multivalued. An easy implication of Definition 2 is that, if x is a demand point, then it is supported by exactly one normalized price. As is evident, kinks, whatever their source, can lead to the failure of the conclusion of Proposition 3 (Fig. 2, right diagram). From the viewpoint of our sufficient conditions for Proposition 3, kinks result in money-metric nondifferentiability for ordered preferences, as in Example 4 in Section 3; for non-ordered preferences, preferences can be everywhere-kinked with differentiable money metrics, as in Examples 2 and 3; what fails in Proposition 3 for those two examples is the condition that the completed preference relation be SUN.[27] Fig. 3 illustrates an example of preferences that are Rubinstein-smooth, but which violate SUN.

Proposition 4. *Suppose that (a) and (b) from Lemma 2 hold.*

(i) *If (d) from Lemma 2 holds and $\succsim_i^c$ is SUN, then $\succsim_i$ is Rubinstein-smooth at $x_i = \tilde{x}_i(0)$ for $i = 1, ..., I$.*
(ii) *If (c) from Lemma 2 holds, $\succsim_i$ is Rubinstein-smooth at $x_i = x_i^0$ for $i = 1, ..., I$, and $p^0 \cdot \tilde{y}'(0) > 0$, then the conclusion of Proposition 3 holds.*

Proof. (i) Consider a consumer i, but in what follows suppress the consumer i subscript. Suppose first that there is a λ^* such that, for $0 < \lambda < \lambda^*$, $\tilde{x}(0) + \lambda z \succ \tilde{x}(0)$ for some $z \in \mathbb{R}^L$. Set $q = p^0$. Since $\tilde{x}(0) \in d(p^0, p^0 \cdot \tilde{x}(0))$, it follows that $p^0 \cdot \tilde{x}(0) < p^0 \cdot (\tilde{x}(0) + \lambda z) = p^0 \cdot \tilde{x}(0) + \lambda p^0 \cdot z$, so $p^0 \cdot z > 0$.

Now suppose that $p^0 \cdot z > 0$. Let $\Lambda = \{\lambda > 0 \mid \tilde{x}(0) + \lambda z \geq 0\}$. By the SP Theorem(a), $\tilde{x}(0)$ maximizes $M(x, p) - p \cdot x$ on $X = \mathbb{R}_+^L$. Since $\tilde{x}(0) \gg 0$ and $M(\cdot, p)$ is differentiable at $\tilde{x}(0)$, the first order condition $\nabla_x M(\tilde{x}(0), p^0) = p^0$ holds. It follows that

$$\frac{d}{d\lambda} M(\tilde{x}(0) + \lambda z, p^0) \mid_{\lambda=0} = p^0 \cdot z > 0.$$

By (a)-(b) and (d), and a slight modification of the proof of Proposition 3, there is a $\lambda^* \in \Lambda$ such that, for $0 < \lambda < \lambda^*$, $\tilde{x}(0) + \lambda z \succ \tilde{x}(0)$.

To prove (ii), set $x^* = \tilde{x}_i(0)$, $q = p^0$, and $z = \tilde{y}'(0)$ in the definition of Rubinstein smoothness to conclude that, for $i = 1, ..., I$, $\tilde{x}_i(0) + \lambda_i \tilde{y}'(0) \succ_i \tilde{x}_i(0)$ for all $\lambda_i \in (0, \lambda_i^*)$ for some $\lambda_i^* > 0$. Normalize the λ_i's so that they sum to 1. □

It is worth noting from part (ii) that, if (a)-(d) of Lemma 2 holds, and each $\succsim_i$ is Rubinstein-smooth at x_i^0, then the conclusion of Proposition 3 holds without imposing SUN. Fig. 3 gives an example of nontransitive preferences that are Rubinstein-smooth, but not SUN.

4.3.2. An illustration: regret theory

As we explain in the next subsection 4.4, both SUN and Rubinstein smoothness rule out all but trivial sorts of incompleteness in some choice models, in particular a class of "multiple-selves" models. A classical nontransitive, but complete, choice theory is *regret theory*, as developed by Loomes and Sugden (1982), Bell (1982), and Fishburn (1982). Although mainly applied to monetary outcomes, it is easily adapted to consumption choices. Imagine a consumer who must choose a consumption plan today, not knowing exactly what its preferences will be.[28] Formally

[27] An implication of the theorem in Schlee (2021) is that, if preferences are reflexive, continuous, monotone, and convex, then preferences are "everywhere kinked " – g is multivalued at every $x \in \mathbb{R}_+^L$ –*only if* they are not ordered.

[28] Models with uncertain future tastes are common; Schmalensee (1972) and Kreps (1979) are classic references. Diecidue et al. (2012), Nasiry and Popescu (2012), and Özer and Zheng (2016) specifically apply regret theory to consumer demand problems, three of a legion of such papers.

suppose that

$$x \succsim z \quad \text{if and only if} \quad \underbrace{\sum_{s=1}^{S} p_s Q\left(u(x,s) - u(z,s)\right)}_{k(x,z)} \geq 0 \tag{10}$$

where $Q : \mathbb{R} \to \mathbb{R}$ is a strictly increasing, skew-symmetric function ($Q(-\xi) = -Q(\xi)$) that is convex on $\mathbb{R}_+$; and $\{1, ..., S\}$ a set of states that determines the consumer's *ex post* preferences, which are represented by $u(\cdot, s)$ in state s.[29] The function Q gives the degree of regret. When Q is affine, then (10) collapses to transitive expected-utility preferences. But if Q is convex on $\mathbb{R}_+$ (but not affine), preferences are not transitive.

Shafer (1974) shows that, if $\succsim$ is complete, closed, strongly convex,[30] but not necessarily transitive, then demand—in the sense of a *best affordable* point—exists; as does a function $k : X \times X \to \mathbb{R}$ satisfying $x \succsim z$ if and only if $k(x,z) \geq 0$, with k *skew symmetric*: $k(x,z) = -k(z,x)$.[31] The sum to the left of the inequality in (10) is a special case of Shafer's (1974) k function. Since strictly convex preferences satisfy SUN (Schlee and Khan, 2022, Appendix Proposition), if (b)-(d) of Lemma 2 hold, then the conclusion of Proposition 3 holds for the regret-theory subset of Shafer's nontransitive consumer; for example, such preferences can satisfy the smoothness assumptions in Example 5.

4.4. Incompleteness and non-rationalizable demand

Although we do not formally impose completeness in Proposition 3, the condition that the completion of a consumer's preference relation satisfies SUN or Rubinstein smoothness is apt to fail with incomplete preferences. In particular, some models of "multiple selves" can violate both.[32] A simple class of multiple-selves models is this: there is a family $\{R^\alpha\}_{\{\alpha\in\mathcal{A}\}}$ of binary relations $R^\alpha \subset X \times X$ such that $x \succ y$ if and only if, for each $\alpha \in \mathcal{A}$, $x P^\alpha y$, with P^α the asymmetric part of R^α. If each P^α is open and if $P^{\alpha'} \neq P^{\alpha''}$ for some α', α'' in $\mathcal{A}$, then the completion of $\succsim$ will fail SUN, and generally some indifference sets will have kinks, as in the right diagram of Fig. 2.[33] And it is easy to find examples of economies with incomplete preferences for which the local welfare measure, $p^0 \cdot \tilde{y}'(0)$, is positive, but the conclusion fails. It fails, for example, in the economy that Rigotti and Shannon (2005) use to illustrate consumer inertia and indeterminacy: their consumers all have incomplete, multiple-selves preferences of the class proposed by Bewley (2002) that has "kinks everywhere;" see Example 3, the left diagram of Fig. 1 and the right diagram of Fig. 2. Bewley preferences violate both SUN and Rubinstein smoothness.

In the last paragraph, the number of selves and how the selves are aggregated do not depend on the constraint set. Other models of multiple selves differ in that they allow either the

29 Bleichrodt and Wakker (2015) survey the influence of regret theory. Diecidue and Somasundaram (2017) elegantly axiomatize this functional form and succinctly review the literature, including applications; Lanzani (2022) is an even more recent, complementary treatment. Savage (1951) proposed a nonsmooth max-min version of regret.

30 $\succsim \subset X \times X$ is *strongly convex* if $x \succsim z$ and $y \succsim z$ with $x \neq y$, then $\lambda x + (1-\lambda)y \succ z$ for every $\lambda \in (0,1)$.

31 An interesting recent application of Shafer (1974) is Alós-Ferrer et al. (2021, p. 1860) in the context of random utility models. Aguiar et al. (2020) extend Shafer's nontransitive consumer.

32 Strotz (1955) is a classic reference.

33 Schlee (2021) considers kinks, intransitivity, and incompleteness. Fishburn (1991) corrects several misunderstandings about implications of intransitivity, particularly regarding money pumps, in the spirit of Machina (1989). And Machina (2001) considers preference kinks for ordered preferences over risks.

number of selves or how the preferences of the various selves are aggregated to depend on the constraint set.[34] A simple case is that, for each constraint set, there is a single binary relation that determines the chosen point, and the binary relation differs across some constraint sets, as in Kalai et al. (2002). They illustrate with Luce and Raiffa's (1957) restaurant example. A person choses chicken, when a restaurant's menu consists of just {steak, chicken}; but chooses steak when informed that the menu is actually {steak, chicken, frog's legs}. The example violates Sen's α axiom that, if something is chosen in a constraint set B, and it is still available in a constraint set $A \subseteq B$, then it must be chosen out of A (Sen, 1971). Clearly the choices in the restaurant example cannot be rationalized by any single binary relation. In many behavioral models, choices similarly cannot be so rationalized.[35]

4.4.1. *A behavioral money metric and Pareto improvements*

As we argue in Schlee and Khan (2022, Section 4.2), not *all* is lost. Bernheim and Rangel (2009) propose a relation that extends the usual revealed-preference relation to choice models that violate the weak axiom. They assume that each of a family of constraints sets is possible, including *all* two-point constraint sets and that choice in any constraint set in the family is nonempty. They define an *unambiguously chosen over* relation thus: *x is unambiguously chosen over y if y is never chosen if x is available*; in particular it implies that the choice out of the two-element set $\{x, y\}$ is precisely x. Their relation is of course irreflexive. In what follows, the reader should think of the family of available constraint sets to be all two-point sets, and all budget sets with positive prices.

Now let $P_i^* \subset X \times X$ be an irreflexive, LNS, and open binary relation, which we take to be the B-R unambiguously-chosen-over relation, amended by LNS and openness. In the Shafer subset of regret-theory, for example, P_i^* equals $\succ_i$, and the completion of P_i^* satisfies SUN; if in addition, the smoothness conditions of Example 5 hold, then the conclusion of Proposition 3 holds. Define

$$d_i^*(p, w) = \{x \in B(p, w) \mid y P_i^* x \Rightarrow p \cdot y > w\}.$$

In some cases, such as the regret model, the consumer's demand equals d_i^*; in general, however, the consumer's choice correspondence, call it $\tilde{d}_i(p, w)$, is a subset of $d_i^*(p, w)$ for each (p, w) with a proper inclusion sometimes; in the language of Pollak (1977), P_i^* *weakly rationalizes* the consumer's demand $\tilde{d}_i$, in the sense of rationalizing a superset of the demand correspondence. As in Schlee and Khan (2022), define a *behavioral money metric* by

$$M_i^*(x, p) = \inf_{x' P_i^* x} p \cdot x'. \tag{11}$$

Obviously, the conclusions of the money-metric SP theorem remain true if M^* replaces M and d^* replaces d. The conclusion of part (a) continues to hold if the consumer's actual choice correspondence, $\tilde{d}$ is sometimes a proper subset of d^*; but part (b) can fail. Proposition 2 makes no

[34] Ambrus and Rozen (2015) usefully categorize various models of multiple selves in their introduction. For the class of models in the last paragraph, the aggregation rule is max-min, since each self must be better off to judge that the person is better off.

[35] Richter (1971) characterizes the set of choices that can be rationalized by some binary relation, possibly non-ordered. For a simple consumer-theory example, suppose that $L = 2$, and that, if $(1, 1)$ is outside the budget set, then the consumer maximizes $u(x) = \min\{x_1, x_2\}$; but when it is in the budget set, the consumer maximizes $U(x) = x_2$. Normalize wealth to 1. At $p = (1/2, 1/2)$, the unique demand is $(0, 2)$; but at $p = (1, 1/2)$, the unique demand is $(2/3, 2/3)$, even though $(0, 2)$ is still affordable at the higher price of good 1.

use of either part of the money-metric SP theorem and so can be used to *rule out* supply changes as strict Pareto improvements, even for demand not rationalizable by any binary relation: simply replace $\succ_i$ with P_i^* and M_i with M_i^* in the statement and proof of Proposition 2.

4.4.2. Price-dependent preferences

Another important class of behavioral models are those with *menu-dependent preferences* in which preferences between two alternatives depend on the constraint set in which they are available, an idea enthusiastically endorsed by Sen (1997). The already-mentioned Luce-Raiffa restaurant example is perhaps more naturally viewed as menu-dependent preferences, rather than as multiple selves. A special kind of menu-dependent preferences in consumer theory is *price-dependent preferences*, sometimes referred to as "prices in the utility function." It is often used to explain *Veblen effects* in consumption, in particular that a higher price of a good might make consuming it more desirable.[36]

As mentioned, Shafer (1974) develops a theory of the nontransitive consumer, in which preferences are complete, continuous, and strongly convex, but not necessarily transitive. Surprisingly, he demonstrates that the demand that exists under his assumptions can also be generated by a price-dependent utility function,

$$u_i(x, p) = \min_{\{x' | p \cdot x' \leq 1\}} k_i(x, x'), \tag{12}$$

where wealth is normalized to 1, and k_i is again Shafer's preference function, namely a function on $X \times X$ that is continuous, skew symmetric, with the sets $\{x \in X \mid k_i(x, z) \geq 0\}$ strictly convex for each $z \in X$. Let R_i^p be the binary relation that $u_i(\cdot, p)$ represents. (Our money-metric SP theorem gives another price-dependent function which generates a demand in the sense of best undominated points, but without any existence claim.) From the viewpoint of the positive theory of demand, Shafer's nontransitive consumer is indistinguishable from a consumer with transitive, but price-dependent, preferences.

As noted, in Example 5 the Bernheim-Rangel-unambiguously-chosen-over relation, namely P_i^*, *equals* $\succ_i$ – the asymmetric part of the nontransitive and strongly convex binary relation $\succsim_i$ that generates consumer i's demand—and $M_i^* = M_i$. The extension of Proposition 3 to this class of price-dependent preferences is immediate.

Corollary 2. *Suppose that, for $i = 1, ..., I$, P_i^* replaces $\succ_i$ in the definition of a strict Pareto improvement and that P_i^* replaces $\succ_i$ and M_i^* replaces M_i in the statement of Proposition 3 and that (b)-(d) hold. Then the conclusion of Proposition 3 holds for an economy of consumers with price-dependent preferences represented by $\{u_i(\cdot, p)\}_{p \in \mathbb{R}^L_{++}}$, where $u_i(\cdot, p)$ is given by (12).*

On letting $\tilde{x}_i(\alpha) = \tilde{x}_i(0) + \alpha s_i \tilde{y}'(0)$ for $s_i > 0$, the conclusion is that $\tilde{x}_i(\alpha)\ P_i^*\ \tilde{x}_i(0)$ for each consumer i for $\alpha > 0$, but small enough; and so, for small-enough $\alpha > 0$, the allocation $\mathbf{x}(\alpha)$ strictly Pareto dominates $\mathbf{x}(0)$ according to the profile of Bernheim-Rangel unambiguously-chosen-over-relations $(P_1^*, ..., P_I^*)$.

[36] Kalman (1968) and Piccione and Rubinstein (2008) develop a theory of price-dependent preferences, motivated in part by such Veblen effects. Bagwell and Bernheim (1996) do not directly assume price-dependent preferences, but provide a foundation for it by assuming that people care directly about status. Pollak (1977) develops some positive implications of price-dependent preferences, but also reflects on welfare interpretations. Cosaert (2018) extends revealed preference methodology to distinguish between price-dependent preferences and the conventional budgetary effects of price changes.

Although demand in this example *is* rationalized by a binary relation, that binary relation is *not* necessarily what matters for welfare. We conclude this subsection by elaborating on this issue. To the point: In what sense can we say that a consumer has become better off when the consumer's preferences are complete and transitive, but price dependent? As Arrow and Hahn (1971, pp. 129-30) write,

> ... when the utility function is no longer a given for the economic system, but instead, varies with sorne of its variables (in this case, price) the significance of Pareto efficiency becomes obscure, since an allocation that is dominated at one set of prices is not dominated at another.

Here we follow Arrow-Hahn in pursuing what Pollak (1977, Section III) calls the *conditional* approach to price-dependent prices: prices affect how consumers rank bundles at those prices, but otherwise don't directly affect welfare except through affecting budget sets.[37] From now on, fix a consumer i, suppress the index i, and normalize i's wealth to equal 1, and let P^p denote the asymmetric part of R^p. As Arrow-Hahn suggest, there is little hope for evaluating welfare changes "in the large." But for our local result, Corollary 2, something more hopeful can be said. Suppose in addition that $k(\cdot, z)$ is C^2, strongly monotone and strongly quasiconcave with no critical points. Then the consumer inverse demand $g : \mathbb{R}^L_{++} \to \mathbb{R}^L_{++}$—given implicitly by $d(g(x), 1) = x$—is nonempty and continuous at each $x \gg 0$. Define $\tilde{p}(\alpha) = g(\tilde{x}(\alpha))$. We will show that the consumer strictly prefers $\tilde{x}(\alpha)$ to $\tilde{x}(0)$ at both $p = p^0$ and at $p = \tilde{p}(\alpha)$, the prices that support each bundle as a demand. First, since $\tilde{p}(\alpha)$ is continuous,

$$\lim_{\alpha\to 0} \tilde{p}(\alpha) \cdot \frac{\tilde{x}(\alpha) - \tilde{x}(0)}{\alpha} = p^0 \cdot \tilde{x}'(0) > 0. \tag{13}$$

Since $\tilde{x}(\alpha)$ maximizes $u(\cdot, \tilde{p}(\alpha))$ on $\{x \in \mathbb{R}^L_+ \mid \tilde{p}(\alpha) \cdot x \leq 1\}$ and, by (13), $1 = \tilde{p}(\alpha) \cdot \tilde{x}(\alpha) > \tilde{p}(\alpha) \cdot \tilde{x}(0)$ for α small enough, $u(\tilde{x}(\alpha), \tilde{p}(\alpha)) > u(\tilde{x}(0), \tilde{p}(\alpha))$ for α small enough. Second, the argument used to prove Proposition 3 can be adapted to show, using an envelope theorem argument, that

$$\frac{d}{d\alpha} u(\tilde{x}(\alpha), p^0)\Big|_{\alpha=0} > 0, \tag{14}$$

so that, for α small enough, $u(\tilde{x}(\alpha), p^0) > u(\tilde{x}(0), p^0)$. It follows that, for α small enough, $\tilde{x}(\alpha)\ P^p\ \tilde{x}(0)$ for both $p = p^0$ and $p = \tilde{p}(\alpha)$, giving a foundation for asserting that the consumer is better off at $\tilde{x}(\alpha)$ than at $\tilde{x}(0)$.

4.4.3. A summing-up

We have presented several examples of nonstandard choice models to illustrate cases in which our results do and do not hold. All meet the conditions in Proposition 2. Although that result is simple, it does illustrate that the money-metric sum can be used to rule out Pareto improvements, a fact that is useful as well as simple. For some examples, the conclusion of Proposition 3 holds, in others it does not. It is clear that kinks, whatever the source, can cause the conclusion to fail. And multiple-selves models in which both the number of selves and their aggregates are independent of the constraint set are apt to generate kinks.

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[37] "In conditional models of price dependent preferences, the objects of choice are commodity bundles, and the preference ordering $R(P)$ depends on prices. In unconditional models, the objects of choice are 'quantity-price situations' " (p. 74).

The example of price-dependent preferences can of course also be viewed as a case of multiple selves or multiple criteria in the sense of Kalai et al. (2002) in which R^p governs the consumer's choice on the budget set $p \cdot x \leq 1$. A sufficient condition for Proposition 3 to extend to the behavioral money metric is that the completion R^{*c} of P^* satisfy Rubinstein smoothness or SUN (for example, R^{*c} is strongly convex). This can hold in a multiple-selves model in which a single binary relation or self governs welfare for a given constraint set, but varies across constraint sets, as in the price-dependent preferences example, or for particular cases of the Kalai et al. (2002) model.

5. Samuelson's 1974 local-concavity conjecture

As noted, Lemma 2 implies that the derivative of the money-metric sum does not depend on how the change in output is allocated to consumers: (the s_i's), giving one sense in which the money-metric sum is neutral with respect to distribution. Here we ask what can be said about higher order, but still local, effects on the sum to the distribution of the change in production.

As already mentioned, Blackorby and Donaldson (1988) prove that the money metric is concave in consumption for every price only if preferences are quasihomothetic (footnote 11); but as we noted in Schlee and Khan (2022, Section 2.3), the subsequent literature is silent about the region of consumption on which the money-metric is not concave. Samuelson (1974, p. 1276) makes an intriguing assertion. Letting x^0 be a point demanded at $(p, w) = (p^0, p^0 \cdot x^0)$, he writes that the money-metric $M(x, p^0)$ is "locally concave in all x that are near to x^0 (and which may as well be restricted to x's that are at least as good as x^0)." But in our notation, what he writes in symbols is that (his equation (50))

$$\nabla_x M(x^0, p^0) \cdot (x - x^0) \geq M(x, p^0) - M(x^0, p^0) \tag{15}$$

for all x in some neighborhood of x^0, and that the Hessian of $M(\cdot, p^0)$ is negative semidefinite at $x = x^0$, namely (his equation (51))

$$v^T \times D_x^2 M(x^0, p^0) \times v \leq 0 \quad \text{for every} \quad v \in R^L, \tag{16}$$

each of which is a version of the notion that $M(\cdot, p^0)$, is *concave precisely at the demanded point* x^0.[38] Samuelson's verbal assertion about concavity in a neighborhood of a demand point has remained an open question since 1974.[39] For what it is worth, it was reflection on Samuelson's conjecture that lead the authors to the money-metric SP Theorem reported in Section 2.3.

To illustrate the importance of Samuelson's conjecture, consider Example 1 from the introduction, an exchange economy with two identical consumers with preferences represented by a continuous, strictly increasing, strictly quasiconcave utility. This example can be set in the context of Section 4.3 by supposing that the utilities are C^1, that $\tilde{y}(0) = 0$, and that p^0 supports equal consumption $\bar{x} = (x + y)/2$ as a competitive equilibrium. Let consumer i get a fraction s_i of the additional production, $\Delta\tilde{y}(\alpha) = \tilde{y}(\alpha) - \tilde{y}(0) = \tilde{y}(\alpha)$ for all $\alpha \in \mathcal{A}$. If the common money metric $M(\cdot, p^0)$ is concave in a neighborhood of $\bar{x}$ and α is small-enough,

38 Khan and Piazza (2011) and Khan and Piazza (2012) discuss different notions of concavity at a point, including that of Gale (1967).

39 We learned of the openness of the question from a referee on Khan and Schlee (2017). The referee encouraged us to resolve the conjecture; at the time, however, we did not know the money-metric SP Theorem which enables us to resolve it at last. It is conceivable that what Samuelson was just seeking to express concavity-at-a-point; but some prominent scholars are curious about the truth of the claim independently of Samuelson's intent.

$$M\big(\bar{x}+s_1\Delta\tilde{y}(\alpha),p^0\big)+M\big(\bar{x}+s_2\Delta\tilde{y}(\alpha),p^0\big)\leq 2M\big(\bar{x}+\tfrac{1}{2}\Delta\tilde{y}(\alpha),p^0\big),$$

with equality if $s_1=s_2$: locally the money-metric sum is maximized if the change in output is allocated equally to the identical consumers. More broadly, as we allude to in the Introduction and in Schlee and Khan (2022), welfare theorists and applied economists abandoned money metrics for welfare analysis precisely because of its nonconcavity, which were perceived to lead to inegalitarian implications beyond equal treatment of equals.[40] If money metrics were locally concave, then these perceived inequities would not arise, at least for small changes.

A rephrasing of Samuelson's verbal conjecture is this: for any point $\{x^0\}=d(p,w)$ with $x^0\gg 0$, there is a some neighborhood of x^0 on which $M(\cdot,p)$ is concave. Call $d(p,\cdot):\mathbb{R}_+\rightrightarrows\mathbb{R}^L_+$ the *wealth-expansion correspondence* at p. We prove that if $M(\cdot,p^0)$ is concave on some neighborhood of a point $x^0\in d(p,p\cdot x^0)$ then $d(p,\cdot)$ is severely restricted on that neighborhood. Recall that Gorman (1961) proved, in a classical setting with single-valued demand, that demand is affine in wealth for each fixed price—$d(p,w)=f(p)+g(p)w$, so that wealth expansion paths are straight lines—if and only if preferences are quasihomothetic (footnote 11); homothetic preferences are the special case in which $f(p)=0$ for every p. Obviously the affine-in-wealth form of the demand function is convex in wealth for each price. For demand correspondences, the natural extension of the affine-in-wealth form is that the *graph* of the correspondence $d(p,\cdot)$ is convex. Since we do not have a utility representation, we simply refer to the restrictive *demand* implication of quasihomotheticity.

The next proposition follows directly from the money-metric SP Theorem in Section 2.3 (the proof uses both the "if" and "only if" parts of that theorem).

Proposition 5. *(Samuelson's Conjecture.) Suppose that $X=\mathbb{R}^L_+$, $\succsim\subset X\times X$ is reflexive and LNS, and that $\succ$ is open. Let $(p,w^0)\gg 0$, $x^0\in d(p,w^0)$, and let N be any convex set in $\mathbb{R}^L$ containing x^0. If $M(\cdot,p)$ is concave on $N\cap X$, then the graph of wealth expansion correspondence at p is convex on $G_N(p)=\{(x,w)\geq 0\mid x\in d(p,w)\cap N\}$.*

Proof. Let $x'',x'\in N$ with $x''\in d(p,w'')$ and $x'\in d(p,w')$ for nonnegative w',w''. We show that, for any $\lambda\in[0,1]$, $\lambda x''+(1-\lambda)x'\in d(p,\lambda w''+(1-\lambda)w')$. Since N is convex, $\lambda x''+(1-\lambda)x'\in N\cap X$; and since $\succsim$ is LNS, $\lambda x''+(1-\lambda)x'$ is itself in the closure of $\{x\in X\mid x\succ\lambda x''+(1-\lambda)x'\}$. So

$$p\cdot(\lambda x''+(1-\lambda)x')\geq M(\lambda x''+(1-\lambda)x',p). \tag{17}$$

The next inequality follows from the concavity of $M(\cdot,p)$ on $N\cap X$; and the equality following it, from the money-metric SP Theorem(a), $x'\in d(p,w')$, and $x''\in d(p,w'')$:

$$M(\lambda x''+(1-\lambda)x',p)\geq\lambda M(x'',p)+(1-\lambda)M(x',p)=\lambda p\cdot x''+(1-\lambda)p\cdot x'. \tag{18}$$

Now (17) and (18) together imply that $p\cdot(\lambda x''+(1-\lambda)x')=M(\lambda x''+(1-\lambda)x',p)$. By the money-metric SP Theorem(b), $\lambda x''+(1-\lambda)x'\in d(p,w^\lambda)$ for $w^\lambda=p\cdot(\lambda x''+(1-\lambda)x')$. By LNS, $w^\lambda=\lambda w''+(1-\lambda)w'$. It follows that the graph of $d(p,\cdot)$ is convex on $G_N(p)$. □

Although the set N could be a small neighborhood of x^0—the case directly relevant for Samuelson's conjecture—it is an arbitrary convex set containing x^0 in the Proposition statement. It can be $\mathbb{R}^L_+$ for example, or equal any *at-least-as-good-as set* for $\succsim$, if $\succsim$ is convex. If

[40] Beyond the references already cited, we mention Fleurbaey (2009, p. 1053) in this respect.

$\succsim$ is complete, transitive, monotone and convex, the first implies that $\succsim$ is homothetic, and the second implies that they are quasihomothetic (on N). As an unintended consequence, Proposition 5 therefore gives a new, much-streamlined proof of the money-metric nonconcavity. The existing, longer proofs of Blackorby and Donaldson (1988) and Khan and Schlee (2017) impose conditions to ensure that the money metric represents a consumer's preferences—which need not hold even if preferences are standard; see Weymark (1985). Proposition 5 dramatically extends the scope of those results by allowing for incomplete or intransitive preferences. And it extends to some behavioral models outside this class, such as Shafer's consumers with transitive, but price-dependent preferences.

Although Samuelson's verbal conjecture is false, something of it can still be salvaged. As we mention in the Introduction, what Samuelson does correctly assert without proof is this: if the money metric is C^2 in consumption (and a demand point x^0 at p^0 is interior), then its Hessian matrix is *negative semidefinite* at (x^0, p^0). Formally

Corollary 3. *If $M(\cdot, p^0)$ is C^2 on an open neighborhood of a demand point $x^0 \in d(p^0, p^0 \cdot x^0)$ with $x^0 \gg 0$, then the Hessian matrix of $M(\cdot, p^0)$ at $x = x^0$ is negative semidefinite.*

Proof. The proof follows from the money-metric SP Theorem(a), Lemma 3, and the necessary second-order condition for an interior maximizer of $M(x, p^0) - p^0 \cdot x$ on $X = \mathbb{R}^L_+$. □

An implication is that a second-order (in consumption) approximation to the money metric function *is* concave. Putting together Proposition 3 and Corollary 3, a second-order approximation of the money metric sum can identify local Pareto improvements (from a valuation equilibrium), while avoiding some obvious inequities in the distribution of the change in output; for example, in Example 1, a change in the total supply would be allocated equally to the two identical consumers. The results together suggest using a second-order approximation to the money-metric sum for local welfare, at least as a back-of-the-envelope calculation.

Blackorby and Diewert (1979) derive a second-order approximation to an expenditure function in prices which provides a foundation for "flexible functional forms" in applied demand analysis. In particular "if two expenditure (distance, indirect and direct utility) functions approximate each other at a point, then each pair of the remaining three representations of preferences also differentially approximate each other at a point" (p. 592). Hammond (1984) develops a second-order approximation to a money-metric in prices, as do Banks et al. (1996) in the context of an empirical analysis of the welfare effects of tax changes.

One reason that these results are useful in applied demand analysis for consumers with ordered preferences is that compensated demands can be deduced from an expenditure function using Shepard's lemma; and the ordinary demands deduced from them by the equality between compensated and ordinary demands. Without ordered preferences, the last equality fails. In line with Deaton (2003, pp. 136-41),[41] our proposed approximation is in *quantities*, rather than prices. The demands can be derived by maximizing $\tilde{M}(x, p) - p \cdot x$, where $\tilde{M}(\cdot, p)$ is a quadratic (second-order approximation to a) money metric. As with the duality approach for ordered preferences mentioned in the last paragraph, the approach requires a set of sufficient and necessary conditions for a function to be a money metric. Honkapohja (1987) characterizes money metrics for ordered

[41] The first-order approximation to a money metric around a demand point x^0 at supporting reference price p^0 is just $M(x(\alpha), p^0) \approx p^0 \cdot x(\alpha)$, Deaton's equation (4), which he converts into an empirically more useful form, his equation (6).

preferences; in ongoing work we characterize money metrics for nonordered preferences, and develop the implied functional forms for demand for quadratic money metrics.

6. Concluding remarks

The money metric of McKenzie-Samuelson was widely estimated and used for welfare analysis in the last quarter of the 20th century. Its use declined as a result of arguments from welfare theorists that the money-metric sum would tend to favor inegalitarian allocations. Our 2022 paper attempted at least a partial rehabilitation of the money-metric sum, an agenda we continue here.

Some unfinished business in this agenda is a systematic comparison of the money metric with other welfare measures from the viewpoint of optimality and equity. Besides the obvious textbook measures, there is the already-mentioned welfare ratio of Blackorby and Donaldson (1987), the benefit function of Luenberger (1992), the distance function surveyed by Deaton (1979), and the two non-additive money-metric functions proposed by Hammond (1994) and Bosmans et al. (2018). In addition, Corollary 3 requires that the money metric be C^2 for nontransitive preferences. In ongoing work, we give preference conditions for this assumption.

Our original interest in the money-metric Saddlepoint Theorem was its potential to give new comparative statics in the consumer's problem. A long-understood problem in applying lattice-theoretic arguments to the consumer's problem is that budget sets are not ordered by the strong-set order. The saddlepoint theorem allows us to convert the constrained optimization problem into an unconstrained one, with multiplier always equaling unity, and thereby bypassing direct imposition of the budget constraint. That the money metric depends on the price allows us to prove that one or more goods are normal at some prices, but not at others, whether or not preferences are ordered Schlee and Khan (2023).

7. Appendix: A proof of money-metric continuity and differentiability

In this Appendix, we turn to the proofs of both the continuity and differentiability of the money metric. Our proof of Proposition 1 relies on the upper hemicontinuity of the compensated demand and continuity of the money metric.

Lemma 4. *Suppose that $\succsim$ is reflexive, locally nonsatiated, that $\succ$ is open, and that R has a closed graph. Then the compensated demand correspondence h is upper hemicontinuous and the money metric M continuous at every $(p, x) \in \mathbb{R}^L_{++} \times X$.*

Shafer (1980), Weymark (1985), and Honkapohja (1987) give proofs of continuity of the money metric and compensated demand for complete and transitive preferences. A novelty of our continuity result is that we do not assume that preferences are either transitive or complete.[42] Shafer (1980, Lemma, part (i)) proved that the money metric is jointly continuous in $(p, x, \succsim)$, where the each preference relation lies in the space of complete, transitive, closed, and LNS preferences, a space endowed with the Hausdorff topology of closed convergence. Completeness is not essential to his proof but transitivity cannot simply be dropped (transitivity is used 4 lines

[42] To be sure, Mas-Colell (1974), Gale and Mas-Colell (1975) and Shafer and Sonnenschein (1975) show the existence of Walrasian equilibrium with nonordered preferences. Their methods of proof make no use of properties of the compensated demand. And hence what we present is not available in the literature.

from bottom of his proof of part (i)). Our proof shows that transitivity of $\succsim$ can be replaced by closedness of R, and completeness replaced by reflexivity, in his argument. Later, Honkapohja (1987) also proved that the compensated demand is a closed correspondence and the money metric is continuous in (p, x) for complete, transitive and closed preferences, and his proof can likewise be altered to weaken transitivity and completeness. Rather than send the reader to either proof with a list of amendments showing how closedness of R and reflexivity of $\succsim$ can substitute for completeness and transitivity, we present a self-contained proof.[43]

Proof of Lemma 4. We will show that the lemma is a consequence of Berge's theorem.[44] Fix $(p^0, x^0) \in \mathbb{R}^L_{++} \times X$. Since prices are strictly positive, there are closed neighborhoods $P \subset S$ of p^0 and N of x^0, and a compact set $K \subset X$ such that $h(p, x) \subseteq K$ for every $(p, x) \in P \times (N \cap X)$. This follows since, for any $(p, x) \in P \times (N \cap X)$ and $y \in h(p, x)$, $p \cdot y \leq M(x, p) \leq p \cdot x \leq \max_{(p',x') \in P \times (N \cap X)} p' \cdot x' =: \Pi$. We can take $K = \{x \in X \mid p \cdot x \leq \Pi \quad \text{for some } p \in P\}$, which is easily shown to be compact. Since R has a closed graph, the correspondence $\mathcal{R} : N \cap X \rightrightarrows K$ given by $\mathcal{R}(x) = R(x) \cap K$ is upper-hemicontinuous at $x = x^0$. Moreover, for $(p, x) \in P \times (N \cap X)$, it follows that $h(p, x) = \text{argmin}_{\{y \in \mathcal{R}(x)\}} p \cdot y$.

We now show that the correspondence $\mathcal{R}(\cdot)$ is lower-hemicontinuous at x^0. Let x^n denote any sequence in $N \cap X$ with limit x^0, and let $y^0 \in \mathcal{R}(x^0)$. We construct a sequence y^n converging to y^0 with $y^n \in \mathcal{R}(x^n)$ for every n. Since $y^0 \in \mathcal{R}(x^0)$, there is a sequence z^m with limit y^0 and $z^m \succ x^0$ for all m. For each n let $\overline{m}(n) = \sup\{m \in \mathbb{N} \mid z^m \in \mathcal{R}(x^n)\}$ and $\underline{m}(n) = \inf\{m \in \mathbb{N} \mid z^m \in \mathcal{R}(x^n) \text{ and } m \geq n\}$. If (i) $\overline{m}(n) = -\infty$ (that is, $z^m \notin \mathcal{R}(x^n)$ for every m), then set $y^n = x^n$; if (ii) $\overline{m}(n) < \infty$, then set $y^n = z^{\overline{m}(n)}$; if (iii) $\overline{m}(n) = \infty$ then set $y^n = z^{\underline{m}(n)}$. Since $\mathcal{R}(x)$ is closed for every $x \in X$ and $\succsim$ is LNS, $y^n \in \mathcal{R}(x^n)$ for every n. And since $\succ$ is open, for every fixed m, there is an n^* with $z^m \succ x^n$ for all $n \geq n^*$, and so it follows that case (i) holds for only finitely-many n. Since $x^n \to x^0$, $\succ$ is open, and $z^m \succ x^0$ for every m, it also follows that $\lim_{n\to\infty} \overline{m}(n) = \infty = \lim_{n\to\infty} \underline{m}(n)$.

Since (ii) or (iii) must hold for infinitely many n, $\lim_{n\to\infty} y^n = y^0$. The correspondence $\mathcal{R}$ is therefore lower hemicontinuous. It follows that $\mathcal{R}$ is continuous.

Now consider any sequence (p^n, x^n) in $P \times (N \cap X)$ with limit (p^0, x^0). Since the dot product $p \cdot x$ is continuous on $P \times (N \cap X)$, the constraint correspondence is continuous on $N \cap X$, and the compensated demand correspondence lies in the compact set K on $P \times (N \cap X)$, the conclusion follows from Berge's Theorem. That the money metric is continuous at (p^0, x^0) follows easily. □

Proof of Proposition 1. By Lemma 3, if $M(\cdot, p)$ is differentiable at $x' \in d(p, p \cdot x')$ with $x' \gg 0$, then $\nabla_x M(x', p) = p$. We will show that

$$\lim_{\|x-x^0\|\to 0} \frac{|M(x^0, p^0) - M(x, p^0) - p^0 \cdot (x^0 - x)|}{\|x - x^0\|} = 0. \tag{19}$$

By the money-metric SP Theorem(a),

$$M(x^0, p^0) - M(x, p^0) \geq p^0 \cdot (x^0 - x), \tag{20}$$

[43] Weymark (1985) proves continuity of the money metric in consumption for more general consumption sets than $\mathbb{R}^L_+$, but does not touch on joint continuity in price and consumption.

[44] For a statement of Berge's theorem, see for example Aliprantis and Border (2006, Theorem 17.31).

for every $x \in X$, so for any $x \in X$ with $x \neq x^0$

$$M(x^0, p^0) - M(x, p^0) - p^0 \cdot (x^0 - x) \geq 0. \tag{21}$$

For $x \in N$, and any selection $\gamma_x \in g(x)$ we have $M(x, \gamma_x) = \gamma_x \cdot x$ and $M(x^0, \gamma_x) \leq \gamma_x \cdot x^0$. From these inequalities find that

$$M(x^0, \gamma_x) - M(x, \gamma_x) \leq \gamma_x \cdot (x^0 - x) \tag{22}$$

which, adding and subtracting the same terms, is the same as

$$M(x^0, p^0) - M(x, p^0) - p^0 \cdot (x^0 - x) \leq (\gamma_x - p^0) \cdot (x^0 - x) - \Gamma(x) \tag{23}$$

where $\Gamma(x) = [M(x^0, \gamma_x) - M(x^0, p^0)] + [M(x, p^0) - M(x, \gamma_x]$. Now restrict the selection $\gamma_x \in g(x)$ to satisfy condition (b) of Proposition 1.

Since $M(x, \cdot)$ is the minimum over a collection of affine functions, it is concave, and so $M(x', p') \leq M(x, p) + y_{(p,x)} \cdot (p' - p)$ where $y_{(\cdot)}$ is any selection from $h(\cdot)$.[45] It follows that

$$\Gamma(x) \geq (\gamma_x - p^0) \cdot y_{(\gamma_x, x^0)} + (p^0 - \gamma_x) \cdot y_{(p^0,x)}. \tag{24}$$

Insert (24) into (23) to find

$$M(x^0, p^0) - M(x, p^0) - p^0 \cdot (x^0 - x) \leq (\gamma_x - p^0) \cdot (x^0 - x + y_{(\gamma_x, x^0)} - y_{(p^0,x)}) \tag{25}$$

Combine (21) and (25) to find

$$|M(x^0, p^0) - M(x, p^0) - p^0 \cdot (x^0 - x)| \leq \Omega(x), \tag{26}$$

where $\Omega(x) = (\gamma_x - p^0) \cdot (x^0 - x + y_{(\gamma_x, x^0)} - y_{(p^0,x)})$. Since $\gamma_x \to p^0$, h is upper hemicontinuous by Lemma 4, and $h(p^0, x^0)$ is single-valued, it follows that $\lim_{x \to x^0} y_{(\gamma_x, x^0)} = x^0 = \lim_{x \to x^0} y_{(p^0,x)}$.[46] The Lipschitz-like condition (b) and the Cauchy-Schwartz inequality imply that

$$\lim_{\|x - x^0\| \to 0} \frac{\Omega(x)}{\|x - x^0\|} = 0, \tag{27}$$

so (19) follows. □

Proposition 1 requires that the compensated demand $h(p^0, x^0)$ be single-valued. The perfect-substitutes example with preferences represented by $u(x) = \sum x_i$ demonstrates that single-valuedness of compensated-demand is not necessary since $M(x, p) = \min\{p_1, ..., p_L\} \sum x_i$. The next corollary extends this example to preferences that are locally quasihomothetic around a demand point x^0, without requiring $h(p^0, x^0)$ to be single-valued.

Corollary 4. *Suppose that all the assumptions of Proposition 1 hold except that $h(p^0, x^0)$ is single-valued. Suppose too that preferences are locally quasihomothetic in the sense that $M(x, p) = a(p) + b(p)v(x)$ on some open neighborhood $Q \subset X \times \mathbb{R}^L_{++}$ of (x^0, p^0), where*

[45] For example, the proof of Proposition 9.24f in Kreps (2013) applies word-for-word here, with just a change in notation.

[46] For this argument about differentiability at a point, the domain of h can be restricted without loss of generality so that the range can be taken to be compact, as in the proof of Lemma 4.

b and v are functions. If in addition b is locally Lipschitz at the point $p = p^0$*, then* $M(\cdot, p^0)$ *is differentiable at* $x = x^0$*.*[47]

Proof. The function Γ in the proof of Proposition 1 here equals $\Gamma(x) = [b(\gamma_x)) - b(p^0)] \cdot [v(x^0) - v(x)]$. By Lemma 4, v is continuous. And since b is locally Lipschitz at the point p^0, the conclusion follows by combining (21) and (23) for this Γ, then taking limits. □

Declaration of competing interest

None declared.

Data availability

No data was used for the research described in the article.

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[47] The function b is locally Lipschitz at the point $p = p^0$ if for some open neighborhood P of p^0, there is a real number $K > 0$ such that, for every $p \in P$, $\|b(p^0) - b(p)\| \leq K\|p^0 - p\|$.

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