CONSISTENT PLANNING

AND

INTERTEMPORAL WELFARE ECONOMICS

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Ph. D. Thesis

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CONTENTS

DETAILED CONTENTS

P R E F A C E.

Much of the latter part of this dissertation is a development of ideas which were originally worked out in collaboration with Professor Mirrlees, and which were set out in our joint article, "Agreeable Plans". Section 8.6. and parts of chapter 10, in particular, rely heavily upon that article. Nevertheless, the dissertation itself does not incoporate any work done in collaboration. It is entirely my own work, except for those results which are individually acknowledged as coming from the work of others.

SUMMARY

Most of the existing literature on intertemporal welfare economics overlooks the possibility of planned choices being altered. In particular, it assumes that individuals' tastes are fixed. This thesis attempts to set out a dynamic welfare theory which allows for changing choices on the part of individuals, but maintains consistent social choice.

The first part is preliminary discussion of static welfare economics. Individual welfare is linked to choices which are in the individual's interest. Social welfare is assumed to be related to individual welfare via a form of constitution. Then certain unanimity principles can be justified, and the additive form of Bergson welfare plausible.

The second part starts by presenting various kinds of dynamic choice. One of these forms, derived by "intertemporal liberalism", is identified with intertemporal welfare. The relationship between individual choice and intertemporal social welfare is discussed, and the need for a general form of dynamic constitution suggested. The theory is related to such problems as changing tastes, unborn generations, and population control.

The third and final part considers the problem of the time horizon. The use of an infinite horizon is seen as a way of avoiding detailed consideration of the horizon, on the assumption that it is in any case very distant. Moreover, the inconsistency that would result from finite horizon planning is avoided. Because of possible paradoxes, an alternative to the standard approach of optimal growth theory is suggested, and its theoretical properties investigated.

The thesis makes use of mathematical language or notation - notably set theory, analysis and topology - wherever this adds precision to the results, or clarifies or shortens the exposition.

INTRODUCTION

Welfare economics is the study of what it is that makes one economic policy desirable and others less desirable. This, at least, is a rough definition which will serve to explain in this introduction what is the subject of this study. Intertemporal welfare economics is concerned with policies which have effects over a significant length of time, and for which different effects at different moments of time have to be considered together. The clearest and most common examples are policies affecting capital accumulation. But many other areas of policy - such as natural resource exploitation, population, pension schemes, the formation of international economic communities - spring readily to mind.

Now, from the work of Irving Fisher, Hicks, Arrow and Debreu, economists have been able to develop a powerful construct for handling many such issues. Commodities at different dates are regarded as different goods. Then, using this apparatus of dated commodities, intertemporal welfare economics is no different, in essence, from static welfare economics. Following Vickrey, we may call this approach to intertemporal welfare economics the *metastatic* approach.

The metastatic approach has by now been well absorbed into economics, and is frequently used to decide practical issues such as whether to undertake a specific investment project. Indeed, most applications of cost-benefit analysis rest on the metastatic approach to intertemporal welfare economics.

Nevertheless, proponents of the metastatic approach have not, on the whole, made very clear what precisely underlies it. It is easy to raise awkward questions. Graaff was very concerned about the time horizon. Other problems arise when tastes change, and because of unborn generations. More seriously still, tastes may not only change, but change in a way which depends upon the economic policy which is adopted. This phenomenon is called an endogenous change of tastes. Similarly, the number and composition of the population may be endogenous. Another problem, which is related to the problem of changing tastes, is that choices over time may be inconsistent. In other words, the plan which an individual makes today is likely to be changed before very long. This has serious implications for intertemporal welfare analysis, as Strotz recognized.

The final problem is perhaps the most serious of all. This is uncertainty. Once again, economists - notably Arrow and Debreu - have developed an apparatus for treating policy issues when uncertainty seems important. But this apparatus - contingent commodities - barely scratches the surface of the problem. What happens as individuals' subjective probabilities change, as they learn by experience? How does one treat uncertainty about future tastes? What about the uncertainty which arises because individuals cannot possibly make all the calculations needed to determine their "rational" behaviour?

To attempt to answer all these questions in a single work would be foolish. Accordingly, I have limited myself by ignoring uncertainty, for the most part. Of course, this means that the work has very little practical relevance unless it so happens that neglecting uncertainty makes little real difference to the analysis of practical issues. That the same is true of virtually all existing work is an inadequate defence. My reason for neglecting uncertainty is simply that easier problems have to be tackled first. My hope is that the present work will facilitate a later treatment of some aspects of welfare economics under uncertainty.

Although intertemporal welfare economics is the subject of this study, I have found it convenient to start by discussing static welfare economics. One reason for this, I suppose, is that the metastatic approach - or a generalization of it - retains its normative appeal even when the new problems arise. In addition, before venturing into new kinds of problems, we should be sure of the scope and aims of welfare economics. To judge from recent writings, many welfare economists are still rather unsure of what precisely they are doing and saying.

The first three chapters, therefore, neglect time - or rather, they fail to make any explicit mention of time and the special problems arising in intertemporal welfare. Chapter 1 explores the relationship between ethics and welfare economics. In particular, it explains the role of the logical exercises which are the heart of most of the succeeding chapters. Chapter 2 links welfare to choice, both for society and individuals, and defines a constitution. Chapter 3 studies specific properties of constitutions, namely unanimity principles, and correspondence to an additive Bergson social welfare function.

Time is first considered explicitly in chapter 4, which considers dynamic choice - i.e. choice when there is scope for changing one's mind. Chapter 5 discusses the relationship between intertemporal welfare and dynamic choice, and also dynamic constitutions.

In chapter 6, the theoretical constructs of chapters 4 and 5 are used to discuss how problems such as changing tastes can be treated in intertemporal welfare economics.

By now, it has become completely standard to treat the horizon problem by considering an infinite horizon. Chapter 7 discusses what this means, and how it may achieve consistency which would otherwise be lacking. Chapter 8 considers the problem of optimal capital accumulation, and illustrates the paradoxes to which the standard approach can lead. These two chapters lay the foundations for the last two chapters, which suggest a different approach to the horizon problem. This approach stems from the recognition that planning is dynamic, and so only a short-term plan need be chosen, provided it is justified by the possibilities of extending it. Such short-term plans are called "overtures", and the new approach is called "overture planning". Chapter 9 introduces the subject, and discusses "overture optimal" and "overture maximal" plans. The relationship to "agreeable plans" is also shown. Chapter 10 contains theoretical results concerning the existence, uniqueness, and characterisation of such plans.

For dynamic choice, the assumption that it should correspond to a utility function, or even to a preference relation, has rather less normative appeal than for static choice. This is illustrated in chapter 4. Partly for this reason, and partly because it seems safer to consider choice explicitly rather than through preference relations, much of the analysis is in terms of fairly general choice functions. These are defined, and the properties used are derived, in appendix 1.

This work is a theoretical investigation, and so much of the discussion in the following chapters is in abstract terms. The chapters and sections are arranged in logical sequence. So applications often follow the theoretical discussion. If the reader prefers to see some applications first, which he can keep in mind in considering the theory, he is advised to turn to chapter 6. This discusses a number of practical problems and relates this work to conventional approaches. For the later chapters, on infinite horizon choice, a suitable practical problem to bear in mind is that of capital accumulation in a one-good growth model, as expounded in section 8.6.

Chapter 1.

ETHICS, VALUE JUDGMENTS, AND WELFARE ECONOMICS.

1.1 Values and Welfare Economics

Before embarking on a theoretical analysis, it is desirable to be clear what the subject of the analysis is, and how the results can be used. After all, it is the practical applications of a theory which should determine the questions to be considered.

Now in the case of welfare economics, the purpose of theoretical analysis is still surprisingly unclear. The confusion can be illustrated by two quotations from fairly recent writings:-

> "It must be concluded that the techniques and results of welfare economics are irrelevant to any evaluation of the desirability of choices made within or by a society. Any appearance to the contrary springs from normative judgments made by the welfare economist without justification. This is so, even though these techniques can throw light on the implications, the consistency, and the effects of normative judgments." (From p. 190 of E. van den Haag (1967))

> "Welfare economics and ethics cannot, then, be separated. They are inseparable because the welfare terminology is a value terminology. It may be suggested that welfare economics could be purged by the strict use of a technical terminology, which, in ordinary speech, had no value implications. The answer is that it could be, but it would no longer be welfare economics. It would then consist of an uninterpreted system of logical deductions, which would not be about anything at all, let alone welfare. As soon as such a system was held to be about anything, for example, welfare or happiness, it would once again be emotive and ethical." (From pp. 79-80 of Little (1957))

Consider an argument of the following type:-

(1) The set of values which, I believe, determine what is right is $\mathcal{V}$.

(2) In this case, the facts which determine the effect of implementing the set of values $\mathcal{V}$, are $\mathcal{F}$.

(3) The logical consequence of applying the values $\mathcal{V}$ in the factual situation $\mathcal{F}$ is that action a should be followed.

(4) Therefore, I believe that a is right.

It is hard to see how any policy recommendation, of the type considered by welfare economists, could be justified except by an argument of this form. For, although the decomposition into values, facts and logical argument may be difficult, none of the above ingredients is really dispensable. Facts alone cannot be cited to justify an action, and so values are involved somewhere. On the other hand, what action is right is not a question which can be settled without any reference to facts whatsoever. Finally, although the move from facts and values, taken together, to a choice of action can be made to seem trivial, it is, strictly speaking, a logical move(1).

We are now in a position also to see more clearly what role values have in welfare economics. Insofar as a welfare economic study leads up to a practical recommendation, it *must* make value judgments at some point. It may explore the consequences for policy of certain values. It may even suggest that, if certain values $\mathcal{V}$ are accepted, then the right policy is a^*. But this is, at most, an invitation to the reader to accept the values $\mathcal{V}$; the study has not, in itself, made any value judgments.

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(1) The decomposition considered here involves the distinction between facts and values, known as Hume's Law. This has been widely disputed by philosophers, but it seems possible to regard the distinction as one which is impossibly hard to make in a universally agreed fashion, rather than as one which cannot even be made in principle. And to see the role of theoretical welfare economics does not really require complete agreement about how to separate facts from values; all that is needed is a recognition that logic can be applied to sets of premises which, together, have both factual and evaluative content.

It may also be hard to tell where logical analysis starts and the specification of facts and values stops. For example, facts can be logically analysed to provide new facts. Is it the premises or results of this logical analysis which are to be regarded as the facts $\mathcal{F}$? Fortunately, what we decide makes little difference.

In this study, there will be no value judgments. It is, therefore, a purely theoretical study. In any practical application, value judgments have to be made. The theory will be developed to try to accommodate as many different kinds of values as is reasonably possible. But full generality has, of course, to be abandoned in order to achieve more useful results.

1.2 Ethics and Economic Policy.

So far, we have considered the *form* of argument which is used in welfare economics, but have not stated what the *subject* of these arguments is.

It hardly needs saying that welfare economics is concerned with economic problems. Broadly speaking, this means that it is concerned with the way in which scarce resources are used and their produce distributed to meet the wants of members of society. But the point about the form of analysis and argument is that it can be used in many other fields besides economics. It can be used to treat ethical questions in politics, law, medicine, psychology, etc. One can talk of welfare theory or of welfare analysis in general. Welfare economics is then that part of welfare theory which is most suitable for treating economic problems.

The type of problem which welfare economics purports to handle is policy recommendation and, closely associated with this, the evaluation of economic systems. But welfare economics is concerned with the *justification* of a given policy recommendation or an evaluation, rather than with recommendation or evaluation *per se*. The reason for this is that given by Barry[1]. A recommendation or evaluation can be made with no justification to support it. But, without the justification, it is of no interest to the welfare economist, who wants to know *why* a recommendation or evaluation has been made.

Justifications are based on values, and they characterise *normative* economics. But, although welfare economics is a branch of normative economics, not all of normative economics is welfare economics. In normative economics, no kind of justification is necessarily excluded.

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(1) Barry's reasons concerned political argument, but are equally applicable to economic argument. See Barry (1965) (pp. 2-3).

But in welfare economics, it is generally accepted that only ethical justifications are admissible. Consequently, purely egoistic norms, and other norms which are not ethical, are excluded.

In fact, in welfare economics, the traditional view is that a policy is justified insofar as it improves the welfare of the society and of its members, taken together. But, for this to tell us what form justification takes, we need to know what welfare means.

1.3 Interests and Welfare.

Welfare means "well-being". That it should be the primary end of economic policy is therefore open to dispute. It is too close to making contentment the goal. A better term might be "interests". It seems to cover just about all aspects of the consequences of a policy. If we claim that a policy is in the interests of an individual *A*, we purport to take account of *A*'s tastes, wants, needs, desires, satisfaction, happiness, pleasure, etc. - both now and in the future. Moreover, "interests" are commonly regarded as the ultimate justification. If it is claimed that policy *x* is right because of what it does for *A*, it is always possible to make the counter-claim that *x* is not (really) in *A*'s (best) interests. But if it is claimed that *x* is in *A*'s *interests*, the only way of countering the claim is to object that *x* is *not* in *A*'s interests and to give a reason. Then, of course, there is likely to be an argument as to what *A*'s interests really are.

A more precise definition of "interests" will be given in the next chapter. *A*'s interests will be defined in terms of the choices which, if they were being made on *A*'s behalf, would be ethically justified. To conform with the usual use of the word by economists, "welfare" will be defined as effectively identical to "interests". It should, however, be remembered that this is a technical sense of the word "welfare", which may well differ from the standard one.

Chapter 2.

WELFARE AND CHOICE.

2.1. Introduction.

In Chapter 1, it was suggested that in welfare economics, a policy is justified insofar as it advances the interests of the society and of its members. However, the term "interests" was not precisely defined. But it was suggested that "welfare" should be understood as effectively the same as "interests".

It is common in welfare economics to identify an individual's welfare with what he sees as his own interests. It is assumed that the individual has an ordinal utility function, representing his choices. One presumes that, in any environment, the individual chooses the option which is most in his interests, as he sees them. It is also the option which maximizes his utility function[1]. Then the individual's welfare is measured by the ordinal utility function. Since welfare is to be identified with interests, this means that the individual's own view of his interests is accepted as proper. To summarize, this common procedure involves three assumptions, namely:-

(1) The individual's choices in the various possible environments correspond to an ordinal utility function.

(2) This utility function represents the individual's interests, as he sees them.

(3) The individual's interests are, in fact, as he sees them.

The first two assumptions force us to consider only those individuals whose choices satisfy rather stringent restrictions. It is common to

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(1) A clear statement of the common economist's view of welfare is contained in Graaff (1957) (pp. 5, 33-34).

label such individuals "rational egoists". It is common to regard individuals who violate assumption (1) as "irrational". Individuals may violate assumption (2) either because they are "irrational" or because they are "altruistic" or "malevolent" - i.e. their utility function takes account of the interests of others as well as of themselves. For the moment, we shall assume that (2) is violated because of irrationality; the rational altruist will be considered later, in sections 2.3. and 2.5.

In fact, it is common to discount an irrational individual - or, at least, to tamper with his choices in some way, so that the result is the choice function which, it is believed, the irrational individual would have had if he had been rational. There is, however, no obvious procedure for tampering with general irrational choices[(1)]. Nor is there a procedure which does not involve judging what interests an individual possesses.

The third assumption involved in the procedure which is common in welfare economics - namely, that the individual's interests are as he seems them - is also, in effect, a judgment about what an individual's interests really are. Nevertheless, it is a judgment to which many economists would be happy to subscribe, for many policy issues[(2)].

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(1) In certain very special cases, there may be. For example, if an individual has "imperfect discrimination" and ignores small utility differences, it is nevertheless possible to discover his utility function from his choices. See Luce (1956).

(2) Indeed, Archibald (1959) has claimed that at least part of welfare economics - the "new" welfare economics which "makes no interpersonal comparisons" - is value-free. He identifies an individual's welfare with his own choices. The result is a "welfare economics" which is "purely descriptive", as he states. But, as Archibald uses the term, "welfare" has been stripped of all its ethical force. There can be no presumption that it is ethically right to promote Archibald-welfare unless certain value judgments are made. These value judgments are essentially assumption (3) above. Without these value judgments, Archibald's version of the "new welfare economics" is a pointless exercise.

Incidentally, it is worth remarking that such judgments about individuals' interests are difficult to categorize: they may be ethical value judgments or they may be factual judgments[1]. Discussion of this is best left until section 2.3.

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(1) Ng (1972) has called such factual judgments "subjective judgments of fact".

These assumptions and judgments are so specific that it seems desirable to move away from them. But then the meaning of "welfare" and of "interests" needs to be re-examined. This is what will be done in the rest of this chapter.

Suppose that a given individual's interests are known. Then we know what, in our opinion, the individual ought to choose in any given environment. This suggests that interests can be defined in terms of choices in various environments - just as, in revealed preference theory, preferences are revealed by the demands of the individual as he faces a varying budget constraint. But, before an exact definition can be given, a more definite concept for describing choices in various environments is needed. This concept is that of a choice function. Choice functions have already been studied by a number of authors(1) but a definition is provided and explained in the following section in order to make clear what is meant here by a choice function, and by certain associated terms. A summary of the definitions, and of various properties of choice functions, is contained in Appendix 1.

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(1) Choice functions, as opposed to preference relations, have been studied principally by Arrow (1963)(pp.19-21), (1959), Uzawa (1960), Herzberger (1971), Sen (1969), (1970a), (1971), Schwartz (1970), Weddepohl (1970), Pattanaik (1971).

2.2. Choice Functions.

In any choice situation, there is a set of options, A, which will be called the *feasible set*. There is also an *agent* who makes the choice. A choice situation is completely characterized by the agent and by the feasible set. The feasible set may also be called the *environment*.

In the end, the agent has to choose a single option from A. This is so, even though the agent may be willing to choose more than one option. If the agent is not forced to choose a single option, in the end, the set of options cannot have been properly specified. Moreover, we assume that each option includes within it a complete description of the consequences of choosing that option.

Given the feasible set A, there is a set of options which the agent is willing to choose. This set, called the *choice set*(1) is written as $C(A)$. Since A is the feasible set, it is meaningless to be willing to choose any option which is not in A. So we require that $C(A)$ must be a subset of A. Then the set $A - C(A)$ is the (possibly empty) set of options which the agent is unwilling to choose from A.

Of course, in the end, the agent must choose a single option from A. Nevertheless, as remarked before, he may be *willing* to choose more than one option. Consequently, $C(A)$ may contain several options. If it does, then a final choice has to be made from $C(A)$. We shall assume that this final choice is left to chance, or to some other external agency.

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(1) Some authors have preferred to reserve the term "choice set" for what is here called the "feasible set". This is unfortunate because it is hard to think of a better term than "choice set" for the set of options which an agent is willing to choose. To describe an element as "eligible" as Weddepohl (1970) does, seems to suggest that the element is a serious candidate for choice, rather than something the agent is willing to choose.

A *choice function* is a mapping whose domain is a set $\mathcal{D}$ of possible environments. It indicates the choices which a given agent is willing to make in those environments. Let $X = \bigcup \{A \mid A \in \mathcal{D}\}$. Then X is the set of options which may be possible, depending upon which environment is the true one. X is called the *underlying set*. The *power set* of X, $\mathcal{P}(X)$, is the set of subsets of X.

Given any $A \in \mathcal{D}$, there is a choice set $C(A)$, which is a subset of A, and so a subset of X. Consequently, a choice function is defined as a mapping $C: \mathcal{D} \to \mathcal{P}(X)$.

But not <u>any</u> mapping $C: \mathcal{D} \to \mathcal{P}(X)$ can be accepted as a choice function. For, if the feasible set is A, then, as already mentioned, $C(A)$ must be a subset of A. In addition, the agent must choose <u>something</u>; to choose nothing at all is not one of the options open to him. Therefore $C(A)$ must not be empty.

It is not very interesting to have a choice function C whose domain consists of a single set A. If the environment were to change from A, nothing could be said. So it is common to define a choice function on as large a domain as possible. The largest possible domain, given the underlying set X, would be the power set $\mathcal{P}(X)$. But, for many of the choice functions studied by economists, there are subsets A of the underlying set for which the choice set $C(A)$ is empty. For example, X may be a real vector space, $C(A)$ may be the set of options in A which maximize a continuous utility function; then it is possible to find infinite sets A for which $C(A)$ is empty. Consequently, we shall not insist that $C(A)$ must be nonempty for every $A \subsetneq X$. But the only compelling reasons for allowing choice sets to be empty apply, it seems, to infinite sets. So we shall insist that any *finite* subset of X has a non-empty choice set[1].

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(1) Herzberger (1971) calls this restriction "regularity".

This still leaves the domain $\mathcal{D}$ undefined. The obvious approach is to have $\mathcal{D} = \{A \subseteq X | C(A) \neq \emptyset\}$ and define $C : \mathcal{D} \to \mathcal{P}(X)$. But this obvious approach causes problems because, before saying anything about $C(A)$, we have first to check that $A \in \mathcal{D}$. The alternative, which will be adopted here, is to define C on the <u>whole</u> of the power set $\mathcal{P}(X)$, and to allow $C(A)$ to be empty for $A \notin \mathcal{D}$.

To summarize then, a choice function is a mapping $C : \mathcal{P}(X) \to \mathcal{P}(X)$, for some set X, such that, for some $\mathcal{D} \subseteq \mathcal{P}(X)$

(1) If $A \in \mathcal{D}$, then $C(A)$ is not empty and $C(A) \subseteq A$.

(2) If A is a finite subset of X, then $A \in \mathcal{D}$.

But since the choice set for environments outside $\mathcal{D}$ is irrelevant, we can arbitrarily set $C(A) = \emptyset$ whenever $A \notin \mathcal{D}$. So, finally, we can ignore $\mathcal{D}$ altogether and define a *choice function* as a mapping $C : \mathcal{P}(X) \to \mathcal{P}(X)$ such that:-

(1) For all $A \subseteq X$, $C(A) \subseteq A$.

(2) If A is a finite subset of X, then $C(A)$ is non-empty.

We shall say that C is a choice function *defined on X*.

The rest of this work is, in effect, a study of choice functions and of their role in theoretical welfare economics. The phrase "choice function" will often be abbreviated to "CF".

2.3. Individual Welfare.

In this section, an individual's interests will be identified with a choice function called the individual's *welfare choice function* - or WCF. The underlying set for this CF is the set of all possible economic policies, which will be denoted by X.

Let i denote the individual whose interests are being considered. Now if i's interests are known, then, in any environment A, the set of options which are (most) in i's interests is known. This set, which we shall denote by $C_i(A)$, has the same properties as a choice set. Indeed, $C_i(A)$ is the appropriate choice set if the only criteria for choice are i's interests. Consequently, there is a choice function C_i, defined on X, such that, given any environment $A \subseteq X$, $C_i(A)$ is the set of options which are in i's interests.[1]

But this does not provide a way of telling what an individual's interests are. As already remarked in section 2.1., what an individual's interests are is a matter of judgment, be it a value judgment or a factual judgment. The orthodox starting point is the choices which the individual himself would be willing to make. But, for various reasons, some of these choices may be overruled as indicators of the individual's interests. The most common reason is that an individual's choices may fail to satisfy the rationality postulates which are regarded as essential. Otherwise, the next two most common reasons are that the individual has inadequate knowledge, or that his choices show insufficient forethought. Inadequate knowledge evidently involves uncertainty and it will therefore not be pursued here.

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(1) Here, it is being assumed that whenever A is a finite subset of X, it is possible to specify some options in A as being in i's interests, given this environment. This seems a mild assumption, given that i's interests are supposed known.

Insufficient forethought is hard to characterize, in general, but in many cases it leads to "naive" dynamic choice of the kind studied in chapter 4.

Let C_i^o denote the individual's own choice function, giving the choices he is willing to make. Let C_i denote his welfare choice function. Let α_i denote those individual characteristics, apart from choices, which affect the judgment of the individual's interests. Then C_i can be regarded as derived from C_i^o and α_i by a functional relationship:-

$$C_i = \phi_i(C_i^o, \alpha_i)$$

As far as welfare economic theory is concerned, it is the WCF C_i which is all that needs to be known; how it is derived is of no importance until practical applications of the theory are being contemplated.

Notice that, as they have been defined above, each individual's interests are effectively independent of the interests of others. This does not preclude external effects of consumption, in which one individual's interests are affected by his neighbour's consumption. For notice that the choice function C_i specifies choices of an entire economic policy. Consequently, C_i effectively represents choices of allocations to all members of society. So, in deciding which policies are in i's interests, the effects of other individuals' consumption on these interests is taken into account.

Altruism, however, is precluded. An altruist is someone whose own choice function C_i^o, giving the choices he himself is willing to make, takes into account not only his own interests but also those of other people. But, as individual welfare takes into account only the interests of the individual, those choices which an altruistic individual makes to further the interests of others have to be disregarded. The same is true whenever an individual's own choice function takes into account the interests of others, as he sees them. Thus, in Sen's celebrated example, the preference of the prude for preventing the lascivious man from reading "Lady Chatterley's Lover", and the preference of the lascivious man for having the prude read the novel, are both to be disregarded in judging the two individuals' interests, because both preferences are formed with the supposed interests of the other individual in mind(1).

This does not mean that altruism has to be disregarded altogether in welfare economics. As will be seen later, altruism can be taken into account in deciding about social welfare. Here, all that is claimed is that the welfare of a genuine altruist - that is, someone who is prepared to advance the interests of others even at the cost of hindering his own interests - is independent of the welfare of other people, including those about whom he cares most. The reason is that what is in an altruist's interests is what should be chosen on his behalf - often what he himself would choose - assuming that the fate of all other individuals is fixed.

In identifying an individual's welfare with his interests, interpersonal comparisons are also not allowed for. Thus, each individual's WCF, C_i may correspond to a (real-valued) welfare function W_i. Interpersonal comparisons involve comparing the relative values of W_i, for one individual i, and W_j, for another individual j. But such

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(1) See Sen (1970a), p. 80 or (1970b), p. 155. The requirement that certain types of choice, regarding other people, should be disregarded has also been recognized by Barry (1965). The type of choice which he requires to be disregarded, "publicly oriented wants", is very similar to choice with the interests of others in mind, as defined here.

relative values have not been given any meaning. In any case, such interpersonal comparisons are only relevant to policy decisions when the interests of more than one individual are to be taken into account simultaneously. So they have no role to play in determining the welfare of a single individual.

2.4. Rationality and Admissible Choice Functions.

So far, any choice function C_i has been allowed as a possible WCF for individual i. By contrast, it has been traditional in welfare economics to allow only choice functions satisfying certain "rationality" postulates. In order of increasing strength, the most common forms of these postulates are as follows:-

1) C_i corresponds to a weak preference relation R_i and to a strict preference relation P_i. P_i is acyclic (binary choice)[1]
2) C_i corresponds to a weak preference relation R_i and to a strict preference relation P_i. P_i is transitive (quasi-transitive preferences, quasi-ordinal choice).
3) C_i corresponds to a weak preference relation R_i which is transitive (ordinal choice)[2].
4) C_i corresponds to a choice indicator function u_i (which is called i's welfare function).

While any of the first three postulates can be made to seem plausible, and the fourth can be derived from the third under weak assumptions[3], they are restrictive conditions. Indeed, in chapter 4, we shall see that some of the approaches to dynamic choice bring about violations of postulate (1), the weakest postulate. So it seems desirable not to insist on any stronger rationality postulates than are absolutely necessary for the development of a potentially useful theory.

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(1) "Binary" is the term used by Herzberger (1971). Sen (1971) prefers "normal".

(2) "Ordinal" is a term due to Herzberger (1971).

(3) For derivations, see Debreu (1954), (1959, ch.4), (1964), Rader (1963), Bowen (1968), Arrow and Hahn (1971).

Of course, if we are looking for generality, no rationality postulates at all should be insisted on. But that would make it virtually impossible to say anything worthwhile. So rationality postulates will be brought in, as is necessary. However, much of the argument will rely on a rationality postulate which is weaker than (1) above.

This postulate is:-

0) Whenever $B \subseteq A$, $B - C_i(B) \subseteq A - C_i(A)$.

It is a postulate which has acquired many names[1]. But none of the existing names seems both short and appropriate. So here a CF satisfying postulate (0) will be called *coherent*[2].

There are many ways of explaining the appeal of coherence. To try another may seem pointless. Nevertheless, remembering that the WCF C_i is meant to represent individual i's interests, it may be worth reconsidering coherence for such CF's.

Suppose that the initial feasible set is A and the options in A which are in i's interests are the members of $C_i(A)$. Then, when the feasible set is A, the members of $A - C_i(A)$ are not in i's interests. Now suppose that the feasible set expands to B, where $B \supsetneq A$. Is there any reason for the members of $A - C_i(A)$ to become in i's intereets, when they were not before? Coherence is the assumption that there is no such reason.

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(1) For a brief discussion of this assumption, and references to earlier discussions, see Sen (1969, p. 384) or (1970a, p.17)

(2) Afriat (1962) defines "coherent" choices, in consumer demand theory, as choices satisfying the strong axiom of revealed preference. (Houthakker (1950)). There should be no confusion, although I use the same term in a different sense.

In this section, various postulates concerning an individual's WCF have been mentioned. In what follows, the postulates which are assumed will vary. In general, there will be, for each individual i, a class $\mathcal{C}_i(X)$ of *admissible choice functions*, defined on the underlying set X. $\mathcal{C}_i(X)$ is the class of choice functions satisfying the postulates which are required.

2.5. Social Welfare and Constitutions.

Welfare economics is concerned with the welfare of all individuals in a society. To make decisions about economic policy, the interests of individuals and (where distinct) of the society as a whole have to be weighed against each other. The resulting decisions purport to take account of all these interests. These decisions correspond to a choice function C on the underlying set of possible economic policies X.

In welfare economics, it is common to talk of "social welfare". Here, social welfare will be identified with a choice function which takes account of the interests of society as a whole. Such a choice function will be called a *social welfare choice function*, or SWCF.

As with individual welfare, the precise definition of SWCF in any economy depends upon what the interests of individuals and of society as a whole are, and how these interests are combined into a choice function. As with individual interests, what constitutes social welfare is a matter of judgment. But, whereas what an individual's interests are could perhaps be a factual judgment, it is hard to see how what social welfare is could be anything other than an ethical value judgment.

The question which will concern us here is the relationship, if any, between social welfare and individuals' interests. This is very similar to the relationship between social choice and individual values which Arrow considered.

Now it would seem that most peoples' conception of social welfare depends upon a number of factors, which might be classified as follows:-

1) Individuals' interests
2) Interpersonal comparisons
3) Preference intensities

4) Community interests (as distinct from individuals' interests)

5) Individuals' ethical values

6) Other ethical considerations.

Of these, individuals' interests have already been explained. Inter-personal comparisions are "as if" choices of which individual one would like to be, in a given environment. Preference intensities are measurements of the relative importance of individuals' interests, either for different individuals facing the same issue, or for the same individual facing different issues, or for both[1]. Both (2) and (3) are related to individuals' interests, but are not completely determined by them.

It has often been claimed that a community may have interests which are related in no obvious way to those of its individual members. It is hard to think of an entirely convincing example of this. Nevertheless it is in principle possible to allow non-individualistic ethical considerations of all kinds to affect the social welfare choice function.

So far, for a given individual, only his interests have been admitted as influencing the conception of social welfare. But one also wishes to admit individuals' ethical values. Thus, most people in this country regard cock-fighting as an unethical sporting activity. Yet this has nothing to do with any individual's interests, since it is surely too far-fetched to claim that significant suffering would be caused among those who object to cock-fighting, and do not watch it, by the knowledge that it is being practised. Another example is punishment. Some people believe that a wrongdoer deserves to be punished, regardless of whether the punishment deters this or any other potential offenders, or is in any way in the interests of some individuals in society. One may or may

(1) See Sen (1970a), chapters 7, 8 and 9.

not wish to respect such ethical values, but a theory which excludes them altogether is clearly limited. So, individuals' ethical values will be admitted as relevant. In doing so, we admit Arrow's individual "values" (as opposed to "tastes")[1] in general, and Harsanyi's "ethical preferences"[2] and the preferences of an altruist, in particular.

The social welfare choice function, or SWCF, is analogous to the Bergson social welfare function. Indeed, when the SWCF corresponds to a choice-indicator function, that function is precisely a Bergson social welfare function.

Now, what is the relationship between social welfare and individuals' interests? It can be only a partial one. If individuals' interests change, the SWCF will change, in general. But the SWCF may change even though no

(1) See Arrow (1963), p. 18

(2) See Harsanyi (1953),(1955) , and Pattanaik (1968), (1971) Ch. 9.

individual's interests change. Nevertheless, in this study, it will be assumed that the SWCF changes only if some individual's WCF changes. This means that none of the other factors influencing social welfare - interpersonal comparisons, measures of preference intensity, community interests, individuals' ethical values, etc. - can change, unless some individuals' interests also change. So all of these factors are assumed to be themselves uniquely determined by the set of individual interests, in the given society being considered. For example, if a previously healthy person became sick, but his and everybody else's interests remained unchanged, the interests of the sick person could not be given extra weight in comparison with those of others. Whether this is serious is not very clear, because it is hard to imagine somebody becoming other than trivially ill without his interests changing.

The assumption that interpersonal comparisons do not change, unless individuals' interests do, rules out welfare analyses based on Rawls' concept of justice[1]. This loss could be rectified by extending the underlying set from X to $X \times I$, where I is the set of individuals[2]. That is, not only choices of economic policy have to be contemplated, but also hypothetical choices of who one wishes to be for a given economic policy.

It should be emphasized that the assumption that the SWCF changes only if some individual's WCF changes does not mean that we have a purely individualistic ethical system; it only means that we shall not be able to say anything about changes in the non-individualistic aspects of the system.

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(1) See Rawls (1958), (1971).

(2) As explained in Sen (1970a), ch. 9.

Given these assumptions, a "constitution" function f can be constructed. Let I denote the set of individuals, which is taken to be either a finite set of the form $\{1,\ldots, n\}$, or (to allow for infinite horizons) the set of all positive integers $\{1, 2, \ldots\}$. Let X be the underlying set. Let $\mathcal{C}_i(X)$ be the class of WCF's for individual i which are admissible. Let $\mathcal{C}(X)$ be the class of admissible SWCF's. Now, given any list of individual WCF's $\langle C_i \rangle_{i \in I}$ (where each $C_i \in \mathcal{C}_i(X)$), there is, by assumption, a uniquely determined SWCF $C = f(\langle C_i \rangle_{i \in I})$. So, as $\langle C_i \rangle_{i \in I}$ varies over the product set $\Pi_{i \in I} \mathcal{C}_i(X)$, a mapping $f: \Pi_{i \in I} \mathcal{C}_i(X) \to \mathcal{C}(X)$ is constructed. Such a mapping is called a *constitution*[1].

The properties of constitutions are the subject of social choice theory. The present work, being concerned with welfare economics, will assume that the constitution is given, and not enquire too closely into the properties underlying it. At least, this is true except for the next chapter. There, we shall consider one particular property which has been generally accepted by economists - the Pareto principle - and also the additive form of Bergson social welfare function which will often be assumed in the later chapters.

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(1) "Constitution" is a term first suggested by Kemp and Asimakopoulos (1952), and later adopted by Arrow (1963, p.105). The term was used to describe a function from lists of individual preference relations to social preference relations - i.e., what is commonly called an "Arrow social welfare function". Here, a constitution is a somewhat more general type of mapping, because general choice functions rather than preferences are being considered. Barry (1965) talks of "social decision procedures", and Sen (1970a) of "social decision rules".

2.6. Conclusion.

It may be helpful to summarize the ingredients of the welfare economic theory which will be employed in later chapters. These are:-

1) An underlying set of options, or possible economic policies - X.

2) A set of individuals in the society - I (which may be finite or else countably infinite).

3) For each $i \in I$, a class of admissible welfare choice functions, C_i, defined on X - $\mathcal{C}_i(X)$. Each C_i, under certain circumstances, represents the interests of individual i.

4) A class of admissible social welfare choice functions, C, defined on X - $\mathcal{C}(X)$.

5) A constitution f: $\Pi_{i \in I} \mathcal{C}_i(X) \to \mathcal{C}(X)$.

Chapter 3.

UNANIMITY PRINCIPLES AND SOCIAL WELFARE.

3.1. Introduction.

In section 2.5., a constitution was defined as a mapping from lists of individual welfare choice functions (WCF's) to social welfare choice functions (SWCF's). No properties were attributed to the constitution. Yet it has been customary in welfare theory to invoke the Pareto principle. In its weak form, this states that a change which makes each individual better off is to be regarded as a favourable change. But its definition requires the assumption that each individual's choice function is binary, and it arrives at a social preference relation which is to be incorporated into the SWCF. Such incorporation is not always possible, unless the SWCF itself satisfies certain rationality postulates.

This chapter starts by exploring how the Pareto principle can be generalized to allow for non-binary individual WCF's. The result will be criteria which, even in the case when individual choice is binary, give more selective social choice sets than the Pareto principle gives. The final section of the chapter considers briefly the assumptions underlying the additive form of Bergson social welfare function which is commonly assumed, and will be used extensively in later chapters.

The one crucial assumption which will be made in this chapter is that the appropriate choice, for any group of individuals, depends only on the interests of those individuals. While this assumption is very much concerned with ethical values, in a given situation, the other assumptions are mostly technical in nature.

3.2. Unanimity Principles - the General Case.

Let I be the set of individuals. Imagine that each individual i in I has a representative who makes choices of economic policy with only i's interests in mind. So i's representative has a choice function C_i, which is i's welfare choice function. Given any feasible set $A \subseteq X$, $C_i(A)$ is, by definition, the set of options which i's representative is willing to choose.

Now suppose that, for the feasible set $A \subsetneq X$, there happens to be an option x such that:-

$$x \in \bigcap_{i \in I} C_i(A).$$

Then each individual's representative is willing to choose x. So x is in the interests of each individual in the society. It does not necessarily follow that x is an acceptable social choice. In section 2.5., we saw how a number of factors, apart from individuals' interests, affect social choice. Thus x may involve unnecessary cruelty to animals; although this may not be against any individual's *interests*, it may still be ruled out on ethical grounds - because most individuals find it ethically unacceptable, for example(1).

Nevertheless, cases in which an option is ethically unacceptable, even though it is in the interests of every individual, seem exceptional. So we shall restrict ourselves to choice situations in which, if x is in the interests of all, then it is an acceptable social choice. Then we

(1) Sen's "Liberal Paradox" is *not* an example in which an option which is in the interests of all is not an acceptable social choice. The only option which is genuinely in the interests of each individual is to have the lascivious man read "Lady Chatterley's Lover". Sen applies the Pareto principle to the individuals' *own choices* and shows, in effect, that the result is counter to the individuals' interests. The trouble seems to arise because the individual's choices extend beyond their own interests. See Sen (1970b) and also section 2.3.

have the *first unanimity principle*, which can be stated formally as follows:-

For each $A \subseteq X$,

$$\text{if } x \in \bigcap_{i \in I} C_i(A)$$

$$\text{then } x \in C(A)$$

where C denotes the SWCF.

This is a restriction on social choice which specifies a sufficient condition for an option to be chosen. Since such unanimity is rare, the condition cannot be made a necessary one, otherwise the social choice set would be empty in most situations. But where unanimity does occur - where $\bigcap_{i \in I} C_i(A)$ is not empty - it is tempting to insist that the condition $y \in \bigcap_{i \in I} C_i(A)$ should be necessary as well as sufficient for y to be chosen. For it is possible to meet the interests of all individuals simultaneously, and yet no option outside $\bigcap_{i \in I} C_i(A)$ does this. This suggests what will be called the *strong second unanimity principle*:-

For each $A \subseteq X$,

$$\text{if } \bigcap_{i \in I} C_i(A) \text{ is non-empty}$$

$$\text{then } C(A) \subseteq \bigcap_{i \in I} C_i(A).$$

Together with the first unanimity principle, this principle ensures that if $\bigcap_{i \in I} C_i(A)$ is non-empty, then $C(A) = \bigcap_{i \in I} C_i(A)$, and so the social choice set is completely determined. But if $\bigcap_{i \in I} C_i(A)$ is empty, anything is possible.

The "strong" second unanimity principle leads to difficulties when coherent social choice is insisted on, as will be seen in the next section. So, later, the principle will have to be weakened.

It will be convenient to express the strong second unanimity principle in the form:-

$$C(A) \subseteq \dot{S}U(A) \quad \text{(each } A \subseteq X\text{)}$$

where:-

$$SU(A) = \begin{cases} \bigcap_{i \in I} C_i(A) & \text{(if this is non-empty)} \\ A & \text{(otherwise)} \end{cases}$$

Then SU is a choice function, as is easily seen.

Although the unanimity principles concern social choice for a *given* list of individual WCF's, they have a natural extension to unanimity principles governing the constitution. Say that a constitution f satisfies the first unanimity principle if:-

For all $\langle C_i \rangle_{i \in I} \in \Pi_{i \in I} \mathcal{C}_i(X)$, and all $A \subseteq X$,

$$\bigcap_{i \in I} C_i(A) \subseteq f(\langle C_i \rangle_{i \in I})(A)$$

That is, if for all admissible individual WCF's, and all choice situations, the intersection of the individual's welfare choice sets is a subset of the social welfare choice set, for the SWCF prescribed by the constitution.

Say that a constitution f satisfies the strong form of the second unanimity principle if:-

For all $\langle C_i \rangle_{i \in I} \in \Pi_{i \in I} \mathcal{C}_i(X)$, and all $A \subseteq X$,

$\bigcap_{i \in I} C_i(A)$ is non-empty

implies $f(\langle C_i \rangle_{i \in I})(A) \subseteq \bigcap_{i \in I} C_i(A)$.

Without extra assumptions, such as coherence, it is hard to say more about the consequences of unanimity principles for social choice, or for the constitution.

This section concludes with a result which has some bearing on admissible choice functions, as defined in section 2.4. In effect, it states that any choice function which is admissible as a WCF for some individual must also be admissible as an SWCF.

Theorem 3.2. Let $f\colon \Pi_{i\varepsilon I}\, \mathcal{C}_i(X) \to \mathcal{C}(X)$ be a constitution satisfying the first, and also the strong second, unanimity principle. Suppose that the choice function $\bar{C}$ for which:-

$$\bar{C}(A) = A \quad (\text{each } A \subseteq X)$$

is a member of $\mathcal{C}_i(X)$, for each $i \varepsilon I$. Then

$$\mathcal{C}(X) \supseteq \cup_{i\varepsilon I}\, \mathcal{C}_i(X).$$

Proof Consider the list of choice functions $\langle C_i^o\rangle_{i\varepsilon I}$,

where $\quad C_k^o \varepsilon \mathcal{C}_k(X)$

and $\quad C_j^o = \bar{C} \ (j \neq k)$

By the unanimity principles, for any $A \subseteq X$,

$$f(\langle C_i^o\rangle_{i\varepsilon I})(A) = C_k^o(A)$$

and so $\quad C_k^o \varepsilon \mathcal{C}(X)$.

The extra assumption, that to be willing to choose anything is admissible as an individual WCF, is hardly severe when it is recalled that a choice function is admissible provided that it satisfies certain rationality postulates (see section 2.4.). There is no implication that such a choice function represents any actual individual's interests.

Because unanimity principles for constitutions are no more than straightforward extensions of those for SWCF's, we shall not consider them explicitly from now on. It will be assumed that the individual WCF's are fixed, and that the unanimity principles govern social choice for this fixed list of individual WCF's.

3.3. Coherent Choice and the Unanimity Principles.

From now on in this chapter, it will be assumed that only coherent SWCF's are admissible. An immediate implication of this is that only coherent individual WCF's are admissible, by theorem 3.2.

It is now possible to demonstrate why the strong second unanimity principle is too strong, using an example due to Pattanaik[1].

Example 3.3.1

There are three individuals, $i = 1, 2, 3$. The underlying set X consists of three options x, y, z. Each individual has a WCF which is quasi-ordinal[2]. C_i corresponds to R_i $(i = 1, 2, 3)$, where:-

$$x\ P_1\ y, \qquad y\ I_1\ z, \qquad z\ I_1\ x$$

$$x\ I_2\ y, \qquad y\ P_2\ z, \qquad z\ I_2\ x$$

$$x\ I_3\ y, \qquad y\ I_3\ z, \qquad z\ P_3\ x$$

Now, in this case, the choice function SU is as follows:-

$SU(\{x, y\}) = \{x\}$, $SU(\{y, z\}) = \{y\}$, $SU(\{z, x\}) = \{z\}$

$SU(X) = X$.

Evidently, SU is incoherent. More seriously, if C is any SWCF satisfying the strong form of the second unanimity principle, then $C(A) \subseteq SU(A)$ for all $A \subseteq X$. Thus, in particular, $C(\{z, x\}) \subseteq SU(\{z, x\}) = \{z\}$. So $x \notin C(\{z, x\})$. If C is also coherent, it follows that $x \notin C(X)$. By symmetry, if C is coherent, then $y \notin C(X)$ and $z \notin C(X)$. But, since C is a choice function, $C(X)$

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(1) See Pattanaik (1971), theorem 8.2., p. 141.

(2) See appendix 1, section A.7.

cannot be empty. This is a contradiction, and something must give - either the strong form of the second unanimity principle, or coherent social choice.

So, if the SWCF has to be coherent, the strong second unanimity principle must be violated in certain circumstances. In what circumstances? Can it be weakened so that it never has to be violated ?

Because social choice is coherent, the strong form of the second unanimity postulate:-

$C(A) \subseteq SU(A)$ (each $A \subseteq X$)

implies something stronger - at least, it does when SU is not itself coherent, which it may not be, as example 3.3.1 illustrates. In fact, consider lemma A.2 of appendix 1.

Define $\hat{SU}(A) = \{x \in A | x \in B \subseteq A$ implies $x \in SU(B)\}$. Then $\hat{SU}$ is a set function with the property that $\hat{SU}(A) \subseteq SU(A)$ (each $A \subseteq X$). Also, if C is a coherent choice function with the property that

$C(A) \subseteq SU(A)$ (each $A \subseteq X$),

then $\hat{SU}$ is a choice function, and

$C(A) \subseteq \hat{SU}(A)$ (each $A \subseteq X$).

It is therefore evident that if there is *any* coherent SWCF satisfying the strong second unanimity principle, then $\hat{SU}$ is such a choice function. Consequently, such a choice function exists if and only if $\hat{SU}$ is a choice function (rather than merely a set function); the condition for this is that $\hat{SU}(A)$ must be non-empty whenever A is finite. This answers the first question.

Before turning to the second question, it is worth noting that, if $\hat{SU}$ is a choice function, then there is a coherent SWCF satisfying the first as well as the strong form of the second unanimity principle. Recall that, if the SWCF is to be coherent, we had better insist that each individual's WCF is coherent. Then, the following lemma shows that $\hat{SU}$ at least, is a coherent choice function satisfying both unanimity principles, providing that it is a choice function, of course.

Lemma 3.3.2. If each C_i is coherent, and if, for any $A \subseteq X$, $\bigcap_{i \varepsilon I} C_i(A)$ is non-empty, then

$$\hat{SU}(A) = \bigcap_{i \varepsilon I} C_i(A).$$

Proof It is enough to prove that $\bigcap_{i \varepsilon I} C_i(A) \subseteq \hat{SU}(A)$. So suppose $x \varepsilon \bigcap_{i \varepsilon I} C_i(A)$, and that $x \varepsilon B \subseteq A$. For any $i \varepsilon I$, $x \varepsilon C_i(A)$, and, because C_i is coherent, $x \varepsilon C_i(B)$. Therefore $x \varepsilon SU(B)$. So, by definition, $x \varepsilon \hat{SU}(A)$, as required.

Now we can consider how to weaken the second unanimity principle so that it never has to be violated to ensure that social choice is coherent. Define the *weak second unanimity principle* as follows:-

For each $A \subseteq X$, if $\bigcap_{i \varepsilon I} C_i(A)$ is non-empty,

then $C(A) \subseteq \bigcup_{i \varepsilon I} C_i(A)$.

Again, it is convenient to express this unanimity principle in the form:-

$$C(A) \subseteq WU(A) \qquad (\text{each } A \subseteq X)$$

where:-

$$WU(A) = \begin{cases} \bigcup_{i\varepsilon I} C_i(A) & (\text{if } \bigcap_{i\varepsilon I} C_i(A) \text{ is non-empty}) \\ A & (\text{otherwise}). \end{cases}$$

Then, evidently, WU is a choice function.

The weak form of the second unanimity principle can be interpreted as follows. Suppose that $\bigcap_{i\varepsilon I} C_i(A)$ is non-empty. Then each individual's interests can be fully met by choosing an option in $\bigcap_{i\varepsilon I} C_i(A)$. The principle states that no option y is to be regarded as acceptable unless it fully meets the interests of at least one individual - i.e. unless, for some $j \varepsilon I$, $y \varepsilon C_j(A)$. To violate this principle would be to disregard individuals' interests entirely, it seems.

The following example shows that it is always possible to find a *constitution* satisfying both the first and the weak form of the second unanimity principle. This is true even if we insist on the SWCF being coherent, because then, by theorem 3.2., each individual's WCF must be coherent too.

Example 3.3.3.

Consider the constitution

$$f(\langle C_i \rangle_{i\varepsilon I})(A) = C_1(A) \quad (\text{each } A \subsetneq X).$$

If C_1 is coherent, the SWCF is always coherent. Both the first unanimity principle and the weak form of the second unanimity principle are clearly satisfied.

This example also shows how weak is "weak" unanimity. It does not rule out the obvious generalization to choice functions of an Arrow where:-

$$WU(A) = \begin{cases} \bigcup_{i\epsilon I} C_i(A) \text{ (if } \bigcap_{i\epsilon I} C_i(A) \text{ is non-empty)} \\ A \text{ (otherwise).} \end{cases}$$

Then, evidently, WU is a choice function.

The weak form of the second unanimity principle can be interpreted as follows. Suppose that $\bigcap_{i\epsilon I} C_i(A)$ is non-empty. Then each individual's interests can be fully met by choosing an option in $\bigcap_{i\epsilon I} C_i(A)$. The principle states that no option y is to be regarded as acceptable unless it fully meets the interests of at least one individual - i.e. unless, for some $j \in I$, $y \in C_j(A)$. To violate this principle would be to disregard individuals' interests entirely, it seems.

The following example shows that it is always possible to find a *constitution* satisfying both the first and the weak form of the second unanimity principle. This is true even if we insist on the SWCF being coherent, because then, by theorem 3.2., each individual's WCF must be coherent too.

Example 3.3.3.

Consider the constitution

$$f(\langle C_i \rangle_{i\epsilon I})(A) = C_1(A) \quad \text{(each } A \subseteq X\text{).}$$

If C_1 is coherent, the SWCF is always coherent. Both the first unanimity principle and the weak form of the second unanimity principle are clearly satisfied.

This example also shows how weak is "weak" unanimity. It does not rule out the obvious generalization to choice functions of an Arrow "dictatorship"[1]. But it does confirm that we have at least succeeded in our task of weakening the second unanimity principle so that, even in combination with the first, it can always be satisfied by *some* coherent social choice function.

Notice that the first unanimity principle is always easy to satisfy, because it can never force a choice set to be empty. From now on, it has almost no role in the analysis, and so is left implicit. It provides a sufficient condition for an option to be a social choice; we shall be more concerned with necessary conditions, which is what the second unanimity principle is, in either of its forms.

There is one obvious question which this section has left untouched. What is the character of coherent SWCF's satisfying the weak form of the second unanimity principle? All that we can say at present is that such an SWCF C must satisfy:-

$$C(A) \subseteq \hat{WU}(A) \quad (\text{each } A \subseteq X)$$

where $\hat{WU}(A) = \{x \in A \mid x \in B \subseteq A \text{ implies } x \in WU(B)\}$ is derived from WU in the same way as $\hat{SU}$ was derived from SU. To say much more for general coherent choice functions would be difficult. Accordingly, extra assumptions will now be considered, which restrict the classes of admissible choice functions, and so allow stronger and, in some cases, more familiar results to be derived.

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(1) Arrow defined "dictatorship" for social welfare functions. See Arrow (1963), definition 6, p. 30.

3.4. Binary Choice and the Pareto Principles.

In this section, it will be assumed that only binary choice functions are admissible as individual WCF's. So, assume that individual i's WCF C_i corresponds to a weak preference relation R_i. Given this assumption, there is a link between the unanimity principles and the more familiar Pareto principles[1].

Consider the weak Pareto principle. Define the relation *WPP* on X by:-

$$a \; WPP \; b \quad \text{iff} \quad \forall i \in I \quad a \, P_i \, b$$

The weak Pareto principle states that, given a set A, $x \in C(A)$ only if there is no $y \in A$ such that $y \; WPP \; x$. So define the relation *WPR* by:-

$$a \; WPR \; b \quad \text{iff} \quad \text{not } b \; WPP \; a.$$

Now, it is easy to see that *WPR* is a weak preference relation, because it is connected and reflexive. In addition, the corresponding strict preference relation *WPP* is acyclic, because each individual's strict preference relation P_i is acyclic. Therefore the relation *WPR* corresponds to a choice function on X; let this choice function be denoted by *WPE*. For any set $A \subsetneq X$, $WPE(A)$ is the set of *weakly Pareto efficient* options in A. The weak Pareto principle states that:-

$$C(A) \subsetneq WPE(A) \qquad (\text{each } A \subsetneq X).$$

Consider now the weak form of the second unanimity principle. Evidently, if $a \; WPP \; b$, then $b \notin WU(\{a, b\})$. Now let A be any subset of X. Because $\hat{WU}$ is coherent, if there exists $y \in A$ such that $y \; WPP \; x$, then $x \notin \hat{WU}(\{x, y\})$, and so $x \notin \hat{WU}(A)$. Therefore:-

$$\hat{WU}(A) \subsetneq WPE(A) \qquad (\text{each } A \subsetneq X).$$

from which it follows that the weak form of the second unanimity principle:

$$C(A) \subsetneq \hat{WU}(A) \qquad (\text{each } A \subsetneq X).$$

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[1] See Arrow (1963), condition P, p. 96; and Sen (1970a), ch. 2.

entails the weak Pareto principle:

$$C(A) \subseteq WPE(A) \qquad \text{(each } A \subseteq X\text{)}.$$

But the weak Pareto principle does not entail the weak form of the second unanimity principle, because $\hat{WU}(A)$ may be a proper subset of $WPE(A)$ as will be seen in example 3.4. below.

Before turning to this example, however, it is convenient first to consider the strict Pareto principle in a similar way. Indeed the results are very similar, except that the strict Pareto principle does not always give rise to a choice function, in the proper sense.

Define the relation SPP on X by:-

$$a \; SPP \; b \quad \text{iff} \quad \forall i \in I \;\; a R_i b \; \& \; \exists j \in I \; aP_j b$$

The strict Pareto principle states that, given a set A, $x \in C(A)$ only if there is no $y \in A$ such that $y \; SPP \; x$. So, define the relation SPR by:-

$$a \; SPR \; b \quad \text{iff} \quad \text{not } b \; SPP \; a.$$

It is easy to see that SPR is a weak preference relation, because it is connected and reflexive. But the corresponding strict preference relation SPP may be cyclic, as example 3.3.1. shows. Therefore the relation SPP may not correspond to a choice function. But we can define a set function $SPE(\cdot)$ on subsets of X as follows:-

$$SPE(A) = \{x \in A \,|\, y \in A \text{ implies } x \; SPR \; y\}$$

For any set $A \subseteq X$, $SPE(A)$ is the set of *strictly Pareto efficient* options in A[1]. The strict Pareto principle states that:-

$$C(A) \subseteq SPE(A) \qquad \text{(each } A \subseteq X\text{)}.$$

On the other hand, consider the strong form of the second unanimity principle. It is evident that, if $a \; SPP \; b$, then $b \notin SU(\{a, b\})$. Now let

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(1) "Strictly" is often dropped, because it is customary to consider only strict Pareto efficienty.

A be any subset of X. Because $\hat{SU}$ is coherent - as a set function - if there exists $y \in A$ such that $y\ SPP\ x$, then $x \notin \hat{SU}(\{x, y\})$, and so $x \notin \hat{SU}(A)$. Therefore:-

$$\hat{SU}(A) \subseteq SPE(A) \qquad \text{(each } A \subsetneq X\text{).}$$

from which it follows that the strong form of the second unanimity principle entails the strict Pareto principle.

That neither Pareto principle entails the corresponding form of the second unanimity principle is demonstrated by the following example:-

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Example 3.4.

The underlying set $X = \{a, b, c, d\}$

The set of individuals $I = \{1, 2\}$

The two individuals have quasi-ordinal choice functions corresponding to the following quasi-transitive preferences:-

$$a\ P_1\ b \quad b\ P_1\ c \quad a\ P_1\ c \quad a\ I_1\ d \quad b\ I_1\ d \quad c\ I_1\ d$$

$$c\ P_2\ b \quad b\ P_2\ a \quad c\ P_2\ a \quad a\ I_2\ d \quad b\ I_2\ d \quad c\ I_2\ d$$

Then $WU(A) = \begin{cases} \{a, c, d\} & \text{(if } A = X\text{)} \\ \{a, c\} & \text{(if } A = \{a, b, c\}\text{)} \\ A & \text{(otherwise)} \end{cases}$

Evidently, then, WU is a coherent choice function, which implies that it is identical to $\hat{WU}$.

Also, $SU(A) = \begin{cases} \{d\} & \text{(if } A = X\text{, or if } A \text{ is any triple including } d\text{)} \\ A & \text{(otherwise)} \end{cases}$

Evidently, too, SU is a coherent choice function, which implies that it is identical to $\hat{SU}$.

On the other hand, there is no pair of options such that $x\ WPP\ y$

or even such that $x \; SPP \; y$. Therefore:-

$$SPE(A) = WPE(A) = A \qquad \text{(for all } A \subsetneq X\text{)}.$$

This example makes it clear that the second unanimity principle, in either of its forms, may be more stringent than the corresponding Pareto principle. But, whenever A is a pair set, then, obviously:-

$$WU(A) = WPE(A) \text{ and } SU(A) = SPE(A).$$

So the unanimity principles fail to correspond to binary choice functions, as the Pareto principles do. Naturally, this makes the unanimity principles rather harder to apply in general.

3.5. Ordinal Choice and the Unanimity Principles.

As remarked in chapter 2, it is common in welfare economics to assume that any social or individual welfare choice function is ordinal - that is, binary, with a transitive weak preference relation. We shall make the same assumption for the rest of this chapter. More specifically, we assume that:-

(1) The SWCF C is ordinal.

(2) Each individual's WCF C_i is ordinal.

Given these assumptions, we have the following result:-

Theorem 3.5. If the SWCF C, and each individual's WCF C_i are ordinal, then:-

(1) Each form of the second unanimity principle is identical to the corresponding Pareto principle.

(2) There exists a constitution satisfying the strong form of the second unanimity principle.

Proof

(1) Since we already know that:-

$\hat{WU}(A) \subseteq WPE(A)$ and $\hat{SU}(A) \subseteq SPE(A)$ (each $A \subseteq X$)

it is enough to show that:-

$WPE(A) \subseteq \hat{WU}(A)$ and $SPE(A) \subseteq \hat{SU}(A)$ (each $A \subseteq X$).

(a) Suppose that $x \notin \hat{WU}(A)$. Then there exists $B \subseteq A$ such that $x \in B$ and $x \notin WU(B)$. So, for each $i \in I$, $x \notin C_i(B)$. Also, there exists $y \in B$ such that for each $i \in I$, $y \in C_i(B)$. Because C_i is ordinal, it follows that $y \; P_i \; x$ (each $i \in I$). Therefore $y \; WPP \; x$, which entails $x \notin WPE(A)$, because $y \in A$. So $WPE(A) \subseteq \hat{WU}(A)$ (each $A \subseteq X$), as required.

(b) The proof that $SPE(A) \subsetneq \hat{SU}(A)$ (each $A \subsetneq X$) is similar.

(2) It is sufficient to specify one such constitution.

Number the individuals in society so that

$I = \{1, 2, 3, \ldots\}$ (it does not matter if I is infinite, provided it is countable). Let the social weak preference relation R be defined, as a function of individual preference relations R_i, as follows:-

$$
\begin{aligned}
a\,R\,b \quad \text{iff} \quad & \forall i \in I \qquad a\,R_i\,b \\
\text{or} \quad & \exists j \in I \ \text{ s.t. } \ a\,R_i\,b \quad (i = 1, 2, \ldots, j-1). \\
& \qquad\qquad \text{and} \quad a\,P_j\,b.
\end{aligned}
$$

Thus, $a\,R\,b$ if $a\,P_1\,b$, or if $a\,R_1\,b$ and $a\,P_2\,b$,

or if $a\,R_1\,b$, $a\,R_2\,b$, and $a\,P_3\,b$, or ....

It is easy to see that R is reflexive, connected and transitive, because each R_i is. It is also obvious that this "lexicographic" constitution satisfies the strict Pareto principle, and so, because of (1), it satisfies the strong form of the second unanimity principle.

Naturally, neither *WPE* nor *SPE* are ordinal choice functions in general. But they must, at least, be binary choice functions. And, in fact, when each individual's WCF is ordinal, it is easy to see that both *WPE* and *SPE* are quasi-ordinal.

3.6. Additive Bergson Social Welfare.

A *Bergson social welfare function* is a choice-indicator function corresponding to an ordinal SWCF. That is, given the SWCF C on the underlying set X, a Bergson social welfare function - or BSWF - corresponding to C is a mapping $W: X \to \mathbb{R}$ with the property that, for each $A \subseteq X$,

$$C(A) = \{x \in A \mid y \in A \text{ implies } W(x) \geq W(y)\}$$

Let us assume that a BSWF exists, and also that each individual's WCF C_i corresponds to a utility function $u_i : X \to R$. Assume too that the unanimity principles are all satisfied - both first and second, the second in its strong form.

Suppose that x and y are two options in X such that, for each $i \in I$, $u_i(x) = u_i(y)$.

Then $\bigcap_{i\in I} C_i(\{x, y\}) = \{x, y\}$.

By the first unanimity postulate, it follows that

$$C(\{x, y\}) \supseteq \{x, y\}, \text{ and so } W(x) = W(y).$$

Therefore the value of the BSWF $W(x)$, for a given option x, is uniquely determined by the list of values of the individual utility functions $\langle u_i(x)\rangle_{i\in I}$, for this option x, and does not depend on x itself. Thus the function $W(\cdot)$ can be written in the form:-

$$W(x) \equiv F(\langle u_i(x)\rangle_{i\in I})$$

for some mapping

$$F : \Pi_{i\in I}\, u_i(X) \to \mathbb{R}$$

By the strict Pareto principle (which is here equivalent to the strong form of the unanimity principle, by theorem 3.5) if $\quad u_i(x) \geq u_i(y) \qquad$ (each $i \in I$)

and $\quad u_j(x) > u_j(y) \qquad$ (some $j \in I$)

then $\quad W(x) > W(y)$.

It follows that F is strictly increasing in u_i, for each $i \in I$.
A BSWF which can be written in this form with F strictly increasing in each argument, is called an *individualistic* or *Paretian* social welfare function(1).

Now, it has been common in the literature of welfare economics to assume a very much more special form of BSWF, namely the additive form:-

$$W(x) = \Sigma_{i \in I}\, u_i(x).$$

This is a restrictive assumption, which has frequently been criticized(2). Nevertheless, it is often useful and is certainly conceptually more simple than more general kinds of BSWF. Also, it is an assumption which will be used in chapter 6. This section will discuss conditions which are sufficient for there to be a BSWF of this additive form. Naturally, it makes use of the theory of separable utility functions(3).

All the relevant results on separability which we have apply only when the underlying set has the form of a Cartesian product, $X = \Pi_{i \in I} X_i$, where $u_i : X_i \to \mathbb{R}$. For u_i to correspond to C_i, it is then clearly necessary to have C_i defined just on X_i rather than on the whole of X. In much of the literature of welfare economics, this is true in any case. X_i is individual i's consumption set, and individual i has interests

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(1) See Graaff (1957), pp. 7 - 10.

(2) See, for example, Samuelson (1947), pp. 226 - 227, who criticized it as unnecessary, or Sen (1973) who criticizes it as being unable to deal adequately with our regard for equity. Also, it excludes Rawls' criterion of justice; see Sen (1970a), ch. 9.

(3) See Debreu (1960), Gorman (1968a), (1968b), Fishburn (1969), Koopmans (1972a).

in only his own consumption - not in that of anybody else. But this rules out all external effects of consumption, and also, analogously, all public goods and common interests in the choice of production plan. So it is surely excessively restrictive to assume that the underlying set is decomposable in this way.

There is, however, an alternative method of arriving at a decomposable underlying set. It involves hypothetical, "as if" options, which could not possible be feasible. Let X be the true underlying set. Suppose that it were possible to choose a different option in X for each individual. Thus we can choose x_1 for individual 1, x_2 for individual 2, etc. Let x_i denote the option for individual i. Then a typical hypothetical option is a list $\langle x_i \rangle_{i \varepsilon I}$ of options, one for each individual. There is now a hypothetical underlying set. It is the Cartesian product $\Pi_{i \varepsilon I} X_i$ where each X_i is just a copy of the original underlying set X. And the original underlying set X, of options which might be feasible, can be identified with the set $\tilde{X}$ of hypothetical options, in which the same basic option is chosen for each individual. Thus:-

$$\tilde{X} = \{\langle x_i \rangle_{i \varepsilon I} | \exists x \,\varepsilon\, X \text{ s.t. } x_i = x \quad (\text{each } i \,\varepsilon\, I)\}$$

Now go on to imagine a hypothetical SWCF C defined on $\Pi_{i \varepsilon I} X_i$ It induces an SWCF $\tilde{C}$ on $\tilde{X}$, and $\tilde{C}$ can be regarded as the "true" SWCF.

Consider any set of individuals $J \subsetneq I$. Suppose that the sublist of options for individuals outside J is fixed; let it be $\langle x_i^{\circ} \rangle_{i \varepsilon I-J}$. The choice function C on $\Pi_{i \varepsilon I} X_i$ induces a choice function

$C_J(\langle x_i^{\circ} \rangle_{i \varepsilon I-J})$ on the underlying set $\{\langle x_i^{\circ} \rangle_{i \varepsilon I-J}\} \times \Pi_{i \varepsilon J} X_i$.

$C_J(\langle x_i^{\circ} \rangle_{i \varepsilon I-J})$ could be regarded as a projection of the choice function C. To be completely explicit, for each $A \subsetneq \{\langle x_i^{\circ} \rangle_{i \varepsilon I-J}\} \times \Pi_{i \varepsilon J} X_i$,

$C_J(\langle x_i^{\,o}\rangle_{i\epsilon I-J})\ (A) = C(A),$

naturally.

When $\langle x_i\rangle_{i\epsilon I-J}$ is fixed at $\langle x_i^{\,o}\rangle_{i\epsilon I-J}$, this means that the options for individuals outside J are fixed, and so there is no opportunity for making choices which advance their interests. Consequently, $C_J(\cdot)$, which is a social welfare choice function, must reflect the interests of individuals in J, and of no others. Speaking loosely, we would say that $C_J(\cdot)$ reflects J's interests. Of course, as we saw with constitutions in section 2.5., C, and so $C_J(\cdot)$, could also reflect things other than the interests of individuals in I; however, we shall neglect this possibility.

Now *everything* which affects an individual i's interests - at least, everything which is affected by the choices we are contemplating - is completely determined by the option x_i. By definition, therefore, individual i's interests are affected only by the choice of option for him - i.e. by the choice of x_i. So the interests of individuals in the set J are completely unaffected by the list of options for individuals outside J. Therefore, the choice function $C_J(\langle x_i\rangle_{i\epsilon I-J})$ which reflects the interests of individuals in J, must be independent of $\langle x_i\rangle_{i\epsilon I-J}$. So there is a well-defined choice function C_J on $\Pi_{i\epsilon J}\, X_i$, such that for any $\langle x_i\ \rangle_{i\epsilon I-J}$ and any $A \subsetneq \Pi_{i\epsilon J}\, X_i$,

$$C(\{\langle x_i^{\,o}\rangle_{i\epsilon I-J}\} \times A) = \{\langle x_i^{\,o}\rangle_{i\epsilon I-J}\} \times C_J(A).$$

We say then that choices on the space $\Pi_{i\epsilon J}\, X_i$ are *separable* - or, alternatively, that J is *separable*(1). In particular, when $J = \{i\}$, there is an individual WCF $C_{\{i\}}$, or just C_i for short, which is defined on X_i - or, in effect, on the original underlying set X. C_i represents i's

(1) See Debreu (1960), Gorman (1968b).

interests, so here we have an alternative way of defining individual welfare, providing we make appropriate assumptions.

In combination with some further, less exceptionable assumptions, this separability property gives us the additivity property we are seeking. Let us spell out the extra assumptions. First, suppose that the hypothetical SWCF on the underlying set $\Pi_{i \in I} X_i$ corresponds to a choice-indicator function $W : \Pi_{i \in I} X_i \to \mathcal{R}$. Here, W is an extension of a BSWF from the original underlying set X to the hypothetical underlying set. Suppose also that the function W is continuous, and that each of the sets X_i is topologically connected (for a suitable topology on the original underlying set X, of which X_i is a copy). Suppose too that there are at least three individuals i for whom there exists an option $x_i \in X_i$ and a set $A_i \subsetneq X_i$ such that $x_i \notin C_i(A_i)$. Finally, suppose that I contains a finite number of individuals.(1)

These extra assumptions, together with the separability property, imply that there is a continuous utility function v_i for each individual i, such that v_i represents C_i, individual i's WCF, and such that

$$V(\langle x_i \rangle_{i \in I}) \equiv \Sigma_{i \in I}\, v_i(x_i)$$

is a BSWF on the hypothetical underlying set $\Pi_{i \in I} X_i$(2).

As is well known, the function V and the list of functions $\langle v_i \rangle_{i \in I}$, are uniquely determined up to a common positive affine transformation - i.e. one of the form:-

---

(1) Infinite sets of individuals raise special problems. In fact, infinite sets of individuals do not arise very naturally unless there is a potentially infinite time-horizon, or else uncertainty. Infinite time-horizons are taken up in chapters 7 to 10.

(2) For proofs see Debreu (1960), Gorman (1968a) (for a calculus proof which is much simpler but rests on stronger assumptions), Gorman (1968b), Fishburn (1969) and Koopmans (1972a).

$$V' = \alpha V + \beta$$

$$v_i' = \alpha v_i + \beta_i \quad \text{(each } i \in I\text{), where } \alpha > 0.$$

The important result is that even the original SWCF on the underlying set X corresponds to an additive BSWF. For, writing $W(x)$ for $V(\langle x_i \rangle_{i \in I})$ when $x_i = x$ (each $i \in I$), we get:-

$$W(x) = \Sigma_{i \in I}\, v_i(x).$$

which is the customary additive form.

A similar form of BSWF has been propounded by Fleming, Harsanyi and Vickrey[1]. But their arguments were based on different grounds. Here, we have effectively only assumed that welfare choices on behalf of any group depend only on the interests of individuals within that group. It seems that those who object to additivity want to bring in more than individuals' interests. Of course, this is neither improper, nor surprising; for example, Rawls is more concerned with interests in the "original position" than with actual interests[2].

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(1) See Fleming (1952), Harsanyi (1953), (1955), Vickrey (1960) and Pattanaik (1968), (1971)(ch. 9).

(2) See Rawls (1970).

3.7. Conclusions.

It is certainly somewhat restrictive to assume that only individual interests matter in considering what is an appropriate social choice. But this restrictive assumption is a common one, and appealing to many. In this chapter, we have seen how far this assumption, by itself, can take us.

In the first place, we can insist on unanimity principles, as defined in section 3.2. If an option x is in everybody's interests, given the feasible set A, then x is an acceptable social choice, given the assumption that only individual interests matter.

The second unanimity principle, in either its strong or weak form, giving necessary conditions for an option to be an acceptable social choice, is akin to the Pareto principle, in the corresponding form. Nevertheless, the two are not identical except in the special case when each individual's welfare choice function is ordinal. In other cases, the second unanimity principle is more stringent than the Pareto principle, in the corresponding form.

In section 3.6., we saw the consequences of assuming also that each individual has a welfare choice function corresponding to a continuous utility function, and that the social welfare choice function corresponds to a continuous Bergson social welfare function. Suppose too that, given any group of individuals J, the choices to be made on J's behalf depend only on the interests of J's members. Finally, suppose that X is - or can be made into - a Cartesian product space $X = \Pi_{i \epsilon I} X_i$, where X_i is the space whose members are the objects of individual i's interests. Then, with one or two extra technical assumptions, the

Bergson social welfare function takes the additive form:-

$$W(x) = \Sigma_{i \epsilon I} u_i(x).$$

Amongst other things, what this result shows is that those who dispute the suitability of an additive welfare function must (virtually) be bringing in considerations other than individual interests - e.g. ethical views about equality, or special consideration for the interests of the under-privileged.

This concludes the discussion of purely static welfare. From now on dynamic problems will be considered.

Chapter 4.

DYNAMIC CHOICE

4.1. Introduction.

Most economic choices have repercussions which last for a considerable time. This is true, in particular, of any investment decision, or of any decision affecting investment.

The choice functions studied in chapters 2 and 3 took no explicit account of time. Nevertheless, it is easy to make some allowance for time within that kind of choice framework. The options consist of policy sequences, giving a detailed listing of the decisions to be made at each moment of time. These policy sequences are like the intertemporal commodity bundles which have arisen in intertemporal economic theory, and which are associated with the names of Irving Fisher, Hicks, Arrow and Debreu(1). Following Vickrey, we may call this approach to choice over time the *metastatic* approach(2).

While the metastatic approach has been an extremely useful tool in intertemporal economics, its power makes it easy to overlook a fundamental problem concerning choice over time. The agent making choices has time to change his mind, and to revise the choices he

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(1) Fisher (1930) realized that incomes in different periods could usefully be regarded as separate commodities - see also de Montbrial (1971) ch. 10. Hicks (1946), ch. 15 suggests regarding commodities at different dates as different commodities, so that:-

> "The problem of maximizing the present value of the production plan is formally identical with the problem of maximizing the surplus of receipts over costs in the static problem of the firm." (pp. 196-197).

Thereafter Arrow (1953) and Debreu (1959) made clear the extension of general equilibrium theory to intertemporal economics.

(2) See Vickrey (1964).

originally thought he was going to make[1]. He may choose the option x, which is a sequence of decisions $(x_1, x_2, \ldots)$. But, a little while later, he may choose another option y which is a different sequence of decisions $(y_1, y_2, \ldots)$. Then, what is his real choice?

The problem of changing choice is important to welfare economics. For economists use welfare theory to consider choice over time. Yet an individual's interests, social welfare, etc., are each identified with a *single* choice function. This is no less true of interests over time, and of welfare over time.

It might be thought that, as a normative theory of choice, welfare economics could get round the problem of changing choice very simply by not allowing it - by postulating that once the (implicit) agent has made his choice, he must stick to it. And, ultimately, this view must be correct. But it merely assumes away the real problem.

Suppose that the individuals in an economy seem to make reasonably sensible choices, except that they are prone to change their minds without any obvious reason. Suppose too that, as welfare economists, we would be prepared to accept these individuals' choices as representing their interests, if it were not for these changes of mind. How are we to proceed? Surely we cannot neglect these individuals' choices altogether, and yet it is not immediately clear how to reconcile their

---

(1) This aspect of choice seems to have been recognized by Strotz first among economists. See Strotz (1956). Some earlier work by Allais recognized that an individual might regret having made certain choices in the past, but did not suggest that an individual might actually revise his choice. See Allais (1947), and, for a more detailed discussion of his work, appendix 2.

different choices at different moments of time. So we are forced to consider changing choices.

Even if the individuals in the economy never change their minds, we may still be forced to consider changing choices. Suppose that we combine the interests of the individuals alive at a given date into a social welfare choice function, via some constitution. Then, at a later date, we do the same again. Because some individuals have died and others have been born since the constitution was last applied, it is unlikely that the new SWCF will be the same as the old. Of course, an obvious way out of this is to apply a constitution at any given moment of time, to the interests of *all* individuals, including the dead and the unborn[1]. But, particularly when choices involve decisions about population, this may not be possible.

To discuss changing choices, we need another theoretical framework. This will be provided by the dynamic choice functions which are the subject of this chapter. Their relationship to "metastatic" choice functions will be explored. In particular, we shall seek a method of reducing changing choices to a single coherent choice function which can be taken as a welfare choice function. Chapters 5 and 6 explore constitutions involving dynamic choice functions, and the application of this kind of constitution to a number of dynamic problems in intertemporal welfare economics.

Before commencing our analysis of dynamic choice, we shall consider a simple example to illustrate some aspects of the problem.

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(1) This way out is not so obvious that it has always been taken. See Inagaki (1970), (1973).

4.2. The Procrastinator Example

A procrastinor wishes to perform a certain action α. Every day, however, he plans to perform α the following day. Consequently α never gets done.

Formally, the procrastinator's options are $x_t (t = 1, 2, \ldots)$, where x_t denotes the performance of α on day t. Included in his options is the option x_∞, which amounts to never performing α.

On day s, his feasible options are:-

either $A_s = \{x_s, x_{s+1}, \ldots, x_\infty\}$

(if he has not already performed α)

or $\{x_r\}$ (if he has already performed α on day r, where $r \leq s$).

The only interesting case is when he has not already performed α by day s, and then his feasible set is A_s. Let C_s denote his choice function on day s. Thus C_s indicates on which days he is willing, on day s, to perform α. For the procrastinator, $C_s(A_s) = \{x_{s+1}\}$. The outcome of his choosing x_{s+1} on day s, for each s, is x_∞ - the action of α never gets performed. This is an outcome which the procrastinator *never* chose, and, in saying on day s that he "chooses" x_{s+1}, he is being naive(1).

It is worth representing the procrastinator's choices diagrammatically, by a "decision tree"(2). The "branches" are the options x_t $(t = 0, 1, 2, \ldots\infty)$. At each "node" n_t, the procrastinator can either choose x_t, or proceed to the node n_{t+1}. Observe that, "growing out" of each node n_t is a "subtree"

(1) This apt term is due to Pollak (1968), p. 202.

(2) The tree analogy was recognized by von Neumann and Morgenstern (1953) p. 66. "Decision tree" is a term used by Raiffa and Schlaifer (1961), p.7.

of the original tree. Moreover, this is also true at each x_t, although there the subtree is trivial. Nodes such as x_t, where no further decisions are possible, will be called "degenerate nodes".

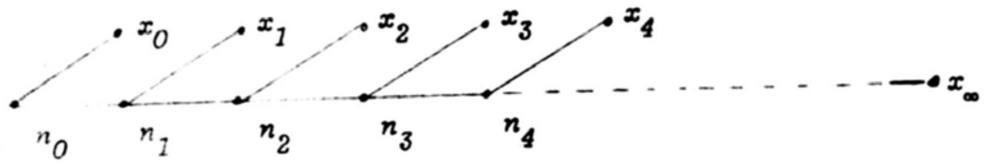
Suppose that the procrastinator is at a given node n_t. Then the set of branches which he can still choose is:-

$$X(n_t) = \{x_s \mid s \geq t\} \cup \{x_\infty\}$$

The set of nodes immediately succeeding n_t is:-

$$Su(n_t) = \{x_t, n_{t+1}\}$$

because x_t is a node, albeit a degenerate one.



The procrastinator provides a particularly clear example of changing choice. Decision trees, such as the procrastinator's, are the basis of general dynamic choice functions, as they will now be defined.

4.3. Dynamic Choice Functions.

In chapter 2, a choice function was defined, on an underlying set X, as a mapping $C: \mathcal{P}(X) \to \mathcal{P}(X)$. For each subset A of X, $C(A)$ is the set of options which the agent is willing to choose from A.

Changing choices, such as those of the procrastinator, obviously have to be described by more than one choice function. But the underlying sets of these choice functions are linked. In fact, they are all subsets of a "decision tree", which will now be defined.

A *decision tree* is a set A of branches. Each branch x of the tree A consists of a sequence of nodes of the tree. Let $x(t)$ denote the node reached at the end of period t on branch x. Then $x = (x(0), x(1), x(2), \ldots)$. The sequence of nodes which determines a branch may be finite or countably infinite.

Let n_0 denote the initial node of the tree. Then every branch must start from n_0. So $x \in A$ only if $x(0) = n_0$.

Let the set of nodes in the decision tree be written as $N(A)$. That is, $N(A) = \{n \mid \exists x \in A, \exists t \,.\, x(t) = n\}$

Given any node $n \in N(A)$, there is a set $A(n)$ of branches which are still possible after node n has been reached. So:-

$$A(n) = \{x \in A \mid \exists t \,.\, x(t) = n\}$$

Say that a node n *precedes* another node n', written as $n \; Pr \; n'$, if there exists a branch x of the tree A such that $x(t) = n$, $x(t') = n'$, where $t < t'$.

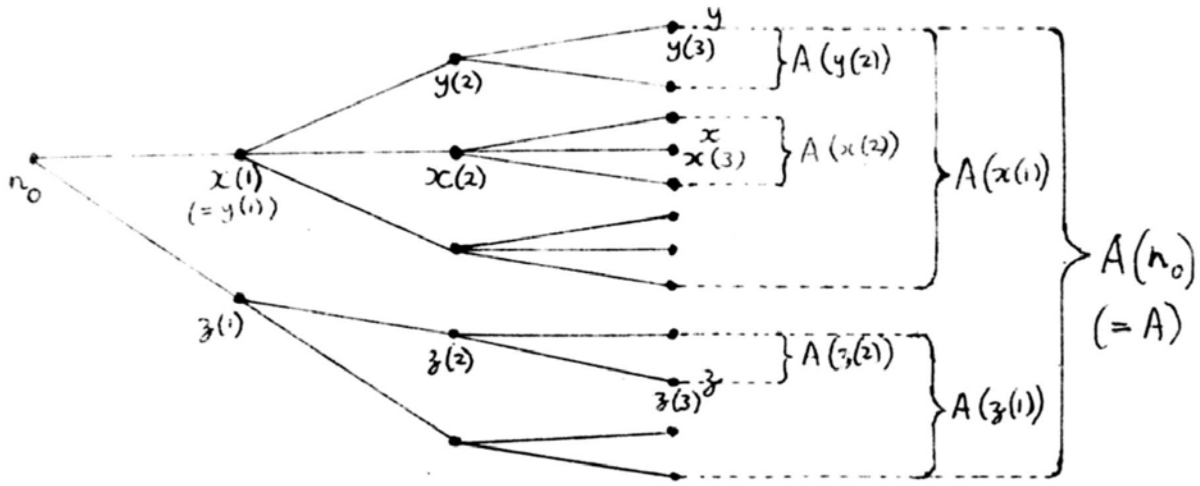
The sets $A(n)$ $(n \in N(A))$, must satisfy the following properties, which have natural interpretations when it is recalled what each $A(n)$ represents:-

(1) $A(n_0) = A$

(2) If $x \in A$, then $x \in A(x(t))$ $(t = 0, 1, 2, \ldots)$

(3) If $n \; Pr \; n'$, then $A(n') \subsetneq A(n)$.

These conditions imply that the set A has a tree structure which can be represented in a diagram as follows:-



The set of nodes within the tree A will be written as $N(A)$. The *tree structure* of A is given by the set of correspondences $A: N(A) \to \mathcal{P}(A)$ where $A(n)$ is the set of options which are still possible after n has been reached.

For any node $n \in N(A)$, say that n' is a *successor* of n, and write $n' \in Su(n)$, if there is a branch $x \in A$ such that $x(t) = n$, $x(t+1) = n'$ (some t)[(1)(2)].

---

(1) Strictly speaking, $n' \in Su(n)$ iff n' is an *immediate* successor of n.

(2) Notice that the decision tree A is, in effect, the "tree-graph" $(N(A), Su)$, in the sense of Berge (1963), p. 27.

Suppose the agent facing the decision tree X is at node n. Then, given any feasible set $A \subseteq X(n)$, there is, presumably a set of options $C(n)(A)$ which he is willing to choose. As A varies over subsets of $X(n)$, we get a choice function $C(n)$ defined on the underlying set X. This suggests the following definition:-

A *dynamic choice function* is a set $\{C(n) | n \in N(X)\}$ of choice functions $C(n)$, defined on the corresponding underlying sets $X(n)$, where X is a decision tree for which the set of nodes is $N(X)$, and the tree structure is given by $X(n)$ $(n \in N(X))$. X is called the *underlying tree* of the dynamic choice function. For brevity, a dynamic choice function will be written simply as $\{C(n)\}$. The choice functions $C(n)$ $(n \in N(X))$ will be called the *components* of the dynamic choice function.

In the typical dynamic choice problem, the feasible set is a subset A of X. But any subset of X inherits a tree structure from X. For if $A \subseteq X$, and $n \in N(A)$, the set of options which are still feasible after reaching the node n is $A(n) = A \cap X(n)$. It is clear too that A itself is a decision tree; it will be called a *subtree* of the underlying tree X. Notice that a subtree can be obtained from the underlying tree simply by cutting off some of the branches, just as a subset can be derived from the underlying set simply by deleting some of the options.

Dynamic choice functions provide a framework in which one can discuss changing choices and their relationship to welfare. As stated in section 4.1., we shall seek a method of reducing changing choices - or, rather, a dynamic choice function - to a single coherent choice function which can be taken as a welfare choice function. But reduction to a single coherent choice function, on the underlying set X, is not quite enough. The result of the reduction process must be another dynamic choice function, otherwise there is a serious danger that the dynamic features of the choice situation will still not be properly taken into account. But it must be a dynamic choice function of which each component is equivalent to the same static choice function . Such a dynamic choice function will be described as "consistent". A precise definition of consistency, and some implications of this definition, follow in the next section. Thereafter, various ways of achieving consistency are explored, and examined for their ability to generate coherent choices.

4.4. Consistent Dynamic Choice.

As suggested above, a dynamic choice function will be defined as consistent if each component is equivalent to the same static choice function. But this equivalence needs to be very carefully specified. The obvious definition is that there should be a choice function $\bar{C}$ on the underlying set X, corresponding to the underlying tree X of the dynamic choice function $\{C(n)\}$, such that:-

$$C(n)(A) = \bar{C}(A) \quad (\text{each } n \in N(X),\ A \subseteq X(n))$$

But this definition is inadequate, as the following example shows:-

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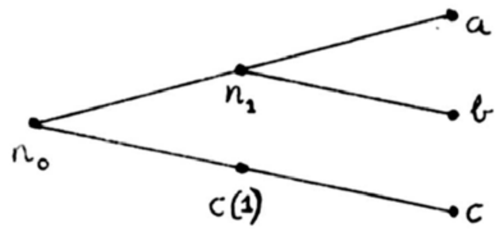
Example 4.4.1.



The underlying tree X is as shown in the diagram. n_1 is used to denote the node $a(1) = b(1)$.

Suppose that $C(n)(A) = \bar{C}(A)$ (each $n \in N(X)$, $A \subseteq X(n)$)

and that $\bar{C}(\{a,b,c\}) = \{a\}$

$\bar{C}(\{a,b\}) = \{b\}$

Then, with $B = \{a,b,c\}$:-

$a \in C(n_0)(B)$, but $a \notin C(n_1)(B(n_1))$

and $b \in C(n_1)(B(n_1))$, but $b \notin C(n_0)(B)$.

So a is chosen at time 0, but not at time 1; b is chosen at time 1, but not at time 0. This is hardly consistent, nor have we really succeeded in making all dynamic choices equivalent to a static choice function. Is it a or b which is "chosen" from B?

Suppose a is unambiguously chosen. Then we need not only

$a \in C(n_0)(B)$ but also $a \in C(n_1)(B(n_1))$.

Suppose b is unambiguously chosen. Then we need not only

$b \in C(n_1)(B(n_1))$ but also $b \in C(n_0)(B)$.

This example helps to suggest the following definition. Suppose that A is the feasible set, and that $x \in C(n)\ (A(n))$. Let n' be any node such that $n\ Pr\ n'$. Then dynamic consistency demands two things.

First, if $x \in A(n')$, so that x is still feasible at the later node, then x must still be in the choice set at n' - i.e. $x \in C(n')(A(n'))$.

Second, suppose that $x \in A(n')$, and that $y \in C(n')(A(n'))$ - i.e. y is a possible choice at n'. For this to be consistent with the earlier choice at n, y as well as x must be a possible choice at n - i.e. $y \in C(n)(A(n))$.

Notice that the second argument only applies if $X(n') \cap C(n)(A(n))$ is non-empty. Otherwise, while the feasible set remains A, the agent never puts himself in the position of having to choose from $A(n')$, and so no choice from $A(n')$ can be inconsistent with his choice from $A(n)$.

To summarize, the dynamic choice function $\{C(n)\}$ on the underlying tree X is *consistent* if, whenever $n\ Pr\ n'$ and $x \in C(n)(A(n)) \cap A(n')$:-

(1) $x \in C(n')(A(n'))$

(2) If $y \in C(n')(A(n'))$, then $y \in C(n)(A(n))$.

An equivalent, but less direct, statement of these conditions for consistency is the following:-

Whenever $n\ Pr\ n'$ and $x \in C(n)(A(n)) \cap A(n')$,

$$C(n')(A(n')) = A(n') \cap C(n)(A(n)).$$

The proof that this is an equivalent statement is straightforward. Another equivalent statement, which is easier to verify for particular cases, is the following:-

Whenever $n' \in Su(n)$ and $x \in C(n)(A(n)) \cap A(n')$,

$$C(n')(A(n')) = A(n') \cap C(n)(A(n)).$$

This is clearly implied by the previous statement. That the converse is true can easily be proved by induction on the number of steps needed to move from n to n'.

The following theorem characterizes consistent dynamic choice more fully and also shows more clearly what was wrong with the first attempt at a definition of consistency.

<u>Theorem 4.4.2</u>

(a) Given any consistent dynamic choice function $\{C(n)\}$ on the underlying tree X, there exists a choice function $\bar{C}$ on X such that:-

(i) For each $n \in N(X)$, $A \subseteq X(n)$,

either $C(n)(A) = \bar{C}(A)$, or $\bar{C}(A)$ is empty.

(ii) For each $n \in N(X)$, $A \subseteq X$,

if $x \in \bar{C}(A)$, and $x \in X(n)$,

then $\bar{C}(A(n)) = A(n) \cap \bar{C}(A)$.

(b) Conversely, given any choice function $\bar{C}$ on X satisfying (ii) above, there exists a consistent dynamic choice function $\{C(n)\}$, on the underlying tree X, defined by :-

$$C(n)(A) = \bar{C}(A) \quad (\text{each } A \subseteq X(n))$$

<u>Proof</u>

(a)(i) Define $\bar{C}(A) = C(n_0)(A)$, whenever $A \subseteq X$, where n_0 is the initial node of the tree A. Then $\bar{C}$ is a choice function.

Suppose $A \subseteq X(n)$, for some $n \in N(X)$. Then $A(n) = A = A(n_0)$. Suppose too that $x \in \bar{C}(A)$. Then $x \in C(n_0)(A)$, so that, by dynamic consistency, since $x \in A(n) \cap C(n_0)(A)$:-

$$C(n)(A(n)) = A(n) \cap C(n_0)(A(n_0))$$

i.e. $C(n)(A) = A \cap \bar{C}(A) = \bar{C}(A)$

(ii) If $x \in \bar{C}(A) = C(n_0)(A(n_0))$, and if $x \in X(n)$,

then $C(n)(A(n)) = A(n) \cap C(n_0)(A(n_0))$

and so $\bar{C}(A(n)) = A(n) \cap \bar{C}(A)$.

(b) Suppose that $\bar{C}$ satisfies (ii), and that $\{C(n)\}$ is defined by

$$C(n)(A) = \bar{C}(A) \quad (\text{each } A \subseteq X(n)).$$

Suppose that $x \in C(n)(A(n))$ and that $n \; Pr \; n'$.

Let B denote $A(n)$. Then $B(n) = B$, $B(n') = A(n')$.

$$\begin{aligned} \text{Now } C(n')(A(n')) &= C(n')(B(n')) \\ &= \bar{C}(B(n')) \\ &= B(n') \cap \bar{C}(B) \\ &= A(n') \cap \bar{C}(n)(A(n)) \end{aligned}$$

as required for dynamic consistency.

The theorem shows that the first attempt to define consistent dynamic choice failed because it overlooked the need for property (ii). Property (ii) is a kind of "rationality" property. Its meaning can be illustrated by lemma A.5.1. of appendix 1, which is restated here for convenience.

Lemma 4.4.3.

The following conditions are equivalent:-

(1) $B \subsetneq A$ and $x \in B \cap C(A)$ together imply that $C(B) = B \cap C(A)$.

(2) C is an ordinal choice function on the family of sets S for which $C(S)$ is non-empty.

Its relationship to property (ii) can be seen by recalling that, if $x \in \bar{C}(A)$ and $x \in X(n)$, then $x \in A$, and so $x \in A(n)$ - i.e. $x \in A(n) \cap \bar{C}(A)$. So, if (1) is satisfied, then $C(A(n)) = A(n) \cap \bar{C}(A)$, as required. In particular, then, if $\bar{C}$ is any ordinal choice function on X, the dynamic choice function $\{C(n)\}$ on X, defined by:-

$$C(n)(A) = \bar{C}(A) \quad \text{(whenever } A \subsetneq X(n)\text{)}$$

is consistent[1].

Now that we know what consistent dynamic choice means, we can proceed to explore ways of removing inconsistency in a given dynamic choice function. Before getting too involved in theoretical detail, however, some of the ways of removing inconsistency will be illustrated in a simple, but fundamental, example.

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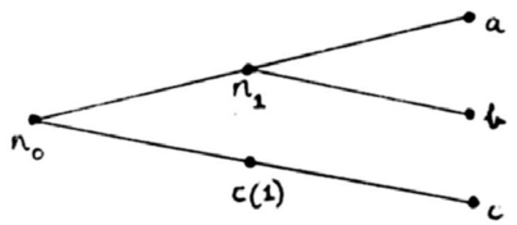
[1] It has been my experience that some people find this result counter-intuitive, and a contradiction of Strotz (1956), who proved that the utility functions he was considering had to have a very special form if they were to correspond to consistent dynamic choice. But this contradiction is only apparent. Given two nodes such that $n' \in Su(n)$, Strotz insisted on a very special relationship between $C(n)$ and $C(n')$ - namely that $C(n')$ was effectively $C(n)$ shifted one period forward in time - a stationarity property. To make this meaningful, there was a similar relationship between $X(n)$ and $X(n')$. It was these stationarity properties which, together with dynamic consistency, gave him such strong results.

4.5. The Potential Addict Example

Consider an individual who is contemplating a mode of behaviour which is potentially habit-forming. More specifically and clearly, suppose the individual is wondering whether or not to start taking an addictive drug. We may presume that the drug gives rise to pleasant sensations, at least initially. The individual would most prefer to take the drug infrequently, or for a short time, so that he enjoys the drug without damaging his health. It is assumed, however, that if the individual starts to take the drug, then he is certain to become an addict, and so to take the drug very much more than he had originally intended, with serious consequences. On the other hand, he can refuse to take the drug at all.

The potential addict has a dynamic choice problem which, given the simplifying assumptions, has the following underlying tree:-



where n_1 is used to denote the node $a(1) = b(1)$, and where the options a,b,c can be described as follows:-

a - take the drug until it is about to impair health, then stop

b - become an addict

c - refuse the drug altogether.

The most plausible dynamic choice function for the potential addict is as follows:-

$C(n_0)$ corresponds to a preference relation R_0 such that:-

$a\ P_0\ b$, $\quad a\ P_0\ c$, $\quad c\ P_0\ b$

$C(n_1)$ corresponds to a preference relation R_1 such that:-

$b\ P_1\ a$.

Evidently, $\{C(n)\}$ is an inconsistent dynamic choice function. Four possible ways of removing this inconsistency are the following:-

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(1) First, the agent can be "naive" or "myopic"[(1)]. At time 0, and node n_0, he "chooses" a. At time 1, and node n_1, the feasible set is $X(n_1) = \{a,b\}$ and his choice is b. The outcome is b, and this is his "choice" from $\{a,b,c\}$.

This procedure can be used to find the naive dynamic choice function $\{C^{\circ}(n)\}$ corresponding to $\{C(n)\}$. Here:-

$$C^{\circ}(n_0)(\{a,b,c\}) = \{b\}, \quad C^{\circ}(n_0)(\{a,b\}) = \{b\},$$

$$C^{\circ}(n_0)(\{a,c\}) = \{a\}, \quad C^{\circ}(n_0)(\{b,c\}) = \{c\}$$

$$C^{\circ}(n_1)(\{a,b\}) = \{b\}$$

as is easily checked. Moreover, $\{C^{\circ}(n)\}$ *is* dynamically consistent. However, the corresponding static choice function $C^{\circ} = C^{\circ}(n_0)$ is obviously incoherent.

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(2) Second, the agent can be "sophisticated".[(2)] He foresees that, if he ever takes the drug, he will then become an addict. So, at time 0 and node n_0, he recognizes that the only two options he can attain, given

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(1) See Strotz (1956) and Pollak (1968).

(2) See Pollak (1968), p. 203. Strotz (1956), p. 173, calls this "the strategy of consistent planning". Since there is more than one way of removing inconsistency, Strotz's term seems to make an excessive claim for one of these ways.

his later choices, are b and c - addiction and refusing the drug altogether. He chooses from the set $\{b,c\}$, and his choice is c - not taking the drug.

This procedure can be used to find the sophisticated dynamic choice function $\{C^*(n)\}$ corresponding to $\{C(n)\}$. Here:-

$$C^*(n_0)(\{a,b,c\}) = \{c\}, \qquad C^*(n_0)(\{a,b\}) = \{b\},$$

$$C^*(n_0)(\{a,c\}) = \{a\}, \qquad C^*(n_0)(\{b,c\}) = \{c\}$$

$$C^*(n_1)(\{a,b\}) = \{b\}.$$

as is easily checked. Moreover, $\{C^*(n)\}$ is *dynamically* consistent. However, the corresponding static choice function $C^* = C^*(n_0)$ is obviously incoherent.

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(3) The third possibility is one which I shall call "intertemporal liberalism". In the first place, the potential addict notices that $b \; P_1 \; a$. Then, he adjusts the choice function $\bar{C} = C(n_0)$ to take account of this, so that, in the end, he has a new dynamic choice function which is consistent. There are, of course, several ways of making the adjustment. One restriction which must be satisfied is that $\bar{C}(\{a,b\}) = \{b\}$. But, as is shown by example 4.4.1. (which is a partially specified version of the potential addict's problem) this does not, by itself, achieve consistency. For consistency, $\bar{C}$ must also satisfy property (ii) of theorem 4.4.2. As lemma 4.4.3. shows, it is sufficient for $\bar{C}$ to be ordinal.

Suppose that $\bar{C}$ corresponds to the preference relation $\bar{R}$, which is transitive. Then the following are the possible adjustments:-

(i) $b \bar{P} a$, $a \bar{P} c$, $b \bar{P} c$

(ii) $b \bar{P} a$, $c \bar{P} a$ $c \bar{P} b$

(iii) $b \bar{P} a$, $c \bar{P} a$, $b \bar{P} c$

(iv) $b \bar{P} a$, $c \bar{P} a$, $b \bar{I} c$

(v) $b \bar{P} a$, $a \bar{I} c$, $b \bar{P} c$.

So the preference relation P_O does indeed have to be altered, and not only by replacing $a P_O b$ by $b \bar{P} a$; consequential changes are also needed.

The special case in which $\bar{C}$ is ordinal will be considered for more general dynamic choice problems later on. So will the use of the term "intertemporal liberalism". But, even now, it is possible to notice some affinity to Sen's use of the term "liberalism".(1) Roughly speaking, "liberalism" involves recognizing that some individual preferences should be decisive, when a social preference relation is being constructed. "Intertemporal liberalism" means that certain preferences in the future should be decisive, when a consistent dynamic choice function is being constructed.

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(4) A fourth suggestion is that the agent should "precommit" himself(2). This means that, somehow, he forces himself to follow the path he chooses initially. But the way this is to be done is left unclear. It seems to involve new options. For example, Odysseus was able to listen to the Sirens without harm because of the careful precautions he took(3). But, if these precautions had not been feasible - if, for example, there had been no rope or twine to bind him to the mast - he could not have

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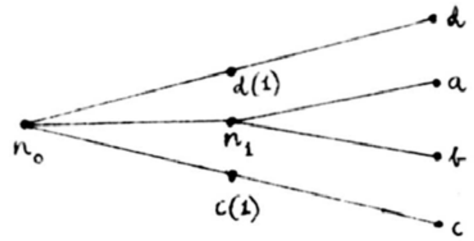
(1) See Sen (1970b), and (1970a), ch. 6.

(2) See Strotz (1956), p. 173, Pollak (1968), p. 202. Blackorby et al. (1973) speak of "preference inheritance".

(3) This striking example is suggested by the quotation at the head of Strotz (1956).

successfully precommitted himself. Again, how is the potential addict to precommit himself, so that he avoids addiction? There must be a new option, d, which involves him taking the drug but being forced to give it up before it harms him. With this new option, the underlying tree becomes:-



d has the same consequences, in terms of enjoyment and continuing health, as does a. But if the agent chooses d, rather than a, he avoids addiction.

It seems that, although precommitment may be an important phenomenon in dynamic choice, it is nothing more than a certain kind of sophistication. For, when *all* the options are taken into account, there is no way of precommitting oneself which is not already specified within the set of options. Accordingly, precommitment cannot be regarded as a way of making dynamic choice consistent after all.

The rest of this chapter is a discussion of these different ways of making dynamic choice consistent. But first some rather drastic assumptions will be made. These have the effect of greatly simplifying the analysis. The seriousness of the assumptions will be discussed later, in section 4.9.

4.6. Some Assumptions and Notation.

To facilitate later analysis, I shall now introduce some simplifying assumptions concerning the underlying tree X, and the dynamic choice function $\{C(n)\}$ which is to be made consistent. The assumptions are as follows:-

(A1) Each branch x of the underlying tree X has at most T nondegenerate nodes. That is, if $x = (x(0), x(1), x(2), \dots)$ is a branch of X, then, for all $t \geq T$, $X(x(t)) = \{x\}$.

This means that on no branch of the underlying tree is there any scope for changing the choice of option after period T has been reached. Such a tree will be called *bounded*, and T will be called a *bound*. T is a bound on the length rather than the number of branches, of course. But notice that a tree with a finite number of branches must be bounded.

(A2) For each node $n \in N(X)$, $C(n)$ is a coherent choice function on the underlying set $X(n)$.

This assumption is straightforward.

(A3) For each node $n \in N(X)$, and each $A \subseteq X(n)$, $C(n)(A)$ consists of precisely one option.

This is certainly a strong assumption, particularly in combination with (A2). Indeed, (A2) and (A3) together imply that each choice function $C(n)$ is ordinal, and corresponds to a *strong* ordering, as is shown in appendix 1, lemma A.5.2. Let $P(n)$ be the strict preference relation corresponding to $C(n)$.

Define a *dynamic strict preference relation* as a set $\{P(n) | n \in N(X)\}$ of strict preference relations $P(n)$, where each $P(n)$ is defined on $X(n)$, for the tree X. Say that $\{P(n)\}$ is *consistent* if, whenever $n \; Pr \; n'$ and $x, y \in X(n')$:-

$$x \; P(n') \; y \quad \text{iff} \quad x \; P(n) \; y$$

In the case when $\{C(n)\}$ is a dynamic choice function, and each component is ordinal, and, for each $A \subseteq X(n)$, $C(n)(A)$ consists of a singleton, we know that, by theorem A.9.3., $\{C(n)\}$ is consistent if and only if $\{P(n)\}$ is consistent.

Let us introduce some further notation. Let X be a decision tree, with structure $\{X(n)\}$. Let $\{C(n)\}$ be a dynamic choice function on X.

Given any node $n \in N(X)$ and any branch $x \in X$, write $n_1(n,x)$ for the node immediately following n on the branch x. That is, if $x(t) = n$, then $x(t+1) = n_1(n,x)$. Also $n' \in Su(n)$ and $x \in X(n')$ if and only if $n' = n_1(n,x)$. And, of course, $Su(n) = \{n' | \exists x \in X(n) \; . \; n' = n_1(n,x)\}$

Given any node $n \in N(X)$, and any $A \subseteq X(n)$, let A_1 or $A_1(n)$ denote the set of nodes immediately succeeding n which can be reached along some branch of A. That is:-

$$A_1(n) = \{n' | \exists x \in A(n) \; . \; n' = n_1(n,x)\}$$

Of course, $X_1(n) = Su(n)$ in this notation.

Also, given any component $C(n)$ of the dynamic choice function, and any $A \subseteq X(n)$, let $C_1(n)(A)$ denote the set of nodes, immediately succeeding n, to which the agent is willing to move along a chosen branch. That is:-

$$C_1(n)(A) = \{n' \in A_1(n) | \exists x \in C(n)(A) \; . \; n' = n_1(n,x)\}$$

Now $C_1(n)$ is not a choice function, because it maps sets of whole branches into sets of single nodes. Nevertheless, it serves to define a choice function $C_1'(n)$ on $X(n)$ as follows:-

$$C'_1(n)(A) = \bigcup\{A(n') \mid n' \in C_1(n)(A)\}$$

So $x \in C'_1(n)(A)$ if and only if, for some $n' \in C_1(n)(A)$, $x \in A(n')$. In other words, x is "chosen" if and only if, after one of the first moves from n which the agent is willing to choose, x is still feasible. Obviously,

$$C(n)(A) \subseteq C'_1(n)(A) \subseteq A \quad (\text{each } A \subseteq X(n))$$

and so $C'_1(n)$ is certainly a choice function.

Nevertheless, it is $C_1(n)$ itself which is more useful in the following analysis.

4.7. Naive Choice

The first method of arriving at consistent dynamic choices, starting from an inconsistent dynamic choice function, is myopic or naive choice. In this section, naive choice will be defined, and it will be shown that it does lead to consistent dynamic choices, although not necessarily to the ones which the agent thinks he is going to make. In addition, it is shown that naivety makes no difference if the dynamic choice function is consistent to start with.

Consider the naive agent at node n of the underlying tree X, facing the feasible set $A \subsetneq X(n)$. The choice set which he believes he has at node n is $C(n)(A)$. Consequently, n' is a node to which the agent might move one period later if and only if, for some branch x of the tree X, and for some time t:-

$$x(t) = n, \quad x(t+1) = n', \text{ and } x \in C(n)(A).$$

i.e. $x(t) = n$, and $x(t+1) \in C_1(n)(A)$.

(using the notation introduced in section 4.6).

And $n'' = x(t+2)$ is a node to which the agent might move two periods later if and only if, in addition:-

$$x(t+2) \in C_1(n')(A(n'))$$

By induction on s, $x(s)$ is a node to which the agent might move at the end of period s if and only if:-

$$x(r) \in C_1(x(r-1))\Big(A(x(r-1))\Big) \qquad (r = t+1,\ t+2,\ \ldots,\ s)$$

Let $C^o(n)(A)$ denote the set of branches which the agent may eventually follow, given his later choices each period. Then what has been shown is that:-

$$x \in C^o(n)(A) \text{ iff, for some time } t,\ x(t) = n, \text{ and:-}$$

$$x(s+1) \in C_1(x(s))\Big(A(x(s))\Big) \quad (s = t,\ t+1,\ t+2,\ \ldots)$$

Notice that $x \in C^O(n)(A)$ iff, when $n' = n_1(n,x)$:-

(i) $n' \in C_f(n)(A)$

(ii) $x \in C^O(n')(A(n'))$.

Now, it seems fairly obvious that the following results will be true:-

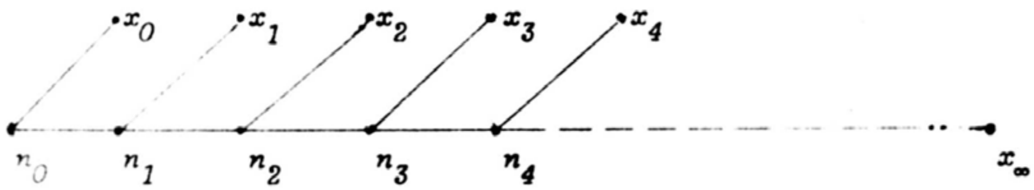
(1) $\{C^O(n)\}$ is a consistent dynamic choice function on the underlying set X.

(2) If $\{C(n)\}$ is consistent, then $\{C^O(n)\}$ is identical to $\{C(n)\}$.

Indeed, these results are valid provided that the underlying tree is bounded - that is, assumption (A1) of section 4.6. is satisfied. But without this assumption, problems can arise, as the following modification of the procrastinator example of section 4.2. shows:-

Example 4.7.1.

The underlying tree X is as in the procrastinator example:-



The agent's choice function at each node n_t is taken to be :-

$$C(n_t)(A) = A - \{x_\infty\} \qquad \text{(each } A \subseteq X(n_t)\text{)}.$$

That is, the agent is assumed not to care when he performs the act a, provided he performs it at some finite time.

It is easy to check that $\{C(n)\}$ is a consistent dynamic choice function on the underlying tree X, and that each component $C(n)$ is ordinal. Indeed, each $C(n)$ corresponds to a utility function $u(n)(\cdot)$ defined as follows:-

$$u(n)(x_t) = 1 \quad (t \text{ finite})$$

$$u(n)(x_\infty) = 0$$

Despite the apparent simplicity and normality of this example, naive choice has some surprising properties.

First, suppose $A = X - \{x_\infty\}$ - that is, performing α at any finite time is feasible, but putting it off forever is not.
Now $x_\infty(t) = n_t$, and so $x_\infty(t+1) \in C_1(x_\infty(t))\left(A(x_\infty(t))\right) \quad (t = 0,1,2,\dots)$.
So, according to the definition of naive choice, $x_\infty \in C^o(n_t)(A)$ (each t), even though $x_\infty \notin A$. That is, $C^o(n_t)(\cdot)$ violates one of the axioms of choice - that $C^o(n_t)(A) \subseteq A$ (each A). So $\{C^o(n)\}$ is not even a dynamic choice function, properly speaking.

Second, $x_\infty \in C^o(n_t)(A(n_t))$, but $x_\infty \notin C(n_t)(X(n_t))$ (each t). That is, naive choice is different from intended choice, even though the intended dynamic choice function $\{C(n)\}$ is consistent.

To surmount this troublesome example, it will be assumed from now on that the underlying tree is *bounded*. Then:-

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Theorem 4.7.2.

$\{C^o(n)\}$ is a consistent dynamic choice function on the underlying tree X.

Proof.

(1) Suppose that A is a finite subset of $X(n)$, for some $n \in N(X)$. Then, whenever n precedes n', $A(n') \subseteq A$, and so $A(n')$ is finite. It follows that $C(n')(A(n'))$ is nonempty, and so, that $C_1(n')(A(n'))$ is nonempty, whenever $n\ Pr\ n'$. Consequently, one can find a sequence of nodes $n_1, n_2, n_3, \dots$ such that $n_1 \in C_1(n)(A)$, and $n_{m+1} \in C_1(n_m)(A(n_m))(m = 1,2,\dots)$
Now, there is a branch $x \in X$ such that $x(t) = n$. Let y be the branch defined as follows:-

$$y(s) = \begin{cases} x(s) & (s = 0,1,\dots t) \\ n_{s-t} & (s = t+1, t+2, \dots) \end{cases} .$$

Then $y \in C^O(n)(A)$, and so this choice-set is nonempty.

(2) Let T be a bound on the length of the branches of the underlying tree X - i.e., suppose that, whenever $x \in X$ and $t \geq T$, then $X(x(t)) = \{x\}$.
Suppose that $x \in C^O(n)(A)$ (some $A \subseteq X(n)$, some $n \in N(X)$).
Then $x(s+1) \in C_1(x(s))\big(A(x(s))\big)$ $(s = t, t+1, \dots)$

$x(t) = n$.

In particular, whenever $s \geq T$,

$x(s+1) \in C_1(x(s))\big(A(x(s))\big)$.

But if $s \geq T$, then $A(x(s))$ is a singleton - $\{y\}$ say.
Now, by induction on s, $x(s) = y(s)$ (each $s \geq T$). Hence x and y must be the same branch. In particular, $x \in A$. This shows that $C^O(n)(A) \subseteq A$.

(3) From (1) and (2), it is clear that $\{C^O(n)\}$ is a dynamic choice function. Call it the *naive* dynamic choice function .

(4) It remains to be shown that $\{C^O(n)\}$ is consistent. Suppose that $n \in N(X)$, $n' \in Su(n)$, $x \in A(n') \cap C^O(n)(A(n))$, and that $x(t) = n$. Then $x(t+1) = n' = n_1(n,x)$.

(i) Now $x(s+1) \in C_1(x(s))\big(A(x(s))\big)$ $(s = t, t+1, \dots)$, and so

$x \in C^O(n')(A(n'))$, certainly.

(ii) Conversely, if $y \in C^O(n')(A(n'))$, then $y(t) = n$, $y(t+1) = n'$, and

$y(s+1) \in C_1(y(s))\big(A(y(s))\big)$. $(s = t+1, t+2, \dots)$.

Since $n' \in C_1(n)(A(n))$, it follows that

$y \in A(n') \cap C^O(n)(A(n))$, as required.

It is worth noting that assumption (A1) - that the tree is bounded, was only needed for part (2) of the proof.

Theorem 4.7.3.

If $\{C(n)\}$ is consistent, then $\{C^{o}(n)\}$ is identical to $\{C(n)\}$ - i.e. naive choice is identical to intended choice.

Proof

(1) The proof will proceed by backward induction. Suppose that T is a [illegible]nd on the length of each branch - i.e. that $X(x(t)) = \{x\}$ whenever [illegible] T. Then, whenever $t \geq T$, $n = x(t)$ and $A \subseteq X(n) = \{x\}$,

$$C^{o}(n)(A) = C(n)(A) = \{x\}.$$

Next, suppose that whenever $n' \in Su(n)$, then $C^{o}(n')$ is identical to $C^{o}(n)$. Let $A \subseteq X(n)$. It is enough to show that $C^{o}(n)(A) = C(n)(A)$.

(2) Suppose that $x \in C(n)(A)$. Let n' denote $n_1(n,x)$.

Then $x \in C(n')(A(n'))$, because $\{C(n)\}$ is consistent. So $x \in C^{o}(n')(A(n'))$, by the induction hypothesis. Since $n' \in C_1(n)(A)$, it follows that $x \in C^{o}(n)(A)$, as required.

(3) Suppose that $x \in C^{o}(n)(A)$. Let n' denote $n_1(n,x)$. Then $n' \in C_1(n)(A)$, and $x \in C^{o}(n')(A(n'))$. So, for some $y \in A(n')$, $y \in C(n)(A)$. Since $\{C(n)\}$ is consistent, it follows that

$C(n')(A(n') = A(n') \cap C(n)(A)$. But $C^{o}(n')(A(n')) = C(n')(A(n'))$ by the induction hypothesis, and so $x \in C(n)(A)$, as required.

Of course, if $\{C(n)\}$ is not consistent, then $\{C^{o}(n)\}$ cannot be the same as $\{C(n)\}$, because $\{C^{o}(n)\}$ is consistent.

The coherence of naive choice will be explored later, in section 4.9.

4.8. Sophisticated Choice.

The second method of removing inconsistency from dynamic choice is to make sophisticated choices. These will now be characterized, and results corresponding to those for naive choice will be proved. But now heavy reliance has to be placed on the assumptions of section 4.6.

A sophisticated agent is one who, in making his choice at a node n of the decision tree, takes into account those choices which he will actually make at nodes which follow n, and recognizes that he can only choose amongst those branches of the decision tree that he will actually follow through to their end.

Let A be the feasible set of options. Suppose that the agent finds himself at node n. Then his feasible set has become $A(n) = A \cap X(n)$, because of the structure of the underlying tree X. But the set of options which are truly available to the agent is a subset of $A(n)$, which depends on his later choices. For, if his later choices veto branch x, he cannot in fact attain x even though $x \in A(n)$.

Let $A^*(n)$ denote the set of branches or options which are attainable, given that the agent is at node n. If $A^*(n)$ were the feasible set, and if the agent faced a purely static choice situation, his choice set would be $C(n)(A^*(n))$ - i.e. he would choose according to the appropriate component of his dynamic choice function $\{C(n)\}$. In fact, the agent faces a dynamic choice situation, in which $A^*(n)$is the attainable set rather than the feasible set. But the sophisticated agent ignores these differences. He regards the choices which he makes later as constraints, just as though they were choices being made by other agents. In fact, he plays a game -

in the von Neumann - Morgenstern sense - with these other agents. This game is one of perfect information.[1] The agent chooses as best he can, given the reactions of these other, fictitious, agents to his choice. Consequently, at node n, the agent has the choice set:-

$$C^*(n)(A(n)) = C(n)(A^*(n)).$$

Under the assumptions of section 4.6., the attainable sets $A^*(n)$ and the sophisticated dynamic choice function $\{C^*(n)\}$ can be found by backward induction, as follows:-

First, if $A \subseteq X(n)$ and $A = \{x\}$, then obviously:-

$$A^*(n) = \{x\} \text{ , and } C^*(n)(A) = C(n)(A) = \{x\}.$$

Given the assumption that $X(x(t)) = \{x\}$ whenever $t \geq T$, this serves to define $\{C^*(n)\}$ whenever $n = x(t)$, where $t \geq T$.

More generally, for $x \in A^*(n)$ to be true, the agent must be willing to follow x from the node $n' = n_1(n,x)$ on, if he starts by going to n'. So it is necessary that $x \in C^*(n')(A(n'))$. But this condition is not, strictly speaking, sufficient. The reason is that whether x is attainable depends not merely on whether the agent is willing to choose x, but on whether the agent *will in fact* choose x. And, ultimately, the agent cannot choose more than one option.

It was precisely to overcome this problem that assumption (A3) of section 4.6. was made. For if $C(n)(A)$ is a singleton, whenever $A \subseteq X(n)$, the problem disappears. As will shortly be shown, there will then always be a unique option which the agent is willing to choose, and so it becomes safe to assume that he will choose that option.

---

(1) See von Neumann and Morgenstern (1953), p. 51 and 112 - 124.

Now define the attainable sets $A^*(n)$ and the sophisticated dynamic choice function $\{C^*(n)\}$ so that:-

$$A^*(n) = \bigcup \{C^*(n')(A(n'))\,|\,n' \in Su(n)\}$$

$$C^*(n)(A(n) = C(n)(A^*(n))$$

(whenever $n \in N(X)$ and $A \subsetneq X(n)$).

$A^*(n)$ has the form above because the agent is free to move to any node $n' \in Su(n)$; once he moves to n', however, his later choices force him to follow $C^*(n')(A(n'))$, which will be a unique branch, as will now be seen. In fact, it is obvious that $C^*(n)(A(n))$ is always a singleton because of assumption (A3) of section 4.6.

The following results show that sophisticated choice is dynamically consistent, and that sophistication is unnecessary if the original dynamic choice function is consistent.

<u>Theorem 4.8.1.</u>

$\{C^*(n)\}$ is a consistent dynamic choice function.

<u>Proof</u>

(1) That $C^*(n)$ is a choice function on $X(n)$, for each $n \in (X)$, is evident because $A^*(n) \subsetneq A(n)$ (each n) and $C^*(n)(A(n)) = C(n)(A^*(n))$. Also, $A^*(n)$ is clearly non-empty whenever A is finite, by backward induction.

(2) If $x \in C^*(n)(A(n))$, and $x \in A(n')$, where $n' \in Su(n)$, then $x \in A^*(n)$, and so $x \in C^*(n')(A(n'))$.

(3) If $y \in C^*(n')(A(n'))$, where $n' \in Su(n)$, then $y \in A^*(n)$. But, by (A3), $C^*(n)(A(n))$ consists of precisely one option. Let this be x. Then, by (2), $x \in C^*(n')(A(n'))$. Again, by (A3), $C^*(n')(A(n'))$ is a singleton, so that $x = y$, and so $y \in C^*(n)(A(n))$.

(4) From (2) and (3), if $x \in A(n') \cap C^*(n)(A(n))$, and $n' \in Su(n)$, then $C^*(n')(A(n')) = A(n') \cap C^*(n)(A(n))$. So $\{C^*(n)\}$ is consistent.

Theorem 4.8.2.

If $\{C(n)\}$ is consistent, then $\{C^*(n)\}$ is identical to $\{C(n)\}$.

Proof.

(1) The proof will proceed by backward induction.

First, if $n = x(t)$, where $t \geq T$, then $X(x(t)) = \{x\}$, and so $C^*(n)(A) = C(n)(A) = \{x\}$ because, if $A \subseteq X(n)$, then $A = \{x\}$.

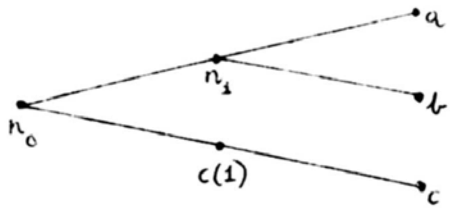
Next, suppose that whenever $n' \in Su(n)$, then $C^*(n')$ is identical to $C(n')$. Let $A \subseteq X(n)$. It is enough to show that $C^*(n)(A) = C(n)(A)$.

(2) Suppose that $x \in C(n)(A)$. Let n' denote $n_1(n,x)$. Then $x \in C(n')(A(n'))$, by consistency. So $x \in C^*(n')(A(n'))$, and therefore $x \in A^*(n)$. Now, by (A2) of section 4.6., we are assuming that $C(n)$ is coherent. Therefore, since $x \in C(n)(A)$, $x \in A^*(n)$, and $A^*(n) \subseteq A$, it follows that $x \in C(n)(A^*(n)) = C^*(n)(A)$, as required.

(3) Suppose that $x \in C^*(n)(A)$. By (A3) of section 4.6., there is a unique y such that $y \in C(n)(A)$. By (2) above, $y \in C^*(n)(A)$. But $x \in C^*(n)(A)$, and $C^*(n)(A)$ is a singleton. Therefore $x = y$, and so $x \in C(n)(A)$, as required.

The assumption that there was a singleton choice set obviously had a crucial role in these proofs. Less obvious may have been the need to invoke coherence, which we did in part (2) of the proof of theorem 4.8.2. The following example illustrates what can go wrong if there is incoherence.

Example 4.8.3.



The underlying tree is as in the diagram above, and $\{C(n)\}$ is defined by the following:-

$$C(n_0)(\{a,b,c\}) = C(n_0)(\{b,c\}) = \{c\}.$$

$$C(n_0)(\{a,b\}) = C(n_0)(\{a,c\}) = C(n_1)(\{a,b\}) = \{a\}.$$

Then $\{C(n)\}$ is dynamically consistent, but $C(n_0)$ is incoherent, because $c \notin C(n_0)(\{a,c\})$, although $c \in C(n_0)(\{a,b,c\})$. In this example, sophisticated choice leads to $A^*(n_0) = \{a,c\}$ (if $A = \{a,b,c\}$), and $C^*(n_0)(\{a,b,c\}) = C(n_0)(\{a,c\}) = \{a\}$, which is *not* the same as $C(n_0)(\{a,b,c\})$.

As with naive choice, if $\{C(n)\}$ is inconsistent, then $\{C^*(n)\}$ cannot be the same as $\{C(n)\}$, because $\{C^*(n)\}$ is consistent.

The coherence of sophisticated choice, together with that of naive choice, will now be explored.

4.9. The Coherence of Naive and Sophisticated Choice.

Now that naive and sophisticated choice have been defined - at least, defined under special circumstances, as specified by the assumptions made in 4.6. - we can set about finding when they are coherent. It turns out to be easiest, in many ways, to treat both kinds of choice together, for this purpose.

We already know, from the potential addict example of 4.5., that both naive and sophisticated choice may be incoherent, no matter how "rational" the components of an inconsistent dynamic choice function may be. On the other hand, if a dynamic choice function is consistent, and satisfies the assumptions of 4.6., then both the naive and sophisticated dynamic choice functions derived from it are, in fact, identical to the original dynamic choice function, and so give coherent choices. This follows from theorems 4.7.3. and 4.8.2.

These special cases tell us something; in particular, they tell us that our problem is to characterize those dynamic choice functions which give rise to incoherent naive choice or to incoherent sophisticated choice.

It turns out that the kinds of choices which led to incoherence in the potential addict example are especially significant. Broadly speaking, unless there is a part of the underlying tree over which the dynamic choice function looks like that of the potential addict, both naive and sophisticated choice are coherent. In fact, given our assumptions, they are not only coherent; they are also identical to each other and are both ordinal. These preliminary remarks will, I hope, serve to make the following steps more plausible.

Recall that the assumptions made in section 4.6. implied that each component of the dynamic choice function $\{C(n)\}$ corresponds to a strong ordering $P(n)$ on the set $X(n)$. Now a generalization of the dynamic choice configuration of the potential addict example is as follows:-

First, define $n(a,b)$, for any pair of branches $a,b \in X$, as the node $n \in N(X)$ such that $a(t) = b(t) = n$, but $a(s) \neq b(s)$ whenever $s \geq t$. So branches a and b have the sequence of nodes $\big(a(0),a(1),\ldots,a(t)\big) = \big(b(0),b(1),\ldots,b(t)\big)$ in common, but separate thereafter.

With this notation, the required generalization of the potential addict example can be described as follows:-

There are three branches a,b,c and two nodes n,n' of the underlying tree X such that:-

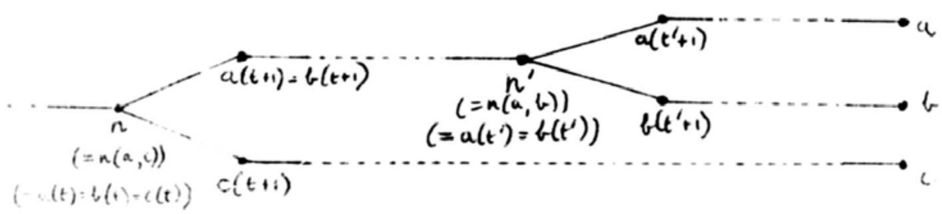
(i) $n = n(a,c) = n(b,c)$

(ii) $n' = n(a,b)$

(iii) $n \; Pr \; n'$

(iv) $a \; P(n') \; b$, $b \; P(n) \; c$, $c \; P(n) \; a$.

Properties (i) to (iii) can be illustrated in the following diagram:-



So the underlying tree is a "stretched" version of that in the potential addict example of section 4.5.

The preferences on such a triple are dynamically inconsistent, of course, because it must also be true that $b\ P(n)\ a$. This type of inconsistency will be seen later to be especially important. Accordingly, we shall say that the dynamic strict preference relation $\{P(n)\}$ is *essentially inconsistent* if there is a triple $\{a,b,c\}$ with properties (i) to (iv) above. Otherwise, we shall say that $\{P(n)\}$ is *essentially consistent*.

By considering the three options a,b,c as in the potential addict example, it is easy to show that any essential inconsistency gives rise both to incoherent naive choice and to incoherent sophisticated choice on the triple $\{a,b,c\}$. Our task now is to prove that under essential consistency, both naive and sophisticated choice are coherent.

Recall that if a coherent choice function has singleton choice sets, then it is ordinal and corresponds to a strong ordering (lemma A.5.2. of appendix 1). So, since the assumptions of section 4.6. imply that both naive and sophisticated choice give rise to singleton choice sets, if they are coherent, they must also be ordinal and correspond to a strong ordering. As will be shown later, the strong ordering they correspond to - if they are coherent - is the following:-

$$x\ P\ y \quad \text{iff} \quad x\ P(n(x,y))\ y$$

It is obvious that P is antisymmetric, in the sense that *either* $x\ P\ y$ *or* $y\ P\ x$ *or* $x = y$. Then we have:-

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Lemma 4.9.1.

P is transitive unless the dynamic preferences $\{P(n)\}$ are essentially inconsistent.

Proof.

Suppose P is not transitive. Then there is a triple of options $\{a,b,c\}$ such that:-

$$a \; P \; b \qquad b \; P \; c \qquad c \; P \; a$$

Evidently, $n(a,b) \neq n(b,c)$, because each $P(n)$ is transitive. Define $n' = n(a,b)$, $n = n(b,c)$. Assume that the options are labelled so that $n \; Pr \; n'$. Then, from the definition of P, it is evident that we have an essential inconsistency.

So, if the dynamic preferences $\{P(n)\}$ are essentially consistent, as we assume they are, then the relation P defined above is a strong ordering. Finally, we show that, in this case, it corresponds to both naive and sophisticated choice.

Lemma 4.9.2.

Suppose that the dynamic choice function $\{C(n)\}$ satisfies the assumptions of section 4.6., and that the corresponding dynamic preference relation $\{P(n)\}$ is essentially consistent. Then naive dynamic choice corresponds to the strong ordering P.

Proof

(1) The proof will proceed by backward induction.

First, if t is large enough, then $X(x(t)) = \{x\}$, and then $C^{o}(n)$ must correspond to P, because any feasible set $A \subsetneq X(n)$ is a singleton anyway.

Suppose that $n \in N(X)$ and that $C^{o}(n')$ corresponds to P whenever $n' \in Su(n)$. Suppose that $A \subsetneq X(n)$.

(2) Now, suppose that $x \in C^{o}(n)(A)$. Let n' be the node immediately succeeding n on the branch x. Suppose too that $x(t) = n$. Then $C(n)(A)$ is a singleton, $\{z\}$ say, where $z(t+1) = x(t+1)$. Suppose that $y \in A - \{z\}$.

For any $\omega \in X(n')$, if $\omega \neq x$, then $\omega \notin C^{\circ}(n')(A(n'))$. So, by the induction hypothesis, $x \; P \; \omega$.

Now, if $y \notin X(n')$, then $y \notin C(n)(A)$, and so $z \; P(n) \; y$. But $n(y,z) = n$, and so $z \; P \; y$. By (a), $x \; P \; z$. Since P is transitive, $x \; P \; y$.

But if $y \in X(n')$, and $y \neq x$, then, by (a),

$x \; P \; y$.

In either case, if $x \in C^{\circ}(n)(A)$, and $y \in A - \{x\}$, then $x \; P \; y$ as required.

(3) Conversely, suppose that $x \in A$ and there exists $y \in A$ such that $y \; P \; x$. Now, if $x \in C^{\circ}(n)(A)$, then, by (2) above, $x \; P \; y$, because $y \in A - \{x\}$. This is a clear contradiction, because P is antisymmetric. So $x \notin C^{\circ}(n)(A)$, as required.

(4) Therefore $C^{\circ}(n)$ corresponds to P, as required.

Lemma 4.9.3.

Suppose that the dynamic choice function $\{C(n)\}$ satisfies the assumptions of section 4.6., and that the corresponding dynamic preference relation $\{P(n)\}$ is essentially consistent. Then sophisticated dynamic choice corresponds to the strong ordering P.

Proof

(1) The proof is very similar to that of lemma 4.9.2., and proceeds by backwards induction. It is enough to show that, when $n \in N(X)$, and $C^*(n')$ corresponds to P whenever $n' \in Su(n)$, then $C^*(n)$ corresponds to P. Suppose that $A \subseteq X(n)$.

(2) Suppose that $x \in C^*(n)(A)$. Then, for some t, $x(t) = n$. Let n' denote $x(t+1)$. Suppose that $y \in A - \{x\}$.

(a) If $y \notin X(n')$, and $y \in A^*(n)$, then, since $\{x\} = C(n)(A^*(n))$, it follows that $x \; P(n) \; y$, and $n = n(x,y)$, so that $x \; P \; y$.

(b) If $y \notin X(n')$ and $y \notin A^*(n)$, then there exists z such that

$\{z\} = C^*(y(t+1))(A(y(t+1)))$. Since $y \notin C^*(y(t+1))\ A(y(t+1))$,

it follows that $z\ P\ y$, by the induction hypothesis, and because

$y(t+1) \in Su(n)$. But now $z \notin X(n')$ and $z \in A^*(n)$, so $x\ P\ z$, by

(a) above. Since P is transitive, $x\ P\ y$.

(c) If $y \in X(n')$, then, since $\{x\} = C(n')(A^*(n')) = C^*(n')(A(n'))$

it follows from the induction hypothesis that $x\ P\ y$, since

$y \in A(n')$.

Therefore, if $y \in A - \{x\}$, then $x\ Py$, as required.

(3) Conversely, suppose that $x \in A$ and there exists $y \in A$ such that $y\ P\ x$. Then $y \in A - \{x\}$. So, if $x \in C^*(n)(A)$, then by (2), $x\ P\ y$. This contradicts antisymmetry of P.

(4) Therefore $C^*(n)$ corresponds to P, as required.

In a sense, the whole of the preceding argument, from the start of section 4.6., has been leading up to a single result. This result can now be set out in full; the proof has already been carried out.

Theorem 4.9.4.

Let $\{C(n)\}$ be a dynamic choice function on the underlying tree X, satisfying the following properties:-

(A1) There exists T such that, for all $x \in X$,

$$X(x(T)) = \{x\}.$$

(A2) For each $n \in N(X)$, $C(n)$ is a coherent choice function on $X(n)$

(A3) For each $n \in N(X)$ and $A \subsetneq X(n)$, $C(n)(A)$ consists of precisely one branch of X.

Let $P(n)$ be the strong ordering to which $C(n)$ corresponds. (It corresponds to a strong ordering because of lemma A.5.2. of appendix 1).

Then both the naive and the sophisticated dynamic choice functions $\{C^{o}(n)\}$ and $\{C^{*}(n)\}$, are well-defined, consistent, and satisfy (A3). A necessary and sufficient condition for both $C^{o}(n)$ and $C^{*}(n)$ to be coherent choice functions, for all $n \in N(X)$, is that there should be no triple $\{a,b,c\}$ of branches of X on which the dynamic strict preference relation $\{P(n)\}$ is essentially inconsistent. If this condition is satisfied, then naive and sophisticated dynamic choice both coincide, and correspond to the strong ordering $P^{(1)}$.

Naturally, even though naive and sophisticated choice may still coincide with each other, when $\{C(n)\}$ is inconsistent, they cannot coincide with $\{C(n)\}$, because both $\{C^{o}(n)\}$ and $\{C^{*}(n)\}$ are consistent - as was seen in sections 4.7. and 4.8.

It is now time to evaluate theorem 4.9.4. In intertemporal welfare economics, as in static welfare economics, our concern is with welfare choice functions, for individuals and for society. These must be well-defined, and ideally, coherent. So, in a dynamic choice situation, one looks for a dynamic WCF corresponding to a single "metastatic", WCF. The question now is how successful are naive and sophisticated choice in providing us with such a WCF. In view of theorem 4.9.4., the answer depends on how restrictive one finds essential consistency.

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(1) Blackorby et. al. (1973), theorem 6, is a result giving conditions under which sophisticated choice from budget sets gives rise to intertemporal demands which maximize an intertemporal utility function. They show, in effect, that the dynamic preference relation must be consistent. Although they consider a somewhat different issue, this result seems to be quite close to theorem 4.9.4.

In fact, it is hard to believe that very many dynamic choice functions will be essentially consistent, unless they are fully consistent. For example essential consistency implies that if a, b are two branches which follow a common path up to node n' and then part - i.e. $n' = n(a,b)$ - and if n is a node preceding n' - i.e. $n \; Pr \; n'$ - and if, finally, $a \; P(n) \; b$, then at least one of the following must be true:-

(1) $a \; P(n') \; b$

(2) There is no $c \in X(n) - X(n')$ such that $a \; P(n) \; c$ and $c \; P(n) \; b$.

Of these, (1) is tantamount to consistency of the dynamic strict preference relation $\{P(n)\}$. On the other hand, (2) is a particularly strong requirement, because if $X(n)$ is a topologically connected set, the condition implies that $P(n)$ cannot be represented by a continuous utility function.

At this point, it is time to assess how the restrictive assumptions (A1) and (A3) affect the conclusion that naive and sophisticated choice are unlikely to be coherent. Suppose that we have a completely general decision tree X, and a general dynamic choice function, whose only restriction is that each component $C(n)$ is coherent - without this restriction, there seems virtually no hope that naive or sophisticated choice will be coherent. Pick any three branches of the decision tree X at random, and consider dynamic choices on this subtree, consisting of just three branches. For coherent naive and sophisticated choice on X, we need coherent naive and sophisticated choice on the triple. On this triple, (A1) is satisfied automatically. (A3) will be unless we happen to have picked a triple for which no choice set contains more than one branch. And if (A3) is satisfied, then what is needed for coherent naive and sophisticated choice *just on this triple*, is an absence of essential inconsistency. Since this must be true on *any* triple, it seems hard to believe that assumptions (A1) and (A3) make very much difference.

For this reason, it seems fairly safe to regard both naive and sophisticated choice as inappropriate ways of achieving consistency in dynamic welfare choice functions. And, of course, this reason is quite independent of any additional ethical arguments which could be advanced against such choice procedures. Such arguments are, moreover, easy to advance. Naive choice, and its synonym, myopic choice, are value-loaded phrases which seem appropriate. Sophisticated choice is virtually an implicit "second-best" approach[1].

Consequently, in the rest of this work, there will be no further consideration of naive or of sophisticated dynamic choice. Instead, the third method of reconciling inconsistent dynamic choices is the one to be used from now on. This, remember, was called "intertemporal liberalism" in section 4.5. After a lengthy but necessary consideration of alternatives, it is time to return to it.

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(1) As suggested by Phelps and Pollak (1968).

4.10. Intertemporal Liberalism.

Naive and sophisticated choice were two possible ways of reconciling inconsistent dynamic choices. But they are not the only two possible ways. Moreover, as was seen in section 4.9., they are unlikely to yield a consistent dynamic choice function with coherent components. A third method of removing inconsistency was suggested in section 4.5., and the term "intertemporal liberalism" put forward as a possible description of the method. This section considers the method for more general dynamic choice situations.

It is convenient to retain assumption (A1) of section 4.6., that the underlying tree is bounded. Indeed, this assumption will be retained almost without exception until chapter 7. On the other hand, assumption (A3), that every choice set is a singleton, no longer has any significant role to play, and so it will be dropped. Assumption (A2), that each component of the dynamic choice function $\{C(n)\}$ is coherent, will be discussed shortly.

The problem we have been facing throughout this chapter is that of converting an inconsistent dynamic choice function $\{C(n)\}$ into a consistent one $\{\hat{C}(n)\}$, of which each component $\hat{C}(n)$ is a coherent choice function. It is this coherence which has been difficult to achieve. So let us now see what methods can possibly give coherence. First, however, we must consider all the possible ways of reconciling dynamic inconsistencies.

What is involved here is a mapping g from dynamic choice functions to dynamic choice functions, with the property that each image dynamic choice function is consistent.

Thus $\{\hat{C}(n)\} = g(\{C(n)\})$

But if $\{\hat{C}(n)\}$ is consistent, then $C(n_0)$ - where n_0 is the initial node of the underlying tree X - determines $\{\hat{C}(n)\}$ uniquely, as in section 4.4. So the mapping can be expressed in the following form:-

$$\hat{C}(n_0) = g(\{C(n)\}).$$

Therefore g aggregates the components of a dynamic choice function into a single choice function on the underlying set X. So g will be called a *dynamic aggregation function* - or DAF.

So far, only dynamic aggregation at node n_0 has been considered. But there is no reason to stop there. At any node $n \in N(X)$, there is a dynamic choice function $\{C(n')\}(n)$ on the underlying tree $X(n)$. Here, $\{C(n')\}(n)$ denotes the dynamic choice function $\{C'(n')\}$ defined on $X(n)$ by:-

$$C'(n')(A) = C(n')(A) \text{ whenever } n \; Pr \; n' \text{ and } A \subseteq X(n').$$

The dynamic choice function $\{C(n')\}(n)$ can also be aggregated to yield a single choice function $\hat{C}(n)$ on $X(n)$, and so a corresponding dynamic choice function on $X(n)$. There is no guarantee *a priori* that the choice function $\hat{C}(n)$ will be consistent with $\hat{C}(n_0)$. Nor is there any reason to regard $\hat{C}(n_0)$ as giving "better" choices than $\hat{C}(n)$, or *vice versa*.

So dynamic aggregation can occur at any node $n \in N(X)$. There are, accordingly, mappings $g(n)$ from the set of dynamic choice functions on $X(n)$ to single choice functions on $X(n)$, for each $n \in N(X)$. Each function $g(n)$ is a dynamic aggregation function.

What we are looking for is a *consistent set of dynamic aggregation functions* $\{g(n)\}$ - one DAF for each $n \in N(X)$ - such that:-

(1) $g(n)$ maps dynamic choice functions $\{C(n')\}(n)$ on $X(n)$ into single choice functions on $X(n)$, thus:-

$$\hat{C}(n) = g(n)(\{C(n')\}(n))$$

(2) $\{\hat{C}(n)\}$ is a consistent dynamic choice function on X, for any given initial dynamic choice function $\{C(n)\}$.

(3) For each $n \in N(X)$, $\hat{C}(n)$ is a coherent choice function.

One more property seems sensible. If the dynamic choice function $\{C(n')\}(n)$ on $X(n)$ is already consistent, there is no point in changing it to achieve consistency. But there may still be a need to change it to achieve coherence; to exclude this possibility, we shall keep assumption (A2) of section 4.6. - that each component $C(n)$ of $\{C(n)\}$ is coherent. Then the following extra restriction on DAF's will be imposed:-

(4) If $\{C(n')\}(n)$ is consistent, then $\hat{C}(n) = C(n)$.

Although this restriction is important in restricting the search for DAF's, it has little role to play in the formal analysis.

First of all, let us verify that there is a consistent set of DAF's satisfying these properties.

Theorem 4.10.

Let X be any bounded underlying tree. Then, for dynamic choice functions with coherent components, there is a consistent set of dynamic aggregation functions satisfying conditions (1) to (4) above.

Proof.

We shall show how to construct a consistent set of DAF's by backward recursion. Let $\{C(n)\}$ be any dynamic choice function with coherent components. It is enough now to find $\{\hat{C}(n)\}$ so that (1) to (4) above are satisfied.

First, since the tree X is bounded, if t is large enough, then $X(x(t)) = \{x\}$ for all $x \in X$. So then if $A \subseteq X(x(t))$, $A = \{x\}$, and $\hat{C}(x(t))(A) = C(x(t)(A) = A$. Obviously (1) to (4) are trivially satisfied so far.

Suppose we have constructed $\hat{C}(n')$ for $n' \in N(X(n))$, $n' \neq n$ so that (1) to (4) are satisfied. If we can now construct $\hat{C}(n)$ so that (1) to (4) are satisfied, then backward induction establishes the theorem.

If $\{C(n')\}(n)$ is consistent, define $\hat{C}(n) = C(n)$. Now, for all $n' \in Su(n)$, it must be true that $\{C(n'')\}(n')$ is consistent, and so, by (4), $\hat{C}(n') = C(n')$. Therefore $\{\hat{C}(n')\}(n)$ is consistent, $\hat{C}(n)$ is coherent, and, indeed, (1) to (4) are all satisfied.

Suppose $\{C(n')\}(n)$ is not consistent. On the set of nodes immediately succeeding n, $Su(n)$, let $C_1(n)$ be any coherent choice function. Such a choice function certainly exists - e.g. $C_1(n)(A_1) = A_1$, for all $A_1 \subseteq Su(n)$. Now define, for each $A \subseteq X(n)$:-

$$A_1(n) = \{n' \in Su(n) \mid \exists x \in A(n) \, . \, n' = n_1(n,x)\}$$

as in section 4.6. Thus $A_1(n)$ is the set of nodes immediately succeeding n which can be reached along some branch of the tree X. Define, too:-

$$\hat{C}(n)(A) = \bigcup \{\hat{C}(n')(A(n')) \mid n' \in C_1(n)(A_1(n))\}$$

So $x \in \hat{C}(n)(A)$ if and only if, when $n' = n_1(n,x)$, then $n' \in C_1(n)(A_1(n))$, and $x \in \hat{C}(n')(A(n'))$ - i.e. if the first move of x from n is chosen by the choice function $C_1(n)$, and then the rest of x is chosen by $\hat{C}(n')$.

It remains only to verify that this $\hat{C}(n)$ is a coherent choice function which is consistent with $\hat{C}(n')$ whenever $n' \in Su(n)$.

$\hat{C}(n)$ is a coherent choice function

First, obviously, $\hat{C}(n)(A) \subset A$ whenever $A \subsetneq X(n)$

Also, if A is finite, then $A_1(n)$ is finite, and so $C_1(n)(A_1(n))$ is non-empty. If $n' \in C_1(n)(A_1(n))$, then $\hat{C}(n')(A(n'))$ is non-empty. So $\hat{C}(n)(A)$ is non-empty.

Finally, suppose that $B \subsetneq A \subseteq X(n)$ and $x \in B - \hat{C}(n)(B)$.
Let $n' = n_1(n,x)$. Now:-

(1) Suppose $x \notin \hat{C}(n')(B(n'))$. Since $B(n') \subsetneq A(n')$, and $\hat{C}(n')$ is coherent, $x \notin \hat{C}(n')(A(n'))$. Therefore $x \notin \hat{C}(n)(A)$.

(2) If $x \in \hat{C}(n')(B(n'))$, then it must be true that $n' \notin C_1(n)(B_1(n))$. Since $B_1(n) \subsetneq A_1(n)$ and $C_1(n)$ is coherent, $n' \notin C_1(n)(A_1(n))$. So again $x \notin \hat{C}(n)(A)$.

Thus $\hat{C}(n)$ is coherent.

$\hat{C}(n)$ is consistent with $\hat{C}(n')$, whenever $n' \in Su(n)$

Suppose that $A \subsetneq X(n)$, $n' \in Su(n)$, and $x \in A(n') \cap \hat{C}(n)(A)$. Then $n' = n_1(n,x)$, and, by definition of $\hat{C}(n)$, $n' \in C_1(n)(A_1(n))$.
It is then clear from the definition of $\hat{C}(n)$ that:-

$$\hat{C}(n')(A(n')) = A(n') \cap \hat{C}(n)(A)$$

as required.

Notice that, in this proof, $C_1(n)$ was *any* coherent choice function on $Su(n)$. So, in general, there are very many consistent sets of DAF's. This is perhaps as well, because $\hat{C}(n)$ was not related to the original dynamic choice function $\{C(n)\}$ in any way, unless $\{C(n')\}(n)$ happened to be consistent. So there is scope still for $\hat{C}(n)$ to reflect, to some extent, the properties of $\{C(n')\}(n)$.

Given a procedure for making a dynamic choice function consistent, we shall say that the procedure is *intertemporally liberal* if it corresponds to a consistent set of DAF's satisfying properties (1) to (4). The term "intertemporally liberal" is used because of the relationship of such procedures to Sen's concept of "liberalism"(1). This relationship will now be explained.

Suppose that f is a constitution, as defined in chapter 2. Thus, f maps lists of choice functions, one for each individual, into a social choice function:-

$$C = f(C_1, C_2, \ldots, C_n).$$

Suppose too that the social choice function is always coherent, no matter what the individual choice functions may be.

Let X be the underlying set, and let $A \subsetneq X$. Individual i is said to be *decisive* over the set A if, whenever $B \subsetneq A$, the social choice set satisfies:-

$$C(B) = C_i(B)^{(2)}$$

Notice that, if i is decisive over A, then, because of coherence, for *any* set $S \subsetneq X$,

if $x \notin C_1(S \cap A)$, then $x \notin C(S \cap A)$,

and so $x \notin C(S)$.

Thus i has an effective veto over elements of A which are not in his interests.

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(1) See Sen (1970a) ch. 6. and Sen (1970b).

(2) "Decisiveness" usually refers to preferences, social and individual - see Arrow (1963), definition 10, p. 52. The definition here is an obvious generalization.

Now, Sen was less concerned with a precise definition of liberalism than with a condition for "minimal liberalism" which he showed to be incompatible with the Pareto principle. Nevertheless, his work suggests that any judgment we make about the "liberalism" of a society should be based on the families $\mathcal{A}_i$ of sets A_i over which i is decisive - for each individual i. To take an obvious example which Sen himself suggests, is each individual decisive in the choice of colour for his own walls?[1]

Now, DAF's are somewhat similar to constitutions. To bring this out, imagine that each component $C(n)$ of the dynamic choice function represents the choice of a different agent - the agent n, say. Then $g(n)$ is a "constitution" to be applied to the choices of agents n' for which $n \; Pr \; n'$.

Of course, each agent has two choice functions, $C(n)$ and $\hat{C}(n)$. And, although each DAF is defined as being applied to sets of the original choice functions $C(n)$, dynamic consistency demands that some of the choice functions $\hat{C}(n)$ must also be taken into account. In fact, we lose nothing by considering more general DAF's of the form:-

$$\hat{C}(n) = h(n)\left(\{C(n')\}(n),\ (\hat{C}(n'))_{n' \varepsilon Su(n)}\right)$$

i.e. where the dependence of $\hat{C}(n)$ on $\hat{C}(n')$, for nodes $n' \; \varepsilon \; Su(n)$, is explicitly recognized.[2]

Now, dynamic consistency requires, amongst other things, that for any $A \subseteq X(n')$, where $n' \; \varepsilon \; Su(n)$:-

$$\hat{C}(n)(A) = \hat{C}(n')(A)$$

---

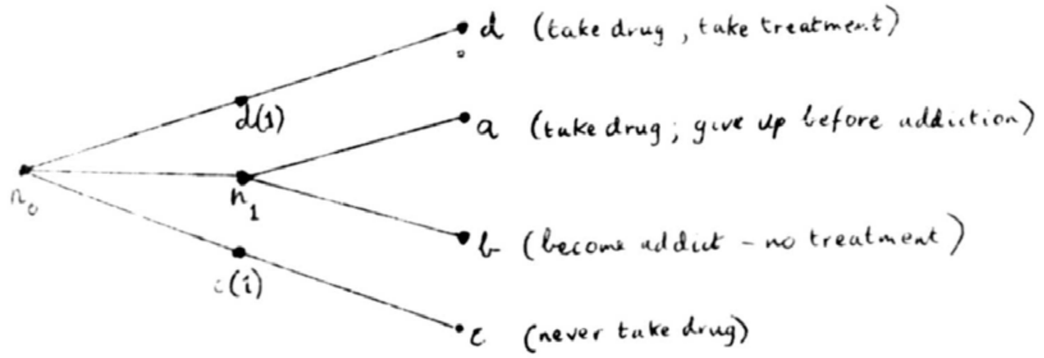
(1) See Sen (1970b), p. 152.

(2) There is no point in taking into account $\hat{C}(n'')$ for nodes n'' which come after members of $Su(n)$, because then $\hat{C}(n'')$ is already determined by $\hat{C}(n')$, for some $n' \; \varepsilon \; Su(n)$, on account of dynamic consistency.

But this means precisely that the agent n', with his *modified* choice function $\hat{C}(n')$, is decisive over the set $X(n')$.

Whether such decisiveness deserves to be labelled "liberalism" is, in the end, a matter of definition. But "intertemporal liberalism" is a convenient term. Moreover, provided that the agent n' takes account of his successors, as he does in using the choice function $\hat{C}(n')$, then choices over $X(n')$ would seem to be his proper concern. For his predecessor n to want to interfere with such choices is illiberal; but n is not being illiberal if he refuses to allow n' to be decisive over any set other than $X(n')$ and its subsets. At least, so it could be argued. So "liberalism" may not be an entirely inappropriate description.

Nevertheless, it is possible for "intertemporally liberal" choices to seem somewhat illiberal. Consider the potential addict example again. Suppose that the potential addict has an extra option, d, which is to write to his doctor to tell the doctor to put the potential addict in care if he does become addicted to the drug. We may suppose, for simplicity, that addicts who are put in care lose their addiction. Then the decision tree is:-



Since $b\ P(n_1)\ a$, it is clear that intertemporal liberalism demands $b\ \hat{P}(n_1)\ a$, by condition (4) on consistent sets of DAF's. Here, of course, $\hat{P}(n)$ is used to denote the strict preference relation to which, it is assumed, $\hat{C}(n)$ corresponds. Then, too, it must be true that $b\ \hat{P}(n_0)\ a$, by dynamic consistency, even though $a\ P(n_0)\ b$. Plausible initial preferences at n_0 might be:-

$a\ P(n_0)\ c\ P(n_0)\ d\ P(n_0)\ b$, with $P(n_0)$ transitive.

With intertemporal liberalism, it seems likely that the agent would still keep the preference $d\ \hat{P}(n_0)\ b$. Since $b\ \hat{P}(n_0)\ a$, and, we assume, $\hat{P}(n_0)$ is transitive, it follows that $d\ \hat{P}(n_0)\ a$. This means that the agent at n_0 prefers to *"force"* his *"successor"* to give up the drug, rather than to let him do it freely - which, in fact, would not occur, of course. This has strong overtones of "illiberalism".

4.11. Uncertain Tastes and Dynamic Consistency.

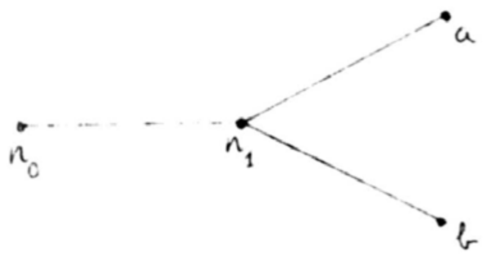
Throughout this chapter, it has so far been assumed that there was a single, well-known, dynamic choice function $\{C(n)\}$. This implies that at node n_0 all future choice functions are known, which is clearly an unrealistic assumption. Indeed, the assumption is so unrealistic that one may wonder whether the search for dynamic consistency is not completely pointless. If future choices are uncertain, how can one be sure of achieving consistency? Why not simply be naive?

Naturally, to discuss this problem fully would involve first of all giving a full discussion of choice under uncertainty, which this study has eschewed. Nevertheless, it is possible to make a few simple remarks, and to show that the quest for consistency is neither completely empty nor completely fruitless.

When making choices under uncertainty, unless he is naive, an agent does not commit himself to a single plan which he carried out without regard to the circumstances. Indeed, typically, he must modify his plan according to the circumstances, since these affect what is feasible, and, by definition, he cannot be sure what circumstances will be in the future. So the agent's future choices must depend upon the circumstances he faces when he puts them into effect. The agent make a *contingent* plan, specifying what choice he will make in each possible situation he may face - or in each "state of the world".

A similar situation arises when an agent is uncertain about his future tastes, or his future choice functions. The agent modifies his choices according to what his future choice functions turn out to be. He makes a contingent plan, specifying what choice he will make for any given configuration of future choice functions - for any "state of mind", one might say.

For example, suppose that the underlying tree is:-



and that there are two possible choice functions at the node n_1:-

$$C^1(n_1)(\{a,b\}) = \{a\}, \quad C^2(n_1)(\{a,b\}) = \{b\}.$$

Then, to be consistent, the agent at node n_0 has to have a contingent choice function $(C^1(n_0), C^2(n_0))$, where:-

$$C^1(n_0)(\{a,b\}) = \{a\}, \quad C^2(n_0)(\{a,b\}) = \{b\}$$

$C^1(n_0)$ indicates the choices which the agent is willing to make if his choice function at n_1 is $C^1(n_1)$; $C^2(n_0)$ indicates the choices which the agent is willing to make if his choice function at n_1 is $C^2(n_1)$. Notice that an agent with this contingent choice function is naive, sophisticated, and intertemporally liberal.

More generally, achieving dynamic consistency by means of a contingent choice function is more complicated, but effectively the same idea prevails. Notice that dynamic consistency is still meaningful, even under uncertainty about future choice functions, and so is intertemporal liberalism. What may not be clear is whether we can ever tell in practice that an agent is being dynamically inconsistent or intertemporally illiberal. Uncertainty gives so much scope for changing choice functions that it is hard to tie the agent down effectively, or so it might seem.

In fact, it is clear that if we merely observe what an agent chooses, there is no way of telling that he is being inconsistent. All we know is that, given a certain feasible set A, the branch the agent chooses is x. That is never inconsistent, in itself. Therefore, to observe inconsistency , we need to know not only which branch the agent chooses, but also the branches which he contemplated choosing at various times. Moreover, to allow for uncertainty, even more must be known; it is necessary to know how the agent's contemplated contingent choices adjust to changes in future choice functions. To see whether such knowledge is ever possible, or the extent to which it helps to identify inconsistency or illiberalism, it is best to discuss some examples.

For the first example, take the procrastinator of section 4.2. If he reports his intention to perform the action α tomorrow, and then fails to carry out this expressed intention, there is a *prima facie* inconsistency. If we face the procrastinator with this inconsistency, he may modify the report of his intentions. He can say that he will perform α tomorrow provided that he feels like it when he wakes up tomorrow morning. Otherwise, he will not perform α tomorrow. Given these intentions for contingent choices, he cannot possibly be dynamically inconsistent. So it is easy for the procrastinator to achieve dynamic consistency; indeed, it is so easy that dynamic consistency itself does little to tell us how the procrastinator might behave when he sees his problem.

For the procrastinator, intertemporal liberalism means the following. Suppose α has not already been performed before day t. Then the man's preference on the morning of day t must be decisive in determining whether it is better to perform α on day t than at some later date. So if there is day t such that α has not been performed before day t, and if the man wakes up on the morning of day t feeling like performing α that day, then α must be performed on day t. It is *not* necessarily true, however, that, if the man wakes up on the morning of day t still not feeling like performing α, then α continues to be put off; it is still possible for the agent to decide to perform α on day t, rather than to risk further serious delay. What intertemporal liberalism excludes is for the agent, when he wakes up on day t, to decide to perform α on day s, where $s > t$. At least, such a decision is only allowed if the agent is *sure* that, when he wakes up on the morning of day s, he will be prepared to perform α that day.

A second example is the potential addict of section 4.5. Dynamic consistency requires him to contemplate, at least, the possibility of addiction, if he starts to take the drug. Intertemporal liberalism requires him to prefer to continue taking the drug if he does become addicted; moreover this preference must apply *before* he becomes addicted as well as after.

A third example is a political party considering its policy on some long-term issue, such as whether to use road vehicles or some modernized railway system as the primary means of transport in the future, or whether to start developing desalination plants to provide fresh water. We may suppose that the present party leaders are old enough to be concerned with only the earlier effects of the policy decision they make, and, in particular, to feel that few of the benefits will be noticeable in their lifetime. The younger members of the party, however, are concerned with all the effects of the policy decision. In addition, it is these younger members who, if the party is in power, will be responsible for implementing the later details of the long-run policy decision. Then the younger party members presumably have a better idea of how the policy will be implemented in the future.

In deciding its policy on such an issue, the party has a dynamic choice problem. It can achieve consistency through intertemporal liberalism. What this means is that the party leaders have to accept the policy recommendations of the younger members, for decisions which will be made far in the future. And "acceptance" means more than consulting the younger members and noting their suggestions. For example, the older members of the party may be tempted to try to find out what their successors wish to do, and then to sabotage those parts of the younger members' plans which they do not like. This would be sophistication rather than intertemporal liberalism.

The last example to be discussed here concerns education. It may be the case that students sometimes have a clearer idea than their teachers of how they might use their learning. Suppose that the teachers are trying to decide a curriculum, and accept that it is only the way in which their pupils can use what they learn which is relevant to the decision. Then, in effect, the teachers are trying to choose a dynamically consistent plan of teaching and the use of imparted knowledge. But it is the pupils who will use the knowledge. Then intertemporal liberalism - or just plain liberalism - requires that the students must play a prominent part in deciding the curriculum. Indeed, the teachers must accept the students' view of how they will use what they learn, unless the teachers happen to believe that the students are bad predictors of what they will do after finishing their education.

These examples have illustrated the force of intertemporal liberalism. Whether it is ethically justified is another question, which brings us back to welfare economics. It is taken up in chapter 5.

4.12. Conclusions.

This chapter set out to do the following:-

(1) Provide a theoretical framework in which changing choices could be discussed

(2) Suggest a method (or methods) of arriving at a single coherent intertemporal choice function which took account of possibly changing choices.

In the attempt to carry out this program, dynamic choice functions were defined. It was seen that, in order to have a dynamic choice function equivalent to a single choice function, dynamic consistency was required. So problem (2) turned into that of eliminating inconsistency from a dynamic choice function, while preserving coherence. Naive and sophisticated choice were considered as possible methods, and rejected - partly because they obviously lacked ethical appeal, but mostly because they failed, in general, to give coherent choices. Finally, intertemporal liberalism was put forward as a property which the method must satisfy. It was defined in a way which guaranteed that the resulting dynamic choice function was consistent and had coherent components. The relationship to Sen's "liberalism" was shown. Finally, its meaning was illustrated by some examples. In particular, it was shown that both dynamic consistency and intertemporal liberalism do mean something, even when there is uncertainty about future choice functions.

In the rest of this work, only intertemporally liberal methods of achieving dynamic consistency will be considered. This is because most of welfare economic theory relies on coherence. Nevertheless, it should be recognized that problem (2) has not really been satisfactorily resolved. In particular, we are a long way from having a uniquely specified method of achieving dynamic consistency.

This chapter has drifted away from a direct discussion of welfare economic problems. But the diversion has been a necessary one; it is possible now to return and to discuss intertemporal welfare economics, making use of the theory of dynamic choice. As will be seen in chapter 6, this theory can shed new light on problems which have sometimes been found perplexing. But first, it is necessary to be clear what is meant by intertemporal welfare - individual and social.

Chapter 5.

INTERTEMPORAL WELFARE AND DYNAMIC CONSTITUTIONS.

5.1. Intertemporal Welfare Choice Functions.

In section 4.1., it was seen that changing choices had to be considered for a conplete understanding of intertemporal welfare economics. Now that changing choices have been fitted into the framework of dynamic choice theory, it is time to return to the specific field of welfare economics. What has to be done now, is to put together the static welfare concepts of chapters 2 and 3, with the dynamic choice functions of chapter 4.

Remember what a welfare choice function is. Individual i's WCF indicates, for any choice situation A, the set $C_i(A)$ of options which are in i's interests. Similarly the social WCF indicates, for any choice situation A, the set $C(A)$ of options which are in the interests of society as a whole. Now a *dynamic* choice situation is described by a set A which is a subset of an underlying *tree* X. Moreover, A inherits the tree structure of X. $A(n)$ is the set of options in A which are still feasible after node n has been reached. As time unfolds, and a path through the tree is followed, there is, for each node n, a set $C_i(n)(A(n))$ of options which are in i's interests, and a set $C(n)(A(n))$ of options which are in the interests of society as a whole.

When intertemporal choices are being considered, it is long-run interests which matter. Also, we continue to assume that there is complete certainty. So nothing unforeseen ever happens. And long-run interests must take account of foreseen changes. This being so, can a plan which is in i's long-run interests today fail to be in his interests tomorrow? Surely, if it ceases to be in his interests, it must be because of some new development which was not foreseen. Similarly, can a plan which is not in i's interests today become in his interests tomorrow - assuming that today we embark on a plan which is in i's interests?

I shall assume that, for all the sets of options to be considered, the answer to both these questions is negative. The implication is that the dynamic choice function $\{C_i(n)\}$, indicating which choices are in i's interests, must be consistent - i.e.:-

$$\text{whenever } n \; Pr \; n' \text{ and } x \in A(n') \cap C(n)(A(n)),$$

$$C(n')(A(n')) = A(n') \cap C(n)(A(n)).$$

This has to be true for each individual i. By a similar argument, the dynamic choice function $\{C(n)\}$, indicating which choices are in the interests of society as a whole, must also be consistent. Then it follows that the interests of individuals and of society can each be identified with a single, "metastatic", choice function - $C_i(n_0)$ for individual i, and $C(n_0)$ for society as a whole - where n_0 is the initial node of the underlying tree X. It is natural to call these choice functions "intertemporal welfare choice functions" - just as the static choice functions corresponding to the interests of an individual or of society were called "welfare choice functions" in chapter 2.

This shows that the metastatic approach to intertemporal welfare economics - the technique of treating the problem as though it were static, but with dated commodities, etc. - can be given a justification. But this is only the beginning rather than the end of the problem. To make use of metastatic techniques, the intertemporal social WCF has first to be found. This might be related to individual intertemporal

WCF's via a constitution, just as a static social WCF might be related to individual static WCF's, as discussed in section 2.5. But does a constitution to be applied to intertemporal WCF's - an "intertemporal constitution" - differ essentially from a "static" constitution? After all, if a long enough time period is being considered, the set of individuals in the society is going to change. And to what extent can we embody individuals' estimates of their own interests into an intertemporal social WCF?

These are the questions to be taken up in the rest of this chapter and also, to some extent, in chapter 6.

5.2. Metastatic Intertemporal Constitutions.

Constitutions were the subject of section 2.5. There it was shown that, under certain assumptions, there exists a mapping f from lists of individual WCF's to social WCF's, such that, if C_i is individual i's WCF, then

$$C = f(\langle C_i \rangle_{i \in I})$$

is the social WCF. A very similar argument might be used to establish the existence of an "intertemporal constitution", when the WCF's indicate intertemporal choice. Such an intertemporal constitution would be "metastatic", in the sense that it treats intertemporal welfare problems as no more than a formal extension of static problems. Where intertemporal constitutions have been considered - for example, in defining intertemporal Pareto efficiency - it is this metastatic approach which is almost invariably used(1). The first question is whether this approach is the right one.

One difficulty occurs because an individual's intertemporal interests may be unclear - very much less clear than his interests at a given moment of time, for example. An individual's WCF is likely to be an amalgam of inconsistent components of a dynamic choice function. How the dynamic aggregation is to be performed is rarely obvious. But this is an inevitable problem in welfare economics anyway, it seems. It has to be faced at some point, before we reach a consistent dynamic social choice function. Why not straight away, while each individual is still being considered separately?

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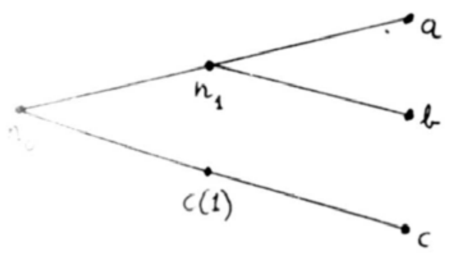
(1) The only exception, it seems, is contained in the work of Allais (1947), which is considered in appendix 2. See also the discussions by Malinvaud (1972) and de Montbrial (1971).

Another difficulty arises when the size and composition of the population is endogenous. What is the welfare choice function of an individual whose very existence depends upon the option society chooses? A question we soon have to face is: Is it in i's interests to be born? Although such a question may not be quite meaningless, there is much to be said for avoiding it, if possible. Surely it is the interests of other members of society which are far more relevant for determining whether i should be born. Of course, once i has been born, his interests come into play. To capture these considerations in an intertemporal WCF for each individual, including many who may never be born, and also in a metastatic constitution, when the list of individuals is itself a matter of choice, is quite hard. Even if it were done, it might not be clear whether the metastatic constitution had the right form. A different approach, more in accord with standard practice and our normal ways of thinking about welfare problems, seems desirable.

The third and last difficulty is, perhaps, much more serious. In section 2.5., we say how the social WCF might depend on a number of things other than individual WCF's - e.g. interpersonal comparisons. Now, in the intertemporal case, might not the social WCF depend not only on the individuals' consistent dynamic WCF's, but also on the inconsistent dynamic choice functions from which these were derived?

Consider the following example, based on the potential addict example of section 4.5. The underlying tree is:-



Imagine two societies, each of n individuals.
For the individuals $i = 2$ to n, the choice functions C_i are the same in each society.

In the first society, individual 1 has the potential addict's dynamic choice function, corresponding to the following preferences:-

$$a\ P(n_0)\ c\ P(n_0)\ b, \quad b\ P(n_1)\ a.$$

As a result of intertemporal liberalism, applied to these preferences, the individual has a WCF corresponding to the following preferences:-

$$b\ P\ a\ P\ c.$$

For example, the drug may be nicotine and the individual may want to smoke so badly that he is prepared to smoke all his adult life rather than never, even though originally he would have liked to give up smoking after a short time.

In the second society, individual 1 has no reservations about smoking, and his original dynamic choice function and his WCF both correspond to the preferences

$$b\ P\ a\ P\ c.$$

Now, if social choice is based on a metastatic constitution, the social choice in each of these two societies must be the same, because each individual has the same intertemporal WCF in each society. There is no way of taking account of individual i's original willingness to be forced to give up smoking, in the first society, unless we are also ready to force him to give up smoking in the second society. This is a serious restriction.

Accordingly, we shall now consider "dynamic constitutions" which generalize "metastatic constitutions". We shall also look further into the restrictiveness of the assumption that there is a metastatic constitution.

5.3. Dynamic Constitutions.

A metastatic constituion takes account of the interests of all the individuals who might, at one time or another, be members of that society. It involves the interests of those who will not be born for many generations to come, and, perhaps, of those who, in the end, are never born. Of course, no constitution works like this in practice. And while reality is not always a reliable guide to the way things ought to be done, in this case it provides a useful suggestion for a first approach towards overcoming the difficulties and deficiencies of metastatic constitutions. For, in practice, insofar as a constitution takes explicit account of individuals' interests, it is usual for it to take account of the interests of only those individuals who are currently alive at any one time. Of course, if this is so, there is almost certain to be inconsistent dynamic social choice, as the set of living individuals changes over time. Accordingly, we shall have to modify standard practice in order to ensure consistency.

First, we need a formal model, in which these ideas can be made precise. Let X be the underlying tree, with tree structure $\{X(n)\}$. To describe the effect of economic policy upon population, it is sufficient to specify the set of individuals who are alive at any given node n of the decision tree; let this set of individuals be $I(n)$. Then the total set of possible individuals is the union of all the sets $I(n)$, i.e.:-

$$I = \cup \{I(n) | n \in N(X)\}$$

Two quite natural assumptions are that there should be no reincarnation, and that individuals are mortal. So, for each individual $i \in I$, and each $x \in X$, *either* there are two finite times $b_i(x)$ and $d_i(x)$

such that, when $n = x(t)$ and $b_i(x) \leq t \leq d_i(x)$, then $i \in I(n)$, or, whenever $n = x(t)$, $i \notin I(n)$. This means that if branch x of the underlying tree is followed, then either individual i never lives at all, or he lives from time $b_i(x)$ to $d_i(x)$. Here, $d_i(x)$ is the time at which i dies; if $b_i(x) > 0$, then $b_i(x)$ is the time at which i is born; but if $b_i(x) = 0$, then i is already alive at time 0, and his birth-date is not specified.

For each $i \in I$, let $\{C_i(n)\}$ denote i's dynamic choice function. Depending upon the precise problem being considered, $\{C_i(n)\}$ can represent various kinds of choice. Two especially important kinds are the following:-

(a) $C_i(n)$ represents i's interests, as he himself sees them, at node n of the decision tree.

Since an individual can hardly perceive his own interests when he is not alive, $C_i(n)$ is defined, strictly speaking, only when $i \in I(n)$. However, it will be notationally convenient to have $C_i(n)$ defined for all n. Now, if $i \notin I(n)$, instead of saying that i cannot perceive his interests, we could alternatively say that he has no interests anyway. And having no interests means, persumably, that the choice function expresses total apathy. So, we shall define $C_i(n)(A) = A$ whenever $A \subseteq X(n)$ and $i \notin I(n)$. Notice that, if $b_i(x) \geq 1$, this is likely to lead automatically to dynamic inconsistency, because $C_i(n_0)(A) = A$ for all $A \subseteq X$, and it is unlikely that the individual will be totally apathetic while he is alive.

(b) $\{C_i(n)\}$ is a consistent dynamic choice function corresponding to i's long run interests. Thus, $C_i(n_0)$ is his intertemporal WCF.

Once again, there are circumstances in which an individual can hardly have any interests, because he could not possibly be born. For example, suppose that there is a node n such that, for any n' for which $n \, P \, n'$, $i \notin I(n')$. Then, at node n, individual i can never come into existence, and so, presumably, he has no interests. Strictly speaking, $C_i(n)$ is not defined at this node. Nevertheless, to avoid having to specify domains of definition, we can say that $C_i(n)$ *is* defined at this node, but takes the apathetic form:-

$$C_i(n)(A) = A \quad (\text{each } A \subseteq X(n)).$$

Another assumption we might make is that dead individuals have no interests in current and future policy choices. This can be expressed by having $C_i(n)$ take the apathetic form whenever n comes after i's death.

Notice, however, that dynamic consistency forces us to acknowledge that the as yet unborn do have interests. Suppose that $i \in I(n)$. Then, typically, $C_i(n)$ will not be apathetic. Yet it must be true that $C_i(n_0)(A) = C_i(n)(A)$ whenever $A \subseteq X(n)$. So, even if i has not been born at n_0, he does have interests at n_0.

Both cases (a) and (b) lead to a dynamic choice function $\{C_i(n)\}$ defined *for all* $n \in N(X)$ - not just for those nodes n such that $i \in I(n)$. But, to express the facts that individuals can only perceive their interests when they are alive, that they cannot have interests if they cannot possibly be born, and our assumption that the dead have no interests, we impose the following restriction:-

Whenever there is no node n' such that $n \; Pr \; n'$ and $i \in I(n')$

$$C_i(n)(A) = A \quad (\text{each } A \subseteq X(n)).$$

This is a minimum restriction; it needs to be strengthened when $C_i(n)$ represents the individual's perception of his own interests.

Now that we have specified the underlying tree, the set of individuals, the times and circumstances in which they are alive, and the choice functions of the individuals, we can start to consider the constitution. Here, we shall start by following the standard practice of taking account of the interests of only those individuals who are currently alive. Let $f(n)$ denote the constitution which will operate at node n of the underlying tree. The relevant set of individual choice functions is $\{C_i(n) \mid i \in I(n)\}$ because, by definition, $C_i(n)$ is the choice function which is relevant in considering i's interests at node n.

Let $\mathcal{C}_i(n)$ denote the class of admissible choice functions for individual i, and let $\mathcal{C}(n)$ denote the class of admissible social choice functions. In each case, of course, it is the class of choice functions which are admissible at node n. Then:-

$$f(n): \Pi_{i \in I(n)} \mathcal{C}_i(n) \to \mathcal{C}(n)$$

A *basic dynamic constitution* is a family of such constitutions $\{f(n) \mid n \in N(X)\}$.

Keep the individuals' dynamic choice functions $\langle\{C_i(n)\}\rangle_{i \in I}$ fixed. Let:-

$$C(n) = f(n)(\langle C_i(n)\rangle_{i \in I(n)}).$$

So $C(n)$ is just the result of applying the component $f(n)$ of the basic dynamic constitution to the list of corresponding components of the individuals' dynamic choice functions. $C(n)$ is itself a component of a dynamic social choice function $\{C(n)\}$. But there is no reason for $\{C(n)\}$ to be consistent; indeed there are good reasons for it to be inconsistent:-

(a) The individuals' dynamic choice functions $\{C(n)\}$ may be inconsistent.

(b) The set of individuals, $I(n)$, whose interests are embodied in $C(n)$, is changing over time.

Especially important is (b). Thus, if n' is a node which comes sufficiently long after n, the sets of individuals $I(n)$ and $I(n')$ will be entirely disjoint. The individuals in $I(n)$ may be totally disinterested in what happens from n' onwards. In other words, if $A \subsetneq X(n')$, then the unanimity principles of chapter 3 lead to completely apathetic social choice:-

$$C(n)(A) = A.$$

On the other hand, individuals in $I(n')$ will be concerned about what happens to them. This gives a clear inconsistency.

In chapter 4, a number of methods of reconciling inconsistent dynamic choices were examined. Here, one wants an intertemporal social WCF,and so the method adopted must yield a coherent choice function. Neglecting trivial exceptions, this means that the method of reconciliation must be intertemporally liberal, as was seen in section 4.10.

Let $\{\hat{C}(n)\}$ be the dynamic social WCF which results when the inconsistencies in $\{C(n)\}$ have been resolved. Let $n'\ Fol\ n$ denote the fact that the node n' lies in the subtree $X(n)$ of branches which are still feasible after node n has been reached. That is:-

$$n'\ Fol\ n \quad \text{iff} \quad n' \in N(X(n)) \text{ iff } \left[n\ Pr\ n'\right] \text{ or } \left[n = n'\right]$$

Let $\{C(n')\}(n)$ denote the set $\{C(n') \mid n'\ Fol\ n\}$ of components of $\{C(n)\}$ Then $\{C(n')\}(n)$ is a dynamic choice function on the subtree $X(n)$. Intertemporal liberalism, as defined in 4.10., means that for each node n, there is a dynamic aggregation function $g(n)$ mapping $\{C(n')\}(n)$ into a choherent choice function $\hat{C}(n)$. Moreover, $\{\hat{C}(n)\}$ must be a consistent dynamic choice function.

Notice that:-

$$\hat{C}(n) = g(n)\Big(\{C(n')\}(n)\Big)$$

$$= g(n)\Big(\{f(n')(\langle C_i(n')\rangle_{i\varepsilon I(n')})\}(n)\Big) \qquad \text{(A)}$$

and that this can be written alternatively as:-

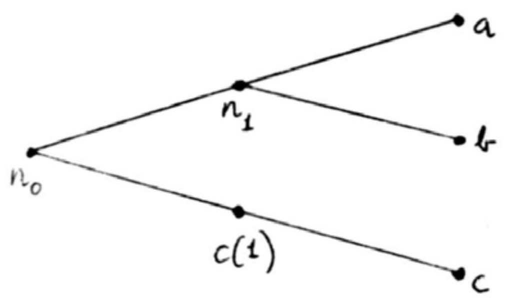
$$\hat{C}(n) = h(n)\Big(\{C_i(n') \mid n'\ Fol\ n,\ i \in I(n')\}\Big) \qquad \text{(B)}$$

In other words, intertemporal liberalism requires that the social WCF at node n must depend on the WCF's of all individuals who could possibly live after n. This seems entirely appropriate.

Nevertheless, the form (A), involving the functions $g(n)$ and $f(n')$, is less general than form (B), involving $h(n)$. Moreover the loss of generality in (A) could be serious, as the following example suggests.

Example 5.3.

The underlying tree is as in the potential addict example:-



Now, however, the branches are different options. One of the important factors affecting the desirability of the United Kingdom's membership of the European Economic Community is the Common Agricultural Policy. It is not implausible to believe that many U.K. citizens would regard membership of the E.E.C. as desirable if and only if the C.A.P. were to be changed to allow lower prices to the consumer, with farm incomes subsidized from the proceeds of increased direct taxation. It is also not implausible to believe that, now the U.K. is a member of the E.E.C., the number of its citizens who would wish to give up membership if the C.A.P. remains unchanged, is declining. Finally, it was suggested that the C.A.P. was much more likely to be changed after the U.K. joined the E.E.C.;

for simplicity, assume that it would definitely not be changed if the U.K. had not joined.

A stylized representation of the U.K.'s decision problem in 1971 is given by the above decision tree. The node n_0 corresponds to the point at which the decision to join the E.E.C. was finally made. The options a, b, c can be described as follows:-

a - join the E.E.C., but leave unless the C.A.P. undergoes a suitable change

b - join the E.E.C., and stay in whatever happens

c - remain outside the E.E.C.

At node n_0, let us assume, everybody prefers to be outside the E.E.C. if the C.A.P. is unchanged, and furthermore, prefers to leave the E.E.C. unless the C.A.P. is changed. That is:-

$$a \;\; P_i(n_0) \;\; b \qquad\qquad (\text{all } i).$$

At node n_1, however, after a spell inside the E.E.C., people change their minds, and become reluctant to leave the E.E.C. once they have entered it. This is despite the absence of any change in the C.A.P Let us assume, then, that:-

$$b \;\; P_i(n_1) \;\; a \qquad\qquad (\text{all } i).$$

So far, option c, of remaining outside the E.E.C., has not been considered. Suppose that the majority prefers at least to try to get the C.A.P. changed, even though first entering and then leaving the E.E.C. is fairly costly. That is, the majority prefer a to c.

On the other hand, suppose that the majority prefer to stay out of the E.E.C., rather than to enter and to accept an unchanged C.A.P. Then the majority prefer c to b.

Suppose that social preferences are determined by majority rule, wherever possible, and the constitution takes a form given by (A). Then the social dynamic preference relation takes the form:-

$$b\ P(n_1)\ a, \quad a\ P(n_0)\ c, \quad c\ P(n_0)\ b, \quad a\ P(n_0)\ c.$$

Now $\{P(n)\}$ is not consistent. According to (A), it must be transformed to a consistent dynamic preference relation $\{\hat{P}(n)\}$ which is consistent. But notice that:-

$$\hat{P}(n_0) = g(n_0)\big(P(n_0),\ P(n_1)\big)$$

in effect. So $\hat{P}(n_0)$ must be independent of the majorities underlying the original dynamic preference relation $\{P(n)\}$. So, too, must the social choice set $\hat{C}(n_0)(\{a,b,c\})$.

This independence, however, is ethically questionable. The strengths of the majorities underlying the preferences $c\ P(n_0)\ b$ and $a\ P(n_0)\ c$ can affect our view of what social choice is appropriate. Let us take an extreme case to illustrate this.

Suppose that we have two societies, S_1 and S_2. In S_1, everybody prefers a to c , but only a small majority prefers c to b. That is, everybody is willing to enter the E.E.C. on the right terms, and there is a large minority in favour of entry even if the C.A.P. cannot be changed. Then one feels that an appropriate social ordering is $b\ \hat{P}(n_0)\ a$, $a\ \hat{P}(n_0)\ c$, $b\ \hat{P}(n_0)\ c$, because it overrules only a small majority. So b - continued membership even if the C.A.P. cannot be changed - emerges as the social choice.

Suppose, on the other hand, that in society S_2, everybody prefers c to b, but only a small majority prefers a to c. That is, nobody at n_0 is willing to enter unless the C.A.P. is changed, and there is a large minority which is against entry whatever happens. Then one feels that an appropriate social ordering is $c\ \hat{P}(n_0)\ b$, $b\ \hat{P}(n_0)\ a$, $c\ \hat{P}(n_0)\ a$, (given the requirement that $\{\hat{P}(n)\}$ must be consistent, and given that $b\ \hat{P}(n_1)\ a$), because it overrules only a small majority. So c - refusal to enter at all - emerges as the social choice.

This example suggests that in deciding what an appropriate consistent dynamic social welfare function $\{\hat{C}(n)\}$ would be, one does want to take into account not only the social choice function $\{C(n)\}$, but also the individual choice functions $\langle C_i(n)\rangle_{i\in I(n)}$ from which each $C(n)$ was derived. That is, form (A) fails to capture all our ethical values, and so we must consider the more general form (B). From now on, it is form (B) which will be assumed.

In the end, then, a *dynamic constitution* is a set of mappings $\{h(n)|n \in N(X)\}$, defined on lists of sets of components of individual WCF's, $\langle\{C_i(n')\}(n)\rangle_{i\in I}$, and such that, if $\hat{C}(n) = h(n)\left(\langle\{C_i(n')\}(n)\rangle_{i\in I}\right)$ (each $n \in N(X)$), then $\{\hat{C}(n)\}$ is a consistent dynamic choice function.

Strictly speaking, this is not the same as (B), because in (B) only some individuals' choice functions were included as arguments of $h(n)$ - namely the choice functions of those individuals who belong to a set $I(n')$, for some n' *Fol* n. But, because of the restrictions placed on the choice functions $C(n)$ - namely, that unless there exists a node n' *Fol* n such that $i \in I(n')$, then $C(n)(A) = A$ for all $A \subseteq X(n)$ - this does not matter. The extra individuals have no interests which are relevant for the constitution $h(n)$, and so it makes no difference whether or not their choice functions are included as arguments of $h(n)$.

It is worth remarking that, if there is just a single individual, then the dynamic constitution reduces to a dynamic aggregation function, of the kind discussed in 4.10., provided that dynamically consistent choices are left alone.

Without a more detailed description of the economy and of the individuals in it, it is hard to say very much more about dynamic constitutions. Clearly we need to turn to applications and specific problems. This will be done, to a limited extent, in chapter 6.

5.4. Conclusions.

The purpose of this chapter has been to show that, in principle, it is possible to devise a welfare theory which takes full account of the dynamic features of the economy. In particular, we have considered ways of aggregating individual dynamic choice functions $\{C_i(n)\}$ into a consistent dynamic social WCF $\{\hat{C}(n)\}$. Two ways of doing this seem to suggest themselves:-

(1)(a) Modify each individual's dynamic choice function $\{C_i(n)\}$ so that it becomes a consistent dynamic WCF $\{\hat{C}_i(n)\}$. Because of consistency, only $\hat{C}_i(n_0)$, for the initial node n_0, really matters.

(b) Aggregate the individual WCF's $\hat{C}_i(n_0)$, via a "metastatic" constitution, into a social WCF $\hat{C}(n_0)$, so that $\{\hat{C}(n)\}$ is consistent.

(2)(a) For each n, aggregate the choice functions $C_i(n)$ of the individuals who are alive at node n into a social choice function $C(n)$ via a constitution $f(n)$, thus:-

$$C(n) = f(n)\left(\langle C_i(n)\rangle_{i\varepsilon I(n)}\right)$$

(b) Modify the dynamic choice function $\{C(n)\}$ so that it becomes a consistent dynamic social WCF $\{\hat{C}(n)\}$.

Neither of these methods, however, is satisfactory. (1) is deficient because individuals' inconsistent choices may reveal something which should be taken into account in aggregating their WCF's $\hat{C}_i(n_0)$ into a social WCF $\hat{C}(n_0)$. Also, the method runs into difficulties when the set of individuals is varying - particularly when it is varying endogenously. Why should the preferences of someone who can never live after node n has been passed affect the social choice at node n?

On the other hand, (2) is deficient because individuals' choices, which are aggregated into the social dynamic choice function $\{C(n)\}$, may reveal something which should be taken into account when $\{C(n)\}$ is modified so that it becomes consistent. This was seen in example 5.3.

Accordingly, a somewhat more complicated form of dynamic constitution is suggested. At each node n, the social WCF $\hat{C}(n)$ depends upon the individual choice functions $C_i(n')$ for all nodes n' following n, and for all individuals i who are alive at n'. Moreover, $\hat{C}(n)$ is constructed so that the dynamic choice function $\{\hat{C}(n)\}$ is consistent.

It probably bears repeating that we have, of course, neglected one of the most important aspects of dynamic welfare - namely uncertainty. This must wait for later work.

The following chapters consider rather more specialized problems. Chapter 6 looks at a number of more specific problems which, it seems, some economists believe are beyond the scope of welfare economics, or else seriously affect its usefulness. Thereafter, we shall explore the consequences of relaxing the assumption that there is an upper bound on the length of the branches of the underlying tree - in other words we shall consider infinite horizons.

Chapter 6.

SOME PROBLEMS IN DYNAMIC WELFARE ECONOMICS

6.1. Endogenous Tastes.

According to the orthodox view of welfare economics, tastes must be exogenous. Therefore, if tastes are moulded by education, by advertising, by receiving information, or in any other way, welfare economists have had little or nothing to say. A forceful statement of this view has recently been made by Gintis[1]. On the other hand, Weizsäcker has recently put forward some tentative suggestions for making welfare judgments when tastes are endogenous[2].

Now, the apparatus of dynamic choice functions which was constructed in chapter 4, and related to welfare in chapter 5, was expressly designed to allow for endogenous tastes. Rather more precisely, it allowed for changing welfare choice functions. If, however, we take the usual step of identifying an individual's welfare with his tastes, the two are equivalent.

It seems therefore that the orthodox view that tastes must be exogenous is mistaken. In fact, it rests on two mistakes. The first is to insist that welfare must be identified with tastes; once this requirement

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(1) See-Gintis (1971)(1972). Also Schoeffler (1952) shows that the point was realized long ago.

(2) See Weizsäcker (1971). Much earlier, Schoeffler (1952) also touched on this problem. Indeed, Schoeffler's suggestion that a change increases an individual's welfare provided the individual desires it both *ex ante* and *ex post*, is one which Weizsäcker adopts. It has been criticized by Harsanyi (1954). Weizsäcker realizes that the criterion can give rise to preference cycles.

is relaxed, changing tastes do not preclude welfare judgments. The second mistake is to believe that the individual's welfare choice function has to be fixed for all time; in fact, there is no reason why it should not be endogenously changing over time, although then the problem of finding a consistent dynamic social WCF has to be faced. This was the problem discussed in chapter 5. As should be apparent from section 4.10., the difficulty is in narrowing down the search for an appropriate social WCF: the difficulty is not, as was often thought, that one cannot find a suitable social WCF at all.

To summarize, the proposal here for handling endogenous tastes is to construct a consistent dynamic WCF in a certain way. For individuals, the method is based on intertemporal liberalism, as suggested in 4.10. For society, a consistent dynamic WCF is constructed by means of a dynamic constitution, as suggested in 5.3.

It is worth comparing the proposal for finding an individual's WCF with von Weizsäcker's discussion of individual welfare. It is clear that he had myopic individuals in mind.(1) In period t, a consumer's tastes depend upon what he consumed in period $t - 1$. But, in making his demands for period t, the consumer ignores the effect of his consumption on his future tastes. In fact, to simplify analysis, suppose that the individual's intertemporal welfare function takes the additive form $\sum_{t=1}^{\infty} u(q_{t-1}, q_t)$. Because the individual is myopic, however, his demands maximize $\sum_{t=1}^{\infty} u(q_0, q_t)$. In other words, he assumes his tastes are always the same. Weizsäcker assumes that the consumer faces the same budget set B each period, so that the consumption stream $(q_1, q_2, \ldots)$ is feasible

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(1) See Weizsäcker (1971), pp. 360 - 363.

iff $q_t \in B$ (each t). So, in effect, the individual chooses q_1 to maximize $u(q_0,q_1)$ subject to $q_1 \in B$[1]. Next period, by the same argument he chooses q_2 to maximize $u(q_1,q_2)$ subject to $q_2 \in B$ - and so on.

It follows that, if q' is the previous period's consumption, then current consumption q maximizes $u(q',q)$. So $u(\cdot,\cdot)$ is the *short-run utility function*.[2]

Weizsäcker's discussion concentrated on long-run demands. q^* is a long-run demand, given the fixed budget set B, if and only if q^* maximizes $u(q^*,q)$ with respect to q, subject to $q \in B$. In other words, q^* is a long-run demand if and only if, should the consumer have consumed q^* in the previous period, he continues to demand q^* in the current period.

An obvious question now, which Weizsäcker discusses in the two good case, is whether long-run demands maximize a long-run utility function. If they do, and $v(q)$ is the long-run utility function, then, by considering choice on finite sets, it follows that $v(\cdot)$ must be such that:-

$v(q) > v(q')$ iff there is a sequence $q^0, q^1, \ldots, q^n$ such that $q' = q^0$, $q = q^n$, and:-

$$u(q^m,q^{m+1}) > u(q^m,q^m) \quad (m = 0 \text{ to } n-1)^{(3)}$$

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(1) Of course, $\sum_{t=1}^{\infty} u(q_0,q_t)$ diverges, but one can use Weizsäcker's overtaking criterion - Weizsäcker (1965) - to argue that the path $q_t = q_1$ (all t), where q_1 is as suggested above, is optimal.

(2) cf. Weizsäcker's "short-run indifference curves" (1971), p. 352.

(3) Weizsäcker (1971), p. 352.

For such a $v(\cdot)$ to exist, $u(\cdot,\cdot)$ has to satisfy certain restrictions. In particular, the long-run strict preference relation P, defined by $q\,P\,q'$ iff $u(q',q) > u(q',q')$, must be acyclic[1].

Suppose there is a long-run utility function $v(\cdot)$. Does it tell us anything which is relevant for welfare judgments? Weizsäcker claimed that it did. He claimed that if $v(x) > v(y)$, then x is superior to y in the long run. Moreover, he claimed that if there is a sequence $q^o, q^1, \ldots, q^n$ such that $u(q^m, q^{m+1}) > u(q^m, q^m)$ $(m = 0$ to $n-1)$, then the sequence of consumption bundles $(q^o, q^1, \ldots, q^n)$ is superior to the *status quo* $(q^o, q^o, \ldots, q^o)$.[2] Are these claims correct?

For a general intertemporal welfare function of the form $\sum_{t=1}^{\infty} u(q_{t-1}, q_t)$, it is easy to see that they are not. x is superior to y in the long run if and only if $u(x,x) > u(y,y)$. It is quite possible that this is true, and also that $u(x,y) > u(x,x)$, $u(y,y) > u(y,x)$, so that $v(y) > v(x)$. Also, it is quite possible to have a sequence $(q_0, q_1, q_2, \ldots)$

for which $\quad u(q_0, q_0) = 0, \quad U(q_0, q_1) = 1$

and $\quad u(q_t, q_t) = -2, \; u(q_t, q_{t+1}) = -1 \quad (t = 1, 2, 3, \ldots)$

Then the sequence $(q_0, q_1, q_2, \ldots)$ is worse than $(q_0, q_0, q_0, \ldots)$ even though each change from q_{t-1} to q_t seems to be an improvement in the short run.

Nevertheless, these examples have assumed that the intertemporal welfare function is given. It may be more pertinent to ask whether long-run preferences can help in deciding what is the appropriate intertemporal WCF for a myopic individual. It is usual to identify individuals'

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(1) Weizsäcker (1971), p. 357.

(2) Weizsäcker (1971), p. 361.

preferences with their welfare; can we identify long-run preferences with long-run welfare?

Notice that any transform $\tilde{u}(q',q)$ of the short-run utility function $u(q',q)$, such that:-

$$\tilde{u}(q',q) = \phi(q', u(q',q))$$

where ϕ is increasing in u, corresponds to the same short-run preferences as u does. In particular,

$$\tilde{u}(q',q) = \psi(q') + u(q',q)$$

for *any* function $\psi(q')$, corresponds to the same short-run preferences. Suppose that $v(\cdot)$ is a long-run utility function corresponding to the short-run utility function $u(\cdot,\cdot)$. Define $\bar{u}(q',q) = v(q') + u(q',q) - u(q',q')$ for all q,q'. Then $\bar{u}(q',q)$ is another short-run utility function. And if we take $\sum_{t\,1}^{\infty} \bar{u}(q_{t-1},q_t)$ as the individual's intertemporal welfare function, then:-

(1) $v(x) > v(y)$ iff $\bar{u}(x,x) > \bar{u}(y,y)$, so $v(\cdot)$ indicates the individual's long-run welfare.

(2) If $u(x_t,x_{t+1}) > u(x_t,x_t)$ $(t = 0,1,2,\ldots)$, then $\bar{u}(x_t,x_{t+1}) > \bar{u}(x_t,x_t)$ and (because $v(x_{t+1}) > v(x_t)$), $\bar{u}(x_{t+1},x_{t+1}) > \bar{u}(x_t,x_t)$ (each t), so that $(x_0,x_1,x_2,\ldots)$ is superior to $(x_0,x_0,x_0,\ldots)$, according to the intertemporal welfare function.

This certainly does not specify the individual's intertemporal WCF uniquely, because by varying the form of the function $v(\cdot)$ which represents long-run preferences, we can vary the intertemporal WCF. And of course, $u(\cdot,\cdot)$ may be varied as well. Nevertheless, it does somewhat restrict the class of possible intertemporal WCF's, which may be helpful.

It should be recalled that the intertemporal WCF must embody value judgments. That an individual's interests are the same as his preferences is a value judgment. That a myopic individual's long-run interests are the same as his long-run preferences, as these are revealed through his short-run myopic choices, is also a value judgment, and one that may be harder to accept. So, although Weizsäcker's suggestion for making long-run welfare assessments when tastes are endogenous is objective, it is far from ethically indisputable.

Throughout the latter part of this section, it has been assumed that long-run preferences exist. In fact, this is a critical assumption, which seems very hard to justify. It is violated, for instance, if my eating fish today makes me less ready to eat fish and more ready to eat meat tomorrow, and if my eating meat today makes me less ready to eat meat and more ready to eat fish tomorrow. An interesting open question is whether the assumption becomes more plausible as the time period lengthens and we consider average demands over the period[(1)].

In conclusion, then, Weizsäcker's long-run preferences, if they exist, may help us in the search for a satisfactory intertemporal WCF for an individual, provided that we are happy to identify his long-run interests with his long-run preferences.

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(1) A question suggested to me by Avinash Dixit. It is related to Gorman's justification of quasi-concave utility functions - see Gorman (1957). Little also suggests looking at average demands over a long period as a way of handling people's desire for variety - Little (1957), pp. 38 - 39.

6.2. Overlapping and Unborn Generations.

Another problem which, it is sometimes thought, limits the applicability of welfare economics, is that many of the people affected by long term plans are not even born when the plans are first being worked out. But again, in our formal analysis in chapter 5, no such problem caused difficulty; so, at the very least, we have succeeded in sweeping any difficulties there might be conveniently out of sight. What precisely might these difficulties be?

One practical difficulty is glaringly obvious. In our planning, we often need to take account of the interests of individuals who are not yet even born, and it is hard to make predictions about these interests. The easiest course is to neglect these interests entirely, and no doubt such neglect does often occur. But this is a comment on the adequacy of practical planning procedures rather than on the limitations of welfare economics. Moreover, a very similar comment applies to planning which overlooks changing tastes. In fact, the problem of predicting the tastes and interests of the unborn seems little more difficult than that of predicting the future tastes and interests of the currently living. Of course, the currently living can be asked what their future tastes are likely to be,whereas the unborn cannot; but are individuals always good predictors of their own future tastes? And what are we to do when our policy choices affect tastes? No - surely no fundamental new difficulty arises because the tastes and interests of unborn individuals have to be predicted.

Much of theoretical welfare economics has been concerned with the relationship between allocations achieved in competitive markets and Pareto efficient allocations. Now, unborn individuals are obviously in no position to undertake forward transactions in an Arrow-Debreu market, and to imagine that other agents might act on their behalf is surely too far-fetched. So, is it not the case that a market allocation need no longer be Pareto efficient because only the currently living can trade in the market? As soon as the question is posed, it becomes clear that, when there are unborn generations, it is unrealistic to assume that there is a single market in which all trades are determined once and for all. Instead, there must be a time-sequence of markets, so that each generation has an opportunity to trade. There must also be a way of transferring purchasing power from one market to the next and *vice versa* - e.g. a bond market. Then, provided that agents have perfect foresight concerning prices, we have a system of markets which is formally equivalent to a single Arrow-Debreu market, in which all individuals participate, including those yet to be born. So the allocation achieved through such a system will be Pareto efficient, granted the usual assumptions. Conversely, a Pareto efficient allocation can be achieved through such a system of markets, provided that lump-sum transfers take place, and all the usual convexity and non-satiation assumptions are valid(1).

Notice that unborn generations can only cause difficulty if the lifetimes of some unborn individuals overlap with the lifetimes of some currently living individuals. Otherwise the unborn generations could be completely ignored. It would be possible to devise a plan which took account of interests of only currently living individuals and which did not specify decisions to be taken after the last currently living

(1) For further details, see Guesnerie (1971) for a discussion of the efficiency of an equilibrium in the sense of Radner (1972). Of course, Arrow (1953) contains a similar discussion for the two-period case, when there is also uncertainty.

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individual had died. This could be dynamically consistent; there would be no need to revise it until a new individual was born, but by that time all currently living individuals would have no interests anyway. In addition, there is no possibility for trade between currently living and as yet unborn individuals, so that an Arrow-Debreu market in which only currently living individuals participate loses nothing. So it is only the combination of overlapping and unborn generations which could really cause trouble.

Nevertheless, we can now see that the difficulties are the practical ones of predicting tastes, interests, and market prices, rather than theoretical ones. Of course, the neglect of uncertainty makes the theory we have developed unrealistic. But, as a first step, it at least shows how one could in principle handle some problems which have troubled economists.

6.3. Endogenous Population

When the number of people in a society is itself affected by the choice of plan - as it is, for example, as soon as measures to encourage birth control are being contemplated, or when a device, such as large numbers of crash barriers, which could reduce the number of deaths in road accidents, is rejected as too expensive - then economists have widely divergent views about what a proper objective is. For example, Rousseau's suggestion, that the well-being of a nation was best indicated by the number of its citizens,(1) would hardly be accepted by Malthus or by most of our contemporaries. Bentham would have us add the utilities of all the members of society, to determine that society's welfare; so the society is better or worse off for the presence of an extra individual according as that extra individual has a positive or negative utility. Some more modern writers have preferred to follow the Average Utility Principle - the welfare of a society is its average utility, or its Benthamite welfare divided by the number of heads. The society then is better or worse off for the presence of an extra individual according as that individual has a utility which is above or below the average(2).

All the above-mentioned criteria were essentially static. Is it better to be in stationary state A, with constant population m, rather than in stationary state B, with constant population n? That is the kind of question such criteria purport to answer. Yet such a question ignores the problems

(1) J.J.Rousseau - Du Contrat Social, Book III, ch. IX. His suggestion is somewhat qualified by a *ceteris paribus* assumption, but he seems not to have regarded this as an important qualification, and it is nowhere explained what precisely was being assumed equal.

(2) For a discussion of this principle and its history, see Rawls (1971) pp. 161 ff.

of transition from one steady state to another. More seriously, perhaps, they fail to capture most of the problem.

Suppose that individual i is born at time t. Quite apart from individual i himself, this obviously affects the welfare of many other individuals, notably i's parents, other relations, the friends i makes during his life, i's spouse, i's children, etc. One might call such effects *direct*. On the other hand, society has another individual to feed, another child to educate, another person who might be expected to contribute valuable labour services, etc. Such effects might be called *indirect*.

The same breakdown into direct and indirect effects is possible when an individual dies.

The direct effects clearly matter more to some individuals than to others. When an individual is born, or dies, benefits and costs are very unevenly distributed between the members of society. But it is the *direct* benefits and costs which are most unequally distributed. The *indirect* benefits and costs are much more evenly distributed through the community.

It is these indirect benefits and costs which seem most usually to be considered by economists. There may be two reasons for this. One is that it is thought much easier to attach an appropriate monetary value to the indirect effects - food, clothing, shelter, education, labour services can all be costed. Another reason is that it is so much easier to consider the effects of a change which influences everybody in much the same way: how does Mr. Average fare if we adopt a certain policy?

So that we can think a little more carefully about appropriate objectives, without worrying about distributional aspects of population changes, I propose to introduce a drastic simplifying assumption. This is that any population change affects all households in the same way. If one household gains a new member, they all do - if one household loses a member, they all do. Moreover, we shall assume that each household has a well defined welfare choice function, and that the household WCF's are to be aggregated into a social WCF. We shall not, however, overlook the direct effects of a population change, which would seem to be the most important.

To simplify still further, assume that each household and each individual lives for a single period. This commits us to non-overlapping generations, which is unrealistic, but makes analysis very much simpler. The conclusions which would be reached in a more general framework are very similar. Let n_{t-1} be the number of individuals alive at the start of period t. In period t, they enjoy their single period of life and have n_t children, who form the next generation. Let x_t be the other variables in period t which are affected by the choice of economic policy, and which affect individuals' welfare. Suppose that a person alive in period t has a WCF corresponding to the utility function $u(x_t, n_{t-1}, n_t)$. In the special case where the individuals' only interests in population *per se* are in the size of their own families, the utility function becomes $u(x_t, n_t/n_{t-1})$, but there is no need to assume this special form.

Suppose for a moment that the population stream is fixed. Suppose too that no choice is possible after time T. Then one is concerned with streams of economic variables $(x_1, x_2, \ldots, x_T)$. Given that all individuals are identical, and if we follow the argument of section 3.6. for an additive Bergson social welfare function, it seems natural just to add individual utilities to give:-

$$W(n_0,\ldots,n_T)(x_1,\ldots,x_T) = \sum_{i\varepsilon I} v_i(n_0,\ldots,n_T;\ x_1,\ldots,x_T)$$

where I is the set of individuals, and v_i is i's utility function. Now assume, as in section 5.2., that i only has interests when he is alive. Define the indicator variable

$$\varepsilon_{it} = \begin{cases} 1 \text{ (if } i \text{ is alive in period } t) \\ 0 \text{ (otherwise).} \end{cases}$$

Given that all individuals have the utility function $u(x_t,n_{t-1},n_t)$ for the period t when they are alive, it is natural to take v_i proportional to $\varepsilon_{it}\ u(x_t,n_{t-1},n_t)$.

Then, since $\sum_{i\varepsilon I} \varepsilon_{it} = n_{t-1}$ (each t), W becomes:-

$$W(n_0,\ldots,n_T)(x_1,\ldots,x_T) = \phi(n_0,\ldots,n_T) \sum_{t=1}^{T} n_{t-1}\ u(x_t,n_{t-1},n_t)$$

Here $\phi(\cdot)$ is the constant of proportionality, which in general depends upon the whole population stream. It is worth noting straight away that we have excluded the following welfare function which has been seen in much of the optimal growth literature, in its infinite horizon form:-

$$W = \sum_{t=1}^{T} u(x_t).\ ^{(1)}$$

This W is the sum over time of *per caput* utility. It has been rejected simply because it seems incompatible with static Bergson social welfare functions, as considered in 3.6.(2)

Up to now, the population stream has been taken as fixed. If the function $\phi(n_0,\ldots,n_T)$ is properly chosen, then, even when it is allowed

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(1) See Koopmans (1965) p. 254, for a discussion of why he , at least, adopts it.

(2) This point has been emphasized by Mirrlees.

to vary, the BSWF must have the same form.

So far, just choice at time 1 has been considered. Now we must consider dynamic choice, and find a consistent dynamic social WCF. Notice first that, if at time s, choice corresponds to the BSWF:-

$$W_s(n_0,\ldots,n_T)(x_s,\ldots,x_T) = \phi(n_0,\ldots,n_T)\sum_{t=s}^{T} n_{t-1}\, u(x_t,n_{t-1},n_t)$$

then consistency is assured. However, in this form of welfare function, the population s generations back affects choice at time s; as s becomes large, this becomes implausible. So we shall introduce the assumption that $W_s(\cdot)$ depends only on $(n_{s-1},\ldots,n_T)$ and derive restrictions on the form of the function $\phi(\cdot)$, using an argument similar in form to Strotz's.(1)

Our extra assumption, then, is that past population size does not directly affect the social WCF at time s. This assumption does not deny that past population has some effect. For example, the number of Georgian houses extant in Britain is of interest to some individuals, at least, and is related to the size of the population in Georgian times, amongst other things. But such effects are allowed for in the list of economic variables x_t. In a similar way, memories of earlier days of less congested life can be allowed for in the list of variables x_t - although such memories are not strictly economic variables. Preferences for family size may depend upon the number of brothers and sisters one grew up with. This would violate the assumption, but could be allowed for without much extra difficulty.

Given our assumption, choice at time s, corresponds to a Bergson social welfare function of the form:-

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(1) See Strotz (1957) p. 172.

$$W_s(n_{s-1},\ldots,n_T)(x_s,\ldots,x_T) = \phi_s(n_{s-1},\ldots,n_T)\sum_{t=s}^{T} n_{t-1}\, u(x_t,n_{t-1},n_t)$$

At time $s+1$, the BSWF is:-

$$W_{s+1}(n_s,\ldots,n_T)(x_{s+1},\ldots,x_T) = \phi_{s+1}(n_s,\ldots,n_T)\sum_{t=s+1}^{T} n_{t-1}\, u(x_t,n_{t-1},n_t)$$

For consistency of choices of $(n_{s+1},\ldots,n_T)$ and $(x_{s+1},\ldots,x_T)$, it is necessary that

$$\phi_s(n_{s-1},\ldots,n_T)\sum_{t=s+1}^{T} n_{t-1}\, u(x_t,n_{t-1},n_t)$$

and

$$\phi_{s+1}(n_s,\ldots,n_T)\sum_{t=s+1}^{T} n_{t-1}\, u(x_t,n_{t-1},n_t)$$

should be proportional, as functions of the choice variables. So there must be a constant $\alpha_s(n_{s-1},n_s)$, for each n_{s-1} and n_s, such that $\phi_s(n_{s-1},\ldots,n_T) = \alpha_s(n_{s-1},n_s)\ \phi_{s+1}(n_s,\ldots,n_T)$.

Starting from $\phi_T(n_{t-1},n_T)$, and working backwards recursively, it follows that:-

$$\phi_s(n_{s-1},\ldots,n_T) = \{\Pi_{r=s}^{T-1}\ \alpha_r(n_{r-1},n_r)\}\ \phi_T(n_{T-1},n_T).$$

So $W(n_0,\ldots,n_T)(x_1,\ldots,x_T)$

$$= \{\Pi_{r=s}^{T-1}\ \alpha_r(n_{r-1},n_r)\}\ \phi_T(n_{T-1},n_T)\sum_{t=1}^{T} n_{t-1}\, u(x_t,n_{t-1},n_t).$$

Notice that the objective:-

$$W = \sum_{t=1}^{T} n_{t-1}\, u(x_t,n_{t-1},n_t)$$

emerges as a special case[1].

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[1] This form of objective has been used by Meade (1955), pp. 82-93, 573-577 and Dasgupta (1969), except that u did not depend directly on n_{t-1} and n_t. Phelps (1966) and Votey (1969),(1972) have discussed implications of parental preferences regarding family size. Mirrlees (1972a), section 7, advocates, for a single period, a welfare function which, when there is certainty, more than one period, and no overlap of generations, becomes

$$\sum_{t=1}^{T} n_{t-1}\, u(x_t,\ n_t/n_{t-1})$$

But the above form is incompatible with that prescribed by the average utility principle, which gives the following BSWF at time s:-

$$W_s = \sum_{t=s}^{T} n_{t-1}\, u(x_t, n_{t-1}, n_t) \Big/ \sum_{t=s-1}^{T} n_t$$

This form has been excluded by our assumption that the BSWF at time s is independent of population before time s, and dynamic consistency. The following example shows much more directly how the average utility principle is dynamically inconsistent:-

Example 6.3.

Suppose that the average utility principle gives rise to the following form of welfare function at times 1 and 2:-

$$W_1 = \sum_{t=1}^{3} n_{t-1}\, u(x_t) \Big/ \sum_{t=1}^{3} n_{t-1}$$

$$W_2 = \sum_{t=2}^{3} n_{t-1}\, u(x_t) \Big/ \sum_{t=2}^{3} n_{t-1}$$

Suppose that there are two paths:-

$$y = (n_0, x_1, n_1, x_2, n_2, x_3) = (1, \bar{x}_1, 1, \bar{x}_2, 1, \bar{x}_3)$$

$$y' = (n'_0, x'_1, n'_1, x'_2, n'_2, x'_3) = (1, \bar{x}_1, 1, \bar{x}_2, 2, \bar{x}_3)$$

where $u(\bar{x}_1) = 3, \quad u(\bar{x}_2) = 0, \quad u(\bar{x}_3) = 1.$

Then $\quad W_1(y) = 4/3, \; W_1(y') = 5/4$

and $\quad W_2(y) = 1/2, \; W_2(y') = 2/3$

so there is an obvious inconsistency.

It is, however, possible to maintain the spirit of the average utility principle, which I take to be homogeneity of degree zero in the population stream. The simplest way of ensuring that each W_s is homogeneous of degree zero is to take:-

$$W_s = (n_{T-1})^{-1} \sum_{t=s}^{T} n_{t-1}\, u(x_t, n_{t-1}, n_t).$$ [1]

But the precise form of the functions $\alpha_T(\cdot)$ and $\phi_T(\cdot)$ is a deep ethical question which will not be pursued here.

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(1) If T really is the end of the world, then $n_T = 0$, presumably.

6.4. The Time Horizon.

So far, throughout the discussion of dynamic choice and welfare, it has been assumed that there is an upper bound T to the length of the branches of the underlying tree X. This means that the number of separate decisions to be made sequentially is bounded. Thus, if economic policy can be changed once a year, then the assumption is that no further decisions can be made after T years. Really, then, we are assuming that after T years, nothing matters. Accordingly, T is called the *horizon*.

In practice, the horizon is uncertain, and there is no guarantee that no decisions will still be outstanding when T is reached. If T is not the true horizon, the original choices, made on the assumption that nothing matters after time T, are likely to be inappropriate.

Suppose that the true underlying tree is X. The effect of assuming that the horizon is T is to truncate the decision tree to $X|T$. The members of this truncated tree - $x|T$ - are equivalence classes within X. Two options a and b of X are regarded as equivalent if they fail to part until after time T - i.e. if $a(t) = b(t)$ $(t = 0$ to $T)$[1]. Given the horizon T, there is a choice function defined on $X|T$. Corresponding to this choice function is another choice function C^T, defined on X, which has the property that if $a|T = b|T$ (i.e. $a(t) = b(t)$ $(t = 0$ to $T))$, then $a|T \in C^T(A)$ iff $b|T \in C^T(A)$ (each $A \subseteq X$).

If the horizon turns out to come after T, this will be realized at some later date - certainly by time T itself. When the realization occurs, choices after time T will be planned. It is extremely unlikely

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(1) Notice that we are assuming that all branches are truncated after the same number of periods. This makes analysis much simpler without losing any real generality.

that these new choices will be the same as the original choices – or rather, failures to choose. By time T, at the latest, choices will be given by a choice function C, defined on X, which differs from C^T. Consequently, the effect of truncating the true decision tree is naive dynamic choice.[1]

Whether this naivety is of importance is not clear. To know whether it is important, we have to assess the effect of naivety. To do this, we must consider the alternative, in which dynamic choice is consistent. This involves extending the time horizon, recognizing that decisions still have to be made after time T. But where does the extension of the time horizon end? It seems that no *finite* extension is certain to make dynamic choice consistent. For this reason, we are led to consider infinite-horizon dynamic choice. This is the subject of the remaining four chapters of this work.

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(1) For an extensive discussion of the horizon problem, see Graaff (1957), ch. 6. Although Graaff knew of Ramsey (1928), (see p. 101n.) he nowhere considers an infinite horizon as a way of surmounting the problem.

Chapter 7.

INFINITE HORIZON CHOICE.

7.1. Introduction

In chapters 4, 5 and 6, it was assumed, as a rule, that the underlying tree X - for the dynamic choice functions being considered - was bounded. That is, there was a finite upper bound H on the number of periods before any branch came to an end - i.e., for any $t \geq H$, and any branch $x \in X$, $X(x(t)) = \{x\}$. Such a number H will be called a *horizon* for the underlying tree. If there is no such bound, the horizon is said to be *infinite*.

The assumption that there is a finite horizon was convenient because it avoided some tricky analytical problems. Nevertheless, as was seen in section 6.4., this assumption is unsatisfactory in many practical policy questions, when policies with extremely long-term consequences are being considered. For, although it seems certain that there is a true finite horizon H, we are extremely uncertain about the precise value of H. If we guess H, and our guess is too small, then eventually our plan will be revised. This is just like naive choice. On the other hand, if our guess of H is too large, naivety is avoided, but the choice we make is still unlikely to be appropriate. There are costs both to underestimating and to overestimating H.

An apparently obvious and superficially attractive way to deal with this problem is to recognize explicitly that there is uncertainty about the true horizon, and to treat the choice problem as one of choice under uncertainty. Suppose we do treat the problem in this way, and suppose that choice under uncertainty corresponds to maximizing the expected value of a von Neumann-Morgenstern utility function. Then we are faced with specifying probabilities p_H for the different possible values of the horizon H.

Now, it seems difficult to specify such probabilities with any confidence. When do we start allowing p_H to be nonzero? Given that the horizon comes after H, what is the conditional probability that the horizon is $H + 1$? It would be pleasant to avoid having to answer such questions. We may be able to do so, if our answers have little effect on the plan we choose. But then the simplest course is to assume, in effect, that there is no uncertainty - which brings us back to the original problem of guessing the horizon.

But perhaps the choice of plan is sensitive to the probabilities p_H. Then we have a case of a choice situation in which the specification of probabilities is both hard and important. The task has to be undertaken. Nevertheless, even when the probabilities are specified, not all the difficulties are over. It could happen that there is still no finite bound to the possible length of the horizon, so that we still have an infinite horizon choice problem.

So, unless we are sure both that there is an upper bound to the true horizon H, and also that our choice of plan is sensitive to the probabilities p_H, there is a case to be made for studying underlying trees whose branches are unbounded in length. Indeed, unless we are sure that there is an upper bound to the true horizon, we are forced to consider such underlying trees . The study of choice functions on such infinite trees is the subject of the remaining four chapters of this work. The remainder of this chapter considers infinite horizon choice quite generally, and prepares the ground for later chapters.

7.2. Finite and Infinite Horizon Choice.

To explore the theory of general dynamic choice functions on infinite underlying trees would be quite hard, but it is fortunately not really necessary. At least, it is not necessary if we maintain the view that there is, in reality, a finite horizon H, but that it is unknown and so we are led to consider infinite horizons. For, if we maintain this view, then we can relate the infinite horizon dynamic choice function to a sequence of finite horizon dynamic choice functions.

Let X be the infinite-horizon underlying tree. For each possible horizon H, we have an equivalence relation E_H on X, defined as follows:-

$$x \, E_H \, y \quad \text{iff } x_t = y_t \ (t = 0 \text{ to } H)$$

Thus, two branches are equivalent if they have failed to separate by time H. Denote a typical equivalence class by $x|H$. Let the set of such equivalence classes be $X|H$ - this is a truncation of the tree X.

Assume that, for each H, there is a coherent and consistent dynamic choice function on $X|H$. Because of consistency, it is enough to specify the choice set $C|H(A)$ for each $A \subseteq X|H$. Then $C|H$ is effectively an ordinary choice function on the underlying set $X|H$. Corresponding to $C|H$ is a choice function C^H on the underlying set X. To define C^H, first define, for each $A \subseteq X$:-

$$A|H = \{x|H \mid \exists y \in A \ \text{ s.t. } \ x|H = y|H\}$$

and then define:-

$$C^H(A) = \{x \in A \mid x|H \in C|H(A|H)\}$$

Then $C^H(\cdot)$ is obviously a coherent choice function on the underlying set X, and one which ignores the distinction between branches which part only after time H.

Now, the choice function which is truly appropriate is C^H, for some horizon H. But, because the horizon is unknown, the choice function which is actually adopted may be an infinite horizon one. Let it be C^*. To be satisfactory, C^* must give choices which are nearly as good as those which C^H gives, for the true H. It seems that the most likely way of achieving this - assuming some sort of continuity - is to have C^* give approximately the same choices as C^H, for the true H. Here, the true H is likely to be large: this is why infinite horizon choice is considered. So, recognizing that H is unknown, we are looking for a choice function C^* which gives approximately the same choices as C^H, provided that H is large; evidently, then C^* will be some sort of limit of the choice functions C^H, as $H \to \infty$.

As a convenient shorthand in the study of limits, we shall adopt the following terminology for properties of infinite sequences. Let $\langle \alpha^H \rangle$ be any infinite sequence of mathematical objects, and let Π denote any property which the objects α^H may have. Then, say that α^H *eventually has property* Π if there exists H_0 such that, whenever $H \geq H_0$, α^H has property Π; say that α^H *frequently has property* Π if there is an infinite subsequence $\langle \alpha^{H_n} \rangle$ of the sequence $\langle \alpha^H \rangle$ such that, for each n, α^{H_n} has property Π.[1]

Notice that α^H frequently has property Π if and only if it is false that α^H eventually has property (not Π). To illustrate the terminology, consider the definition of a limit of a sequence of points $\langle x^H \rangle$ in a topological space X : x^* is a limit of $\langle x^H \rangle$ if and only if, for any neighbourhood N of x^*, x^H eventually belongs to N.

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(1) See Kelley (1955), p. 65.

7.3. Limits of Choice Functions.

There is no single straightforward definition of the limit of a sequence of choice functions, as will become apparent in the following discussion. Different definitions fit different circumstances. Moreover, none of the definitions specify a unique limit, in general; instead, there are two limits, one of which gives an upper bound on choice sets, and the other a lower bound. There is a parallel here with the unanimity principles of chapter 3: indeed, limits are derived on a kind of "eventual unanimity" principle.[1]

The discussion will be facilitated if we concentrate on special cases, characterized by particular assumptions. Two special cases are considered: it is only these special cases which are relevant for the remainder of this work.

We are concerned with the limit C^* of a sequence of choice functions $\langle C^H \rangle$ all defined on the same underlying set X. The assumptions in the first special case are:-

(1) Each choice function C^H is ordinal.

(2) C^* is quasi-ordinal.

To assume that C^* is ordinal is too stringent. For example, if it happens that, for some choice functions C_1 and C_2,

$$C^H(A) = \begin{cases} C_1(A) & \text{(each } A \subseteq X) \ (H \text{ odd}) \\ C_2(A) & \text{(each } A \subseteq X) \ (H \text{ even}) \end{cases}$$

(where $C_1(A) \neq C_2(A)$, for some $A \subseteq X$)

then any limit C^* should be some amalgam of C_1 and C_2. In chapter 3, and the discussion of unanimity principles, we found it reasonably straightforward

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(1) See footnote (2) of p.7/6.

to amalgamate ordinal choice functions into a quasi-ordinal choice function. But to amalgamate ordinal choice functions into an choice function is much more difficult; for example, Arrow's Impossibility Theorem shows that we must violate his axiom of Independence of Irrelevant Alternatives, in general[1]. The parallel is close enough to suggest that there are likely to be similar difficulties in finding an ordinal limit to a sequence of choice functions, and that we should do well to look only for quasi-ordinal limits, at least to start with. Also, at the end of the chapter, there is a brief discussion of ordinal limits.

Even if we do confine ourselves to quasi-ordinal limits, and make the assumptions (1) and (2) above, we shall not generally succeed in specifying the limit choice function C^* uniquely . Instead, for each $A \subsetneq X$, we shall specify two sets $C_1^*(A)$ and $C_2^*(A)$ such that, for the "true" limit choice set $C^*(A)$ (whatever it may be), $C_1^*(A) \subsetneq C^*(A) \subseteq C_2^*(A)$. Here the sets $C_1^*(A)$ and $C_2^*(A)$ will be specified so that the following assumption is certainly satisfied:-

(3) If there exists sets B_1, $B_2 \subsetneq A$ with B_1 non-empty, such that $B_1 \subsetneq C^H(A) \subsetneq B_2$ eventually, then $B_1 \subsetneq C^*(A) \subsetneq B_2$.

This would seem to be no more than a reasonable restriction on our definition of the limit of a sequence of choice functions. For, surely, if the members of B_1 are chosen for all large H, they should be chosen in the limit as the horizon tends to infinity. And if only members of B_2 are chosen for all large H, then only members of B_2 should be chosen in the limit as the horizon tends to infinity[2].

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(1) See Arrow (1963).

(2) Notice too the parallel with unanimity principles. This explains the allusion at the start of this section to "eventual unanimity".

We can ensure that (3) is always satisfied by specifying the lower and upper limits $C_1^*(A)$ and $C_2^*(A)$ so that:-

(4) If there exist sets B_1, $B_2 \subseteq A$, with B_1 non-empty, such that $B_1 \subseteq C^H(A) \subseteq B_2$ eventually, then $B_1 \subseteq C_1^*(A) \subseteq C_2^*(A) \subseteq B_2$.

This property will be used to define $C_1^*(A)$ and $C_2^*(A)$, for each $A \subseteq X$.

Suppose that, for each H, the ordinal choice function C^H corresponds to the preference relations P^H and R^H.

Now we shall look for an appropriate limiting strict preference relation P^* and for a corresponding proper weak preference relation R^*, as defined in section A.8 of the appendix. Recall that R^* must have the following two properties:-

(5) R^* is reflexive and transitive (but not necessarily connected)

(6) $x\ P^*\ y$ iff ($x\ R^*\ y$ and not $y\ R^*\ x$).

In fact, there are two alternative definitions which arise naturally, and we shall have to consider both of them.

First, suppose that $x\ R^H\ y$ eventually. Then $\{x,y\} \supseteq C^H(\{x,y\}) \supseteq \{x\}$ eventually, and so, by (4) above, $x \in C_1^*(A)$. Therefore, we define R_1^* so that:-

(7) $x\ R_1^*\ y$ iff $x\ R^H\ y$ eventually.

Because each R^H is reflexive and transitive, (5) is certainly satisfied; (6) will also be satisfied if and only if P_1^* is defined so that:-

(8) $x\ P_1^*\ y$ iff $[(x\ R^H\ y$ eventually$)$ and not $(y\ R^H\ x$ eventually$)]$

iff $[(x\ R^H\ y$ eventually$)$ and $(x\ P^H\ y$ frequently$)]$

Obviously, P_1^* is asymmetric and transitive, so that it is a strict preference relation corresponding to a quasi-ordinal choice function.

So, as a first attempt, we might take as upper and lower limits the set of maximal and optimal elements in each set A, defined as in section A.8 of the appendix:-

$M_1^*(A) = \{x \in A \mid y \in A$ implies not $y \; P_1^* \; x\}$

$O_1^*(A) = \{x \in A \mid y \in A$ implies $x \; R_1^* \; y\}$

However, as the following example shows, there exists a choice function C^* satisfying (3) and (4), for which $C^*(A) \subseteq M_1^*(A)$ (each $A \subseteq X$) is false.

Example 7.3.1.

Suppose that $X = \{a,b\}$ and that

$a \; P^H \; b$ (H odd)

$a \; I^H \; b$ (H even)

Then $a \; P_1^* \; b$, so that $M_1^*(X) = \{a\}$

But, if we define $C^*(X) = \{a,b\}$, it is clear that (3) is satisfied.

So, second, suppose that $x \; P^H \; y$ eventually.

Then $\{x\} \subseteq C^H(\{x,y\}) \subseteq \{x\}$ eventually, and so, by (4), $C_2^*(\{x,y\}) \subseteq \{x\}$

Therefore, we define P_2^* so that:-

(9) $x \; P_2^* \; y$ iff $x \; P^H \; y$ eventually.

Obviously, P_2^* is transitive and asymmetric, because each P^H is, and so P_2^* is a strict preference relation corresponding to a quasi-ordinal choice function.

Notice that $P_2^* \subseteq P_1^*$, and that the two relations may not be identical - in example 7.3.1., $a \; P_1^* \; b$, but it is false that $a \; P_2^* \; b$.

It is hard to characterize any proper weak preference relation R_2^* corresponding to P_2^*. We do know, however, that one such relation must exist; as in section A.8 of the appendix, we can define $R_2^* = P_2^* \cup I_2^*$, where:-

$$x\ I_2^*\ y \quad \text{iff} \quad \left[(x\ P_2^*\ z \ \text{ iff } \ y\ P_2^*\ z) \text{ and } (z\ P_2^*\ x \ \text{ iff } \ z\ P_2^*\ y)\right]$$

Fortunately, no characterization of R_2^* is required for our purposes. What concerns us is the following result, which shows that we have succeeded in putting the appropriate bounds on the limit choice function C^*.

Theorem 7.3.2.

Let $\langle C^H \rangle$ be an infinite sequence of ordinal choice functions on the underlying set X. Let C^H correspond to the preference relations P^H and R^H $(H = 1, 2, \ldots)$. Let C^* be a quasi-ordinal choice function on X, corresponding to the preference relations P and R (where R is connected, but may be improper - so $x\ R\ y$ iff (not $y\ P\ x$)). Then C^* has the property that:-

(3) If there exist sets B_1, $B_2 \subseteq A$, with B_1 non-empty, such that $B_1 \subseteq C^H(A) \subseteq B_2$ eventually, then

$$B_1 \subseteq C^*(A) \subseteq B_2$$

if and only if $R \supseteq R_1^*$ and $P \supseteq P_2^*$

Proof

(A) Suppose that C^* is a quasi-ordinal choice function, and that $R \supseteq R_1^*$, $P \supseteq P_2^*$. Suppose also that $B_1 \subseteq C^H(A) \subseteq B_2$ eventually, where B_1 is non-empty.

(i) Suppose $x \in A - B_2$. Then $x \notin C^H(A)$ eventually. Suppose $y \in B_1$. Then $y \in C^H(A)$ eventually. As each C^H is ordinal, it follows that $y\ P^H\ x$ eventually, so that $y\ P_2^*\ x$. By hypothesis, $P \supseteq P_2^*$, and so $y\ P\ x$, which implies that $x \notin C^*(A)$

So we have shown that $C^*(A) \subseteq B_2$.

(ii) Suppose $x \in B_1$. Then $x \in C^H(A)$ eventually. So, given any $y \in A$, $x\ R^H\ y$ eventually, and $x\ R_1^*\ y$. It follows that $x\ R\ y$, and so $x \in C^*(A)$.

So we have shown that $B_1 \subseteq C^*(A)$.

(B) Suppose that C^* is a quasi-ordinal choice function, and that, for each $A \subseteq X$, if $B_1 \subseteq C^H(A) \subseteq B_2$ eventually, where B_1 is non-empty, then $B_1 \subseteq C^*(A) \subseteq B_2$.

(i) Suppose that $x\ R_1^*\ y$ - i.e. $x\ R^H\ y$ eventually. Then $\{x\} \subseteq C^H(\{x,y\}) \subseteq \{x,y\}$ eventually, and so $\{x\} \subseteq C^*(\{x,y\})$. Therefore $x\ R\ y$.

(ii) Suppose that $x\ P_2^*\ y$ - i.e. $x\ P^H\ y$ eventually. Then $C^H(\{x,y\}) = \{x\}$ eventually, and so $C^*(\{x,y\}) = \{x\}$. Therefore $x\ P\ y$.

The subscripts 1 and 2 are now unnecessary. We shall always consider R_1^* as the weak preference relation, and P_2^* as the strict preference relation, so write P^* for P_2^*, and R^* for R_1^*. Thus:-

$x\ R^*\ y$ iff $x\ R^H\ y$ eventually

$x\ P^*\ y$ iff $x\ P^H\ y$ eventually.

Define $O^*(A)(= O_1^*(A)) = \{x \in A | y \in A \text{ implies } x\ R^*\ y\}$

$M^*(A)(= M_2^*(A)) = \{x \in A | y \in A \text{ implies not } y\ P^*\ x\}$

$O^*(A)$ is the set of optimal elements of A, and $M^*(A)$ is the set of maximal elements of A.

It is important to notice that R^* is *not* necessarily the weak preference relation corresponding to P^*; it is possible that $x\ P^*\ y$ is false even though $x\ R^*\ y$ and not $y\ R^*\ x$. Indeed, example 7.3.1. is an instance of this possibility. Nevertheless R^* and P^* do have many of the properties of associated preference relations:-

(10) (a) P^* is asymmetric and transitive

(b) R^* is reflexive and transitive

(c) If $x\ P^*\ y$, then $x\ R^*\ y$ and not $y\ R^*\ x$.

Theorem 7.3.2. tells us that, if C^* is defined so that (3) is satisfied, then:-

(11) $O^*(A) \subseteq C^*(A) \subseteq M^*(A)$ (each $A \subseteq X$).

This is the result which will be extensively used in later work.

Notice that $M^*(\cdot)$ is a quasi-ordinal choice function on the underlying set X; it corresponds to the strict preference relation P^*, which is transitive. In general, however, $O^*(\cdot)$ is not a choice function, because $O^*(A)$ may well be empty even if A is finite, as the following example shows[1]:-

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Example 7.3.3.

$X = \{x,b\}$ is the underlying set.

Suppose $a\ P^H\ b$ (H even)

$b\ P^H\ a$ (H odd)

Then neither $a\ R^*\ b$ nor $b\ R^*\ a$ is true, and so $O^*(X)$ is empty.

So far, throughout this section, we have maintained the assumption that each C^H is ordinal. But, in chapter 9 later, we shall be examining double limits of choice functions C_T^H. Since a limit choice function, in general, is not ordinal, we are therefore forced to consider the limit of a sequence of quasi-ordinal choice functions. We shall not give a full discussion of this second special case, but shall proceed directly

(1) We cannot use example A.8.2. of appendix 1, because R^* does not correspond to P^* in the way required.

to state the following result. It is a weakened form of theorem 7.3.2., and is proved in the same way.

Theorem 7.3.4.

Let $\langle C^H \rangle$ be an infinite sequence of quasi-ordinal choice functions on the underlying set X. Let C^H correspond to the preference relations P^H and R^H $(H = 1,2,\ldots)$. Let C^* be a quasi-ordinal choice function on X, corresponding to the preference relations P and R (where R is connected, but may be improper).

Define the relations R^*, P^* as follows:-

$x \, R^* \, y$ iff $x \, R^H \, y$ eventually

$x \, P^* \, y$ iff $x \, P^H \, y$ eventually

Then, C^* has the following property only if $R \supsetneq R^*$ and $P \supsetneq P^*$:-

If there exist sets B_1, $B_2 \subsetneq A$, with B_1 non-empty, such that $B_1 \subset C^H(A) \subsetneq B_2$ eventually, then

(a) $B_1 \subsetneq C^*(A)$

(b) $C^*(A) \subsetneq B_2$

Conversely, if $R \supsetneq R^*$, then part (a) of the above property is satisfied.

The following example shows that even if $P \supsetneq P^*$, part (b) of the property stated may not be satisfied:-

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Example 7.3.5.

Let $X = \{x,y,z\}$. Suppose that the preference relations R^H, I^H, P^H are as follows:-

$x \, P^H \, y, \quad x \, I^H \, z, \quad y \, I^H \, z \quad (H \text{ odd})$

$y \, P^H \, x, \quad z \, P^H \, x, \quad z \, P^H \, y \quad (H \text{ even}).$

Then, if $P = P^*$, we have $C^*(X) = \{x, y, z\}$.

But, for all H, $\{z\} \subseteq C^H(X) \subseteq \{x, z\}$.

Despite this difficulty, we shall continue to assume that, for all $A \subseteq X$,

$$O^*(A) \subseteq C^*(A) \subseteq M^*(A)$$

where $O^*(\cdot)$ and $M^*(\cdot)$ are defined precisely as they were before. The difficulty is that our upper bound $M^*(A)$ on $C^*(A)$ may be too large; this should be borne in mind before elements of the maximal set are accepted as suitable choices. But it is a problem which has to be faced anyway, even when each C^H is ordinal.

7.4. Infinite Horizon Choice and Additive Utility.

Suppose that each finite horizon choice function $C|H$ is defined on a product set $X|H = \Pi_{t=1}^{H} X_t$, and that it corresponds to an additive utility function $\sum_{t=1}^{H} u_t^H(x_t)$. Then C^H is defined on the infinite product set $X = \Pi_{t=1}^{\infty} X_t$, and corresponds to the same additive utility function.

Given these assumptions, the preference relations R^* and P^* assume a form which has been much used in the theory of optimal growth. Moreover, it is possible to modify them so that the limit choice function has an extra important property - it is continuous, in a certain sense, provided that each utility function $u_t^H(\cdot)$ is continuous. This continuity is achieved at a certain cost in terms of satisfying property (3) of the previous section. But a slightly modified form of this property will still be satisfied.

Before turning to infinite horizon choice, however, we shall note that it is usually possible to simplify the additive utility functions to just $\sum_{t=1}^{H} u_t(x_t)$, where $u_t(x_t)$ is independent of H.

For let H' be any horizon beyond H - i.e. $H' > H$. Then C^H corresponds to $\sum_{t=1}^{H} u_t^H(x_t)$, and $C^{H'}$ to $\sum_{t=1}^{H'} u_t^{H'}(x_t)$. Now $C^{H'}$ defines a choice function $C^{H'}|H$ on the underlying set $X|H = \Pi_{t=1}^{H} X_t$ and this choice function corresponds to $\sum_{t=1}^{H} u_t^{H'}(x_t)$. So, in effect, there are two choice functions on $X|H$ - C^H and $C^{H'}|H$. Now, to get an additive utility function corresponding to $C^{H'}$, we have to assume that choices on $X|H$ are independent of what happens after time H. Let us extend this assumption so that, if the horizon is not before time H, then choices on $X|H$ are independent of the horizon. In particular, $C^{H'}|H$ and C^H must be identical choice functions, and

$\sum_{t=1}^{H} u_t^H(x_t)$ and $\sum_{t=1}^{H'} u_t^{H'}(x_t)$ must be equivalent utility functions. Since this is true for every H and H' such that $H' > H$, it must be the case that there is a sequence of utility functions $u_t(\cdot)$ such that, for each H, C^H corresponds to $\sum_{t=1}^{H} u_t(x_t)$ - where $u_t(\cdot)$ is independent of H.[(1)]

Now we can return to infinite horizon choice. It is convenient to introduce the following notation:-

(1) $W^H(x) = \sum_{t=1}^{H} u_t(x_t)$

Thus, W^H is a utility function on the underlying set X, and W^H corresponds to C^H.

In this case, the preference relations of section 7.3. assume a familiar form; they become versions of the overtaking criterion first propounded by Weizsäcker[(2)]. For:-

(2) $x\ P^*\ y$ iff $W^H(x) > W^H(y)$ eventually

$$(\text{iff } \exists H_0\ \forall H \geq H_0 \quad \sum_{t=1}^{H} \left[u_t(x_t) - u_t(y_t)\right] > 0)$$

and:-

(3) $x\ R^*\ y$ iff $W^H(x) \geq W^H(y)$ eventually

$$(\text{iff } \exists H_0\ \forall H \geq H_0 \quad \sum_{t=1}^{H} \left[u_t(x_t) - u_t(y_t)\right] \geq 0)$$

Although it is both familiar and also extremely useful, it is possible to question whether this criterion needs a minor modification. Consider the following example:-

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(1) Brock (1970b) and Koopmans (1972b) reach this conclusion in essentially the same way.

(2) Weizsäcker (1965) p. 85. Weizsäcker worked in continuous rather than in discrete time, but his criterion is obviously the continuous time version of P^*, as defined here.

Example 7.4.1.

$X = \prod_{t=1}^{\infty} X_t$ is the underlying set.

Let $A = \{a^1, a^2, \ldots\}$, a countably infinite set.

Let $u_t(a_t^n) = \begin{cases} -\ 2^{-n-1} & (t=1) \\ \ \ 2^{-n-t} & (t>1) \end{cases}$

so that $W^H(a^n) = -\ 2^{-n-H}$

Then $a^m \; P^* \; a^n$ iff $m > n$. Therefore $M^*(A)$ is empty. Nevertheless, for each $a^n \in A$, $\lim_{H\to\infty} W^H(a^n) = 0$,

Also, consider any option $x(\varepsilon) \in X$, where:-

$$x_t(\varepsilon) = a_t^1 \quad (t>1)$$

and $\quad u_1(x_1(\varepsilon)) = u_1(a_1^1) + \varepsilon \quad (\varepsilon>0)$

Then, no matter how small ε may be, we have:-

$$x(\varepsilon) \; P^* \ldots \; P^* \; a^n \; P^* \; a^{n-1} \; P^* \ldots \; P^* \; a^2 \; P^* \; a^1.$$

Example 7.4.1. is troubling because it is not clear that we should regard a^m as better than a^n when $m > n$. And, perhaps more troubling, P^* fails an obvious test of continuity. For we could define $x(\varepsilon)$ in example 7.4.1. above so that $x(0) = a^1$, and so that $x(\varepsilon)$ is a continuous function of ε. Then, if $n > 1$, we have $a^n \; P^* \; x(0)$, and yet $x(\varepsilon) \; P^* \; a^n$ for all $\varepsilon > 0$. This kind of discontinuity is likely to lead to empty choice sets more often than is comfortable or reasonable.

Accordingly, the Weizsäcker overtaking criterion will be modified so that it has some basic continuity properties at least. This forces us to consider continuous preference relations.

Let X be any topological space, and let R, P be respectively weak and strict preference relations defined on X. Say that:-

(4) R is *continuous* (as a weak preference relation) if, whenever $x^n \to x$ and $y^n \to y$ as $n \to \infty$, and $x^n\ R\ y^n$ eventually, then $x\ R\ y$.

(5) P is *continuous* (as a strict preference relation) if, whenever $x^n \to x$ and $y^n \to y$ as $n \to \infty$, and $x\ P\ y$, then $x^n\ P\ y^n$ eventually.

Both these definitions are standard - at least when R is identical to (not P), and when R is a weak ordering[1]. But they remain useful even when R is neither connected nor transitive.

Now, we can hardly expect the preference relations R^* and P^*, defined in (2) and (3) above, to be continuous, unless the utility functions $u_t(\cdot)$ are continuous. So assume that each X_t is a topological space, and that $u_t : X_t \to \mathcal{R}$ is a continuous function.

Returning to example 7.4.1., the issue raised is a property which might be called "continuity within finite product subspaces". Formally, a *finite product subspace* of X is a subset

$$\Pi_{t=1}^{H} X_t \times \Pi_{t=H+1}^{\infty} \{\bar{x}_t\}$$

$\left(\text{for some } \langle \bar{x}_t \rangle_{t=H+1}^{\infty}\right)$, this subset being endowed with the product topology. A sequence $\langle x^n \rangle$ of points of X *converges to x^* within a finite product subspace* if:-

(a) $\exists H\ \forall t > H, \quad x_t^n = x_t^* \ (n = 1,2,\ldots)$

(b) $\forall t \le H, \quad x_t^n \to x_t^*$ as $n \to \infty$.

So, in a finite product subspace, only a finite number of components x_t can change from a single value $\bar{x}_t$; for convergence within a finite product subspace, the sequence $\langle x^n \rangle$ must lie in some finite product subspace.

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(1) i.e. reflexive, connected and transitive. For equivalent definitions, see Koopmans (1972a), p. 60 - for example.

Now we can define continuity of preference relations, along the lines of (4) and (5) above. Say that R and P are *continuous within finite product subspaces* if,

whenever (i) $x^n \to x$ within a finite product subspace

(ii) $y^n \to y$ " " " "

then (a) if $x^n \; R \; y^n$ eventually, then $x \; R \; y$

(b) if $x \; P \; y$, then $x^n \; p \; y^n$ eventually.

It is evident that this form of continuity is weaker than that of continuity in the product topology on X. Yet example 7.4.1. shows that R^* and P^* fail to be continuous within finite product subspaces. It is easy to modify the definitions of the infinite-horizon preference relations so that they become continuous, in this sense. Indeed, define $\bar{R}$ and $\bar{P}$ as follows:-

(6) $x \; \bar{R} \; y$ iff $\liminf_{H \to \infty} \left[W^H(x) - W^H(y) \right] \geq 0$

(7) $x \; \bar{P} \; y$ iff $\liminf_{H \to \infty} \left[W^H(x) - W^H(y) \right] > 0$

Notice that these definitions are unambiguous provided that each W^H is well-defined up to some common positive affine transformation - i.e. a transformation of the form

$$W^{H\,\prime} = \alpha \, W^H + \beta_H \quad \text{(each } H\text{)}$$

where α is a positive scalar. In particular, the definitions are unambiguous when each W^H is additive. But, more generally, ambiguity is quite possible.

Evidently, $\bar{R}$ is reflexive and transitive, and if $x \; R^* \; y$, then $x \; \bar{R} \; y$. Also, $\bar{P}$ is asymmetric and transitive, and if $x \; \bar{P} \; y$, then $x \; P^* \; y$. That $\bar{R}$ and $\bar{P}$ are continuous within finite product subspaces is immediate from the following:-

Lemma 7.4.2.

If x^n converges to x, and y^n converges to y, both within a finite product subspace, then

$$\liminf_{H \to \infty} \left[W^H(x^n) - W^H(y^n)\right] \text{ tends to}$$

$$\liminf_{H \to \infty} \left[W^H(x) - W^H(y)\right], \text{ as } n \to \infty,$$

provided that each $W^H(\cdot)$ is continuous.

Proof

By hypothesis, there exists T such that, whenever $t > T$, $x_t^n = x_t$ and $y_t^n = y_t$. Therefore, if $H > T$

$$W^H(x^n) - W^H(y^n)$$

$$= \sum_{t=1}^{H} \left[u_t(x_t^n) - u_t(y_t^n)\right]$$

$$= \sum_{t=1}^{T} \left[u_t(x_t^n) - u_t(y_t^n)\right] + \sum_{t=T+1}^{H} \left[u_t(x_t) - u_t(y_t)\right]$$

$$= \left[W^T(x^n) - W^T(y^n)\right] + \left[W^H(x) - W^H(y)\right] - \left[W^T(x) - W^T(y)\right]$$

So $\liminf_{H \to \infty} \left[W^H(x^n) - W^H(y^n)\right]$

$$= \liminf_{H \to \infty} \left[W^H(x) - W^H(y)\right] + \left\{\left[W^T(x^n) - W^T(x)\right] - \left[W^T(y^n) - W^T(y)\right]\right\}$$

$$\to \liminf_{H \to \infty} \left[W^H(x) - W^H(y)\right] \text{ as } n \to \infty,$$

because $W^T(\cdot)$ is certainly continuous.

It is worth noticing that, if $W^H(x) \to W^*(x)$ and $W^H(y) \to W^*(y)$ as $H \to \infty$, then:-

$$x \bar{R} y \quad \text{iff} \quad W^*(x) \geq W^*(y)$$

$$x \bar{P} y \quad \text{iff} \quad W^*(x) > W^*(y).$$

In particular, if, for each $x \in X$, there exists $W^*(x)$ such that $W^H(x) \to W^*(x)$ as $H \to \infty$, then $W^*(\cdot)$ is a continuous utility function corresponding to $\bar{R}$ and $\bar{P}$. It is easy to construct an example in which $W^*(\cdot)$ does *not* correspond to R^* or, indeed, to P^* - see example 7.4.4. below.

It is convenient to have names for the relations $\bar{P}$ and $\bar{R}$. So, if $x \bar{P} y$, say that x *strictly overtakes* y; if $x \bar{R} y$, say that x *catches up to* y[1].

Notice that, even if $x \bar{R} y$ and not $y \bar{R} x$, it may be false that $x \bar{P} y$:-

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Example 7.4.3.

Suppose that $x, y \in X$ and that:-

$$W^H(x) = \begin{cases} 1 & (H \text{ odd}) \\ 0 & (H \text{ even}) \end{cases}$$

$$W^H(y) = 0 \quad (\text{all } H).$$

Then $\liminf_{H \to \infty} \left[W^H(x) - W^H(y)\right] = 0$, so that

$x \bar{R} y$, but not $x \bar{P} y$.

Nevertheless,

$$\liminf_{H \to \infty} \left[W^H(y) - W^H(x)\right] = -1,$$

so that $y \bar{R} x$ is false.

One would like to prove a result similar to theorem 7.3.2. But the property:-

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(1) See Gale (1967), p.3., McFadden (1967), p. 28, and Brock (1970a), p. 275. I have used "strictly overtakes" instead of "overtakes by a finite amount".

(8) If B_1 is non-empty, and $B_1 \subseteq C^H(A) \subseteq B_2$ eventually, then

$B_1 \subseteq C^*(A) \subseteq B_2$

would certainly have to be modified, as can be seen from example 7.4.4. below. Since an appropriate modification is not intuitively clear, nor is it obvious what is the weakest possible modification consistent with continuity, we shall go no further into this.

Example 7.4.4.

Suppose that $x, y \in X$, and that

$$W^H(x) = 2^{-H}, \quad W^H(y) = 0 \quad (H = 1,2,\ldots)$$

(a) Then $W^*(x) = \left(\lim_{H\to\infty} W^H(x)\right) = W^*(y) = 0.$

So we have an example here in which $x\ P^*\ y$, although $W^*(x) = W^*(y)$ – which shows that P^* may fail to correspond to a utility function, and also that P^* is not identical to $\bar{P}$ (because $x\ \bar{I}\ y$).

(b) Here if $A = \{x,y\}$, and $B_1 = B_2 = \{x\}$, then $B_1 \subseteq C^H(A) \subseteq B_2$ for all H, and yet if C^* corresponds to the strict preference relation $\bar{P}$, $C^*(A) = A$, contradicting (8).

(c) Suppose $B_1 \subseteq C^H(A)$ eventually, implies that $B_1 \subseteq C^*(A)$. Even so, here we have an example in which, if $C^*(A) = \{x\}$, this property is satisfied, and yet $y \notin C^*(A)$ although, for all $z \in A$, $y\ \bar{R}\ z$.

Finally, define:-

$\bar{O}(A) = \{x \in A \mid y \in A \text{ implies } x\ \bar{R}\ y\}$

$\bar{M}(A) = \{x \in A \mid y \in A \text{ implies (not } y\ \bar{P}\ x)\}$

$\bar{O}(A)$ is the set of optima in A, and $\bar{M}(A)$ is the set of maxima in A. We assume now that, for each $A \subseteq X$, the infinite horizon choice set $C^*(A)$ satisfies:-

$$\bar{O}(A) \subseteq C^*(A) \subseteq \bar{M}(A).$$

Whereas $\bar{M}(\cdot)$ is a choice function, $\bar{O}(\cdot)$ may not be, because for some finite sets A, $\bar{O}(A)$ may be empty.

All these remarks are similar to those at the end of section 7.3., and the justifications are virtually identical.

Later, too, we shall have cause to examine double limit choice functions when the choice functions C_T^H correspond to additive utility functions. But this examination is best left to chapter 9, which is where it is first used.

7.5. Dynamic Consistency of Infinite Horizon Choice.

In the earlier sections of this chapter, we have concentrated on limits C^* of a sequence of choice functions $\langle C^H \rangle$ defined on the common underlying set X. But, in doing this, we have rather lost sight of our original objective, which was to define a choice function C^* on an infinite-horizon underlying *tree* X, given the sequence of choice functions $\langle C^H \rangle$ defined on finite-horizon trees. As soon as we start to consider choice on underlying trees, we have to consider dynamic choice. So far, we have not done this, in order to concentrate on a single issue. Now it is time to re-examine the infinite-horizon choice functions of the earlier sections, to see how well they fare in dynamic choice situations.

Suppose now that, for each horizon H, there is a dynamic choice function $\{C^H(n)\}$ on the underlying tree X. Then each component $C^H(n)$ is a choice function on $X(n)$. Given the sequence $\langle C^H(n) \rangle$, with the node n fixed, there is, in general, at least one limit choice function $C^*(n)$ defined on $X(n)$. Consequently, there is at least one dynamic choice function $\{C^*(n)\}$ with the property that each of its components is a limit of the sequence of the corresponding components of the finite-horizon dynamic choice functions $\{C^H(n)\}$. $\{C^*(n)\}$ is then said to be a *limit dynamic choice function*.

If it is accepted that infinite-horizon underlying trees are to be considered because the horizon is unknown but distant, then it should now be clear that what is required as a dynamic welfare choice function is a limit dynamic choice function. Because any dynamic welfare choice function is expected to be consistent, this raises the question of whether a limit dynamic choice function is consistent. What will be shown here is that, if each $\{C^H(n)\}$ is consistent - as surely it must be if it is the dynamic welfare function which is appropriate when the horizon is H - then there exists a limit dynamic choice function $\{C^*(n)\}$ which is consistent. This is true whichever kind of limit is being considered - the limits of section 7.3. or those of 7.4. A stronger result can hardly be expected, because the definition of a limit choice function is partly ambiguous, and so it is possible to pick the limits $C^*(n)$ of different components perversely, and to get inconsistency as a result.

To show that there is a consistent limit dynamic choice function, much use is made of section A.9. of appendix 1, and of theorem A.9.4 in particular. So first, the following result, noteworthy in itself, is proved.

Lemma 7.5.1.

Suppose that, for each H, $\{C^H(n)\}$ is a consistent dynamic choice function on X, such that $C^H(n)$ is ordinal, for each $n \in N(X)$. Let $R^H(n)$, $P^H(n)$ be the weak and strong preference relations corresponding to $C^H(n)$. Define $R^*(n)$ and $P^*(n)$ as in section 7.3.:-

If $\quad a,b \in X(n)$

then $\quad a\ R^*(n)\ b$ iff $a\ R^H(n)\ b$ eventually

and $\quad a\ P^*(n)\ b$ iff $a\ P^H(n)\ b$ eventually.

Then $\{R^*(n)\}$ and $\{P^*(n)\}$ are consistent dynamic preference relations.

Proof

First, it is evident that, since $\{C^H(n)\}$ is consistent, so are $\{R^H(n)\}$ and $\{P^H(n)\}$, because if $a,b \in X(n)$, then:-

$$a \in C^H(n)\ (\{a,b\}) \text{ iff } a \in C^H(n_0)(\{a,b\})$$

(where n_0 is the initial node of X).

Now, if $a,b \in X(n)$

then $a \; R^*(n) \; b$

iff $a \; R^H(n) \; b$ eventually

iff $a \; R^H(n_0) \; b$ eventually (because $\{R^H(n)\}$ is consistent)

iff $a \; R^*(n_0) \; b$.

So $\{R^*(n)\}$ is consistent. The proof that $\{P^*(n)\}$ is consistent is virtually identical.

An analogous result can be proved for the limit preference relations of section 7.4. But first, notice that if $\{C^H(n)\}$ is consistent, where $C^H(n)$ represents choice on $X(n)$, and if $C^H(n)$ corresponds to an additive utility function, then this additive utility function must have the form:-

$$W^H(n)(x) \equiv \sum_{t=s+1}^{H} u_t(x_t)$$

where n is the node reached at time s along branch x. This is because $C^H(n_0)$ corresponds to

$$W^H(n_0)(x) \equiv \sum_{t=1}^{H} u_t(x_t),$$

and, whenever $x \in X(n)$, the initial sequence $(x_1,x_2,\ldots,x_s)$ is fixed. Thus $W^H(n)$ depends only on the time s at which node n is reached, and the temporal utility functions u_t are completely independent of n. Moreover, each component $C^H(n)$ of the dynamic choice function also depends only on the time s at which n is reached, and so $\{C^H(n)\}$ can be written simply as $\{C_s^H\}$. Now:-

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Lemma 7.5.2.

Suppose that, for each H, $\{C^H(n)\}$ is consistent and corresponds to $W_s^H(x) = \sum_{t=s+1}^{H} u_t(x_t)$, where n is the node reached at time s along branch x. Define $\{\bar{R}(n)\}$ and $\{\bar{P}(n)\}$ as in section 7.4.:-

If $a, b \in X(n)$

then $a \;\bar{R}(n)\; b$ iff $\liminf_{H \to \infty} \left[W_s^H(a) - W_s^H(b)\right] \geq 0$

and $a \;\bar{P}(n)\; b$ iff $\liminf_{H \to \infty} \left[W_s^H(a) - W_s^H(b)\right] > 0$

Then $\{\bar{R}(n)\}$ and $\{\bar{P}(n)\}$ are consistent dynamic preference relations on X.

Proof

The proof is as straightforward as that of lemma 7.5.1., and so is omitted.

Given these two results, and theorem A.9.4. of appendix 1, it is immediately clear that the following two existence theorems are true:-

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Theorem 7.5.3.

If the assumptions of lemma 7.5.1. are satisfied, and if, whenever $A \subsetneq X(n)$:-

$O^*(n)(A) = \{x \in A \mid y \in A$ implies $x \; R^*(n) \; y\}$

$M^*(n)(A) = \{x \in A \mid y \in A$ implies not $y \; P^*(n) \; x\}$

then there is a consistent dynamic choice function $\{C^*(n)\}$ such that, whenever $A \subset X(n)$, $O^*(n)(A) \subsetneq C^*(n)(A) \subsetneq M^*(n)(A)$.

Theorem 7.5.4.

If the assumptions of lemma 7.5.2. are satisfied, and if, whenever $A \subsetneq X(n)$:-

$$\bar{O}(n)(A) = \{x \in A \mid y \in A \text{ implies } \quad x\,\bar{R}(n)\,y\}$$

$$\bar{M}(n)(A) = \{x \in A \mid y \in A \text{ implies not } \ y\,\bar{P}(n)\,x\}$$

then there is a consistent dynamic choice function $\{\bar{C}(n)\}$ such that, whenever $A \subseteq X(n)$, $\bar{O}(n)(A) \subseteq \bar{C}(n)(A) \subseteq \bar{M}(n)(A)$.

If these theorems were not true, the applicability of infinite horizon choice would be circumscribed. But, reassuring as they may be, there is little guidance as to how to construct a consistent dynamic choice function, in general. This gives significance to two special cases, in which it is obvious what an appropriate consistent dynamic choice function is. The cases are:-

(1) When $O^*(n)$, or $\bar{O}(n)$, is a choice function, for each n.

(2) When, for the true feasible set A, and for each node $n \in N(A)$, $M^*(n)(A(n))$, or $M(n)(A(n))$, is a singleton.

Also, if A is the true feasible set, and for each $n \in N(A)$, $O^*(n)(A(n))$, or $\bar{O}(n)(A(n))$, is non-empty, then at least we have a non-empty set of options, which is a subset of the appropriate choice set, whatever that may be. So it is possible to select an option in the knowledge that the selection is appropriate, granted all the other assumptions of course.

Consequently, in chapter 10 there will be some discussion of these special cases.

7.6. Conclusions

This chapter has considered some problems which arise because the time-horizon is not known. It is commonly thought that the time-horizon is likely to be very distant. Consequently, it may be appropriate to consider infinite-horizon choice - to pretend that the world will last for ever. This may be because no bound can be put on the horizon, or it may be because infinite-horizon choice is a good and convenient approximation to what would be chosen anyway, were there a bound to the horizon.

Whichever of these two reasons applies, the most appropriate infinite-horizon choice set would seem to be the limit, as the horizon H tends to infinity, of the choice set which is proper if H is the true horizon. It turns out that such limit choice sets are not always well-defined. Nevertheless, in section 7.3., two limit choice functions were defined - giving upper and lower bounds to the choice set.

Section 7.4. considered the important special case in which each finite horizon choice function C^H corresponds to an additive utility function:-

$$W^H(x) = \sum_{t=1}^{H} u_t(x_t)$$

In this case, the limit choice functions of section 7.3. correspond to Weizsäcker's overtaking criterion. But, in order to get continuous preference relations - continuous within finite product subspaces at least - the preference relations $\bar{R}$ and $\bar{P}$, weak and strong respectively, were defined as follows:-

$$x \bar{R} y \text{ if } \liminf_{H \to \infty} \left[W^H(x) - W^H(y) \right] \geq 0$$

$$x \bar{P} y \text{ if } \liminf_{H \to \infty} \left[W^H(x) - W^H(y) \right] > 0$$

It is this special case, and these preference relations, which are considered in later chapters.

One of the reasons for considering infinite horizon choice is to avoid the dynamic inconsistency which is bound to arise if too short a time horizon is assumed initially. Accordingly, it is important to be sure that infinite horizon choice is dynamically consistent. In section 7.5., the most that could be done was to show that there exists a consistent dynamic infinite horizon choice function. It was not, however, successfully characterized, and it seems that the most which can be found easily are upper and lower bounds to the choice sets. Because the choice set is clear when the upper bound gives a single element choice set, uniqueness is an important question, which is explored in later chapters.

Now we are in a position to consider infinite horizon choice in the kind of problem to which it has been most applied by economists - the problem of planning capital accumulation. This is the subject of the next chapter, and of chapters 9 and 10, in effect.

Chapter 8.

ON OPTIMAL CAPITAL ACCUMULATION.

8.1. Capital Accumulation and Welfare.

The choice of appropriate policies governing capital accumulation is perhaps the principal problem in intertemporal welfare economics. For it is very clearly an economic problem, and it is also one in which long-run considerations seem extremely important. It is also the problem to which infinite-horizon choice theory has been most applied. Indeed, it is the problem to which, in effect, the whole of optimal growth theory is addressed. Moreover, many of the important developments in infinite horizon choice theory were designed to deal with this problem[(1)].

This chapter will examine how welfare economics can be applied to the choice of a plan of capital accumulation, and a major difficulty will be observed.

Policies concerning capital accumulation matter chiefly because the affect the opportunities which individuals have for consumption, and for work or leisure. Accordingly, any such policy has to be justified indirectly, by referring to the consumption etc. which results from such a policy. This is not to deny that capital accumulation may have other results in which individuals have direct interests - pollution, the aesthetic merits of buildings and bridges, etc.[(2)] So, in considering the choice of such policies, the most appropriate underlying set is the set of all possible streams of capital, consumption, labour, leisure, etc.

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(1) Examples are Ramsey (1928), Samuelson and Solow (1956), Weizsäcker (1965). Other developments are considered in chapter 9. Also, there is some consideration of infinite horizon choice in the mathematical statistics literature. See Blackwell (1965), (1967), Łoś (1967).

(2) An analysis allowing capital accumulation to have direct effects on individual welfare occurs in Kurz (1968).

Suppose that there is a finite number, m, of different kinds of capital good which the economy is capable of producing at one time or another. Let $k(t)$ denote the m-vector of capital good quantities which are available at the end of period t for use in period $t+1$.[1] Let $c_i(t)$ denote the vector of net quantities of goods received by individual i in period t. Here, $c_i(t)$ includes not only consumption, but also the supplies of services and endowments which consumer i provides; we make the usual convention of treating supplies *by* i as negative supplies *to* i. If individual i is not alive in period t, define $c_i(t) = 0$. Then write $c(t)$ for the list $\langle c_i(t)\rangle_{i \in I}$. Although, as we have just emphasized, $c(t)$ includes lists of supplies of labour etc., it is convenient to refer to $c(t)$ simply as "consumption", and this will be done in the following sections.

Given the capital stock vector $k(t-1)$ at the start of period t, the choice of $c(t)$ evidently affects what capital stock vector $k(t)$ can be left over for use in period $t+1$. Let $Y(t)$ denote the set of possible triples $\big(k(t-1), k(t), c(t)\big)$ for period t - it is the technology at time t.

The economy is constrained at time t, in the choice of consumption (and current resource usage) $c(t)$, not only by technology. Other constraints arise because people need food, rest, warmth, and shelter, and can only provide limited services. In addition, there may be constraints on the distribution of goods and services between individuals. Such constraints may be technological, if it is a question of providing adequate transport. But there is also the problem of creating institutions to arrange such distribution. For example, if allocation has to take place through

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(1) As always, time is assumed to be discrete because that makes the theoretical analysis easier.

markets, and if no lump sum transfers of purchasing power are possible, there is a constraint on how goods can be allocated. In general, such constraints depend upon the capital stock $k(t-1)$. Let $Q(t,k(t-1))$ be the set of feasible $c(t)$, given the capital stock $k(t-1)$. If there were just one individual, $Q(t,k(t-1))$ would be the individual's consumption set. With many individuals, it is the Cartesian product of the individual's consumptions sets, less those allocations of consumption which are ruled out because they cannot be effected by the institutions of the economy.

A development path is a time-sequence of capital and consumption vectors, $\langle k(t),c(t)\rangle_{t=1}^{\infty}$. The initial capital stock $k(0)$ is given by history. Then a path $\langle k(t),c(t)\rangle_{t=1}^{\infty}$ is feasible if and only if:-

(a) $\left(k(t-1),k(t),c(t)\right) \in Y(t) \quad (t = 1,2,\ldots)$

(b) $c(t) \in Q(t,k(t-1)) \quad (t = 1,2,\ldots)$

Assume now that for each finite horizon H, there is a social welfare choice function C^H. Assume too that this choice function C^H corresponds to a Bergson social welfare function which is doubly additive - additive across both time periods and individuals. Thus C^H corresponds to:-

$$V^H \equiv \sum_{i\in I} \sum_{t=1}^{H} v_i^H \left(c(t),k(t-1),k(t),t\right).$$

This form of function allows for those consumption externalities which occur within a single period. Also, the capital stock at the beginning and end of each period is allowed to be of concern to individuals during that period. In section 7.4, it was argued that each function $v_i^H(\cdot)$ should be independent of H. Accepting this argument, C^H corresponds to:-

$$V^H \equiv \sum_{i\in I} \sum_{t=1}^{H} v_i \left(c(t),k(t-1),k(t),t\right).$$

So the problem of choosing an optimal policy for capital accumulation is reduced to that of infinite-horizon choice with an additive objective function for each finite horizon - i.e. the kind of problem discussed in section 7.4. However, it is possible to simplify the problem to some extent, at least in a formal sense.

8.2. On Choosing Streams of Capital.

As in section A.10 of appendix 1, we can introduce partial choice. Here, we can choose a stream of capital first, and leave the choice of a stream of consumption until later. Indeed, for each horizon H, there is a well-defined social welfare choice function D^H, on the underlying set whose members are feasible streams of capital $k|H = \langle k(t)\rangle_{t=1}^{H}$. And, by theorem A.10.2., D^H corresponds to the utility functions:-

$$W^H(k|H) = \max \{V^H(c|H,k|H) \mid (c|H,k|H) \text{ feasible}\}.$$

For any given pair $k(t-1),k(t)$, define:-

$$v\big(k(t-1),k(t),t\big) = \max \{\textstyle\sum_{i\varepsilon I} v_i\big(c(t),k(t-1),k(t),t\big) \mid (k(t-1),k(t),c(t)) \;\varepsilon\; Y(t),\; c(t) \;\varepsilon\; Q\big(t,k(t-1)\big)\}$$

Then evidently,

$$W^H(k|H) = \textstyle\sum_{t=1}^{H} v\big(k(t-1),k(t),t\big).$$

We have succeeded in recasting the choice problem as one of a choice of a stream of capital. There is no need to consider consumption explicitly at all, since it has no direct effect on the dynamics of the problem. Also, notice that the additive form of the Bergson social welfare function has been preserved[(1)]. The recasting is a considerable simplification for theoretical analysis.

[(1)] The problem of optimal growth has been considered in this way by, amongst others, Gale (1967). Also McKenzie (1968) has a similar approach, replacing the utility function $v(k(t-1),k(t),t)$ by a set of feasible triples $(k(t-1),k(t),u(t))$ - $u(t)$ being utility at time t. Presumably, this was to emphasize the similarity of consumption turnpikes with additive utility, to the more usual turnpikes which arise from maximizing a purely terminal objective.

From now on, the underlying set X will be the set of feasible capital streams $\langle k(t)\rangle_{t=1}^{\infty}$. To accord with earlier notation, a typical capital stream will be denoted by x, and its component vectors, each period, by $x(t)$ (rather than $k(t)$). $x|H$ will denote $\langle x(t)\rangle_{t=1}^{H}$. The choice function D^H for the horizon H corresponds to the *H-horizon utility function* $\sum_{t=1}^{H} v(x(t-1),x(t),t)$.

The underlying set - or feasible set - X, can be found as follows. For each period t, define $X(t)$ as the set of input/output pairs of capital stock vectors $(x(t-1),x(t))$ which are both technologically possible, and also satisfy the constraints which arise because of the individuals in the economy. Thus:-

$$X(t) = \{(x(t-1),x(t)) \mid \exists c(t) \in Q(t,\ x(t-1)) \text{ s.t. } (x(t-1),x(t),c(t)) \in Y(t)\}$$

Then, recalling the condition for feasibility in section 8.1., it is clear that:-

$$X = \{x = \langle x(t)\rangle_{t=1}^{\infty} \mid (x(t-1),x(t)) \in X(t) \quad (t = 1,2,\ldots), \quad x(0) = \bar{x}(0)\}$$

where $\bar{x}(0)$ is the historically given initial capital stock.

The assumptions made so far have little economic content and, apart from that of additivity, they have been extremely weak. Now come a number of extra assumptions, of which most are commonplace in the theory of optimal growth.

8.3. Assumptions Concerning the Feasible Set.

Let $x = \langle x(t) \rangle_{t=1}^{\infty}$ be any capital stream. Here, each $x(t)$ is a vector of capital good quantities. We assume:-

(A.1) For each t, $x(t)$ is a member of a finite-dimensional Euclidean space, and $x(t) \geq 0$.[1]

There are two economic assumptions underlying (A.1):-

(a) For each period t, there are only a finite number of different kinds of capital good which can be produced by the end of the period, ready for use in period $t + 1$.

(b) No negative quantity of any capital good is possible.

(a) is almost self-explanatory. But notice that it rules out a continuum of quality variations. In particular, if goods are distinguished by location, amongst other things, then the number of possible locations must be finite. So the assumption is restrictive, although hardly seriously restrictive.

Assumption (b) is clearly appropriate for physical capital goods. But an open economy may hold negative amounts of certain types of financial capital, by being in debt to foreigners. To allow for this, $x(t)$ could be defined as the excess of capital over the minimum possible negative amount - if there is a limit to borrowing. Alternatively, the constraint $x(t) \geq 0$ can be replaced by $x(t) \geq - m(t)$, for some $- m(t) \leq 0$. Neither alteration has a significant effect.

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(1) The following notation is used for vector inequalities:-

$x \geqq y$ iff $x_i \geq y_i$ (each i)

$x \geq y$ iff ($x \geqq y$ and $x \neq y$)

$x > y$ iff $x_i > y_i$ (each i).

(A.2) For each t, $X(t)$ is a closed set, in the Euclidean topology.[1]

$X(t)$ is the set of feasible input/output pairs at time t. If $X(t)$ is a finite set, then it must be closed anyway. More generally, $X(t)$ will be an infinite set. But this is a mathematical abstraction designed to surmount the problems which arise when there are indivisibilities. $X(t)$ will be an infinite approximation to the true set of input/output pairs - e.g. it may be the convex hull. It is always possible to make this approximation a set which is closed in the Euclidean topology, since the difference between a set and its closure can hardly affect the validity of the approximation.

For each capital stock vector a and each time-period t, define the set $B(a,t)$ as the set of output stocks which are feasible, given that the input stock is a . Thus:-

$$B(a,t) = \{b \mid (a,b) \in X(t)\}$$

Define also, for each pair a,b of stock vectors, and for each pair s,t of time periods satisfying $s < t$, the set $X(a,b,s,t)$ as the set of capital streams $x = \langle x(t)\rangle_{t=1}^{\infty}$ which are feasible, and for which $x(s) = a$, $x(t) = b$. Thus:-

$$X(a,b,s,t) = \{x \in X \mid x(s) = a,\ x(t) = b\}.$$

(A.2) implies that all the sets $B(a,t)$ and $X(a,b,s,t)$ are closed, and also that X is closed in the product topology.[2] In addition, the

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(1) The Euclidean topology is the one induced by the Euclidean metric $d(a,b) = \{\sum_{i=1}^{n} (a_i - b_i)^2\}^{\frac{1}{2}}$ where $a = (a_1,\dots,a_n)$, $b = (b_1,\dots,b_n)$ and n is the dimension of the space.

(2) In the product topology, a sequence $x^n \to x$ as $n \to \infty$ if and only if, for each t, $x^n(t) \to x(t)$ as $n \to \infty$. This can serve as a definition - see Kelley (1955), pp. 88-93.

correspondencies, $B(\cdot,t)$ and $X(\cdot,\cdot,s,t)$ are both upper semi continuous, because they have closed graphs.[1]

(A.3) $B(a,t)$ is compact (each a,t).

Because of (A.2) this means no more than that the set of stock vectors b which can be produced from a given stock a at time t, within a single period, is bounded. Since production needs physical power, and physical power is available in limited quantities at any one time - given the capital stock a, at any rate - the assumption seems quite reasonable. It has important consequences, as will now be seen.

Whenever $s \leq t$, define:-

$$F(s,a,t) = \{b \;\Big|\; \exists \langle x(r)\rangle_{r=s-1}^{t} \text{ s.t. } x(s-1) = a, \;\; x(t) = b,$$
$$(x(r-1),\, x(r)) \in X(r) \;\; (r = s, s+1, \ldots, t)\}$$

So $F(s,a,t)$ is the set of stocks at time t which can feasibly be reached, starting with a at the start of period s. Evidently, $F(t,a,t) = B(a,t)$ An especially important instance of $F(s,a,t)$ arises when $s = 1$ and $a = \bar{x}(0)$. Then $F(1,a,t) = G(t)$ (say) is the set of stocks at time t which can be reached starting from the historically given stock $\bar{x}(0)$.

The following result is useful:-

Lemma 8.3.1.

For all $s \leq t$ and all $a \in G(s-1)$, $F(s,a,t)$ is compact.

Proof.

The proof proceeds by induction on m, where $m = t - s$. If $m = 0$, then $F(t,a,t) = B(a,t)$, which is compact, by (A.3).

Suppose the result is true for a given $m \geq 0$. Then when $t - s = m + 1$:-

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[1] See Debreu (1959), p. 18, or Berge (1963), p. 111.

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$$(x(r-1),\, x(r)) \in X(r) \;(r = s, s+1, \dots, t)\}$$

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(1) See Debreu (1959), p. 18, or Berge (1963), p. 111.

$F(s,a,t) = \bigcup\{B(b,t-1) \mid b \in F(s,a,t-1)\}$

But $F(s,a,t-1)$ is compact, by the induction hypothesis, and $B(\cdot,t-1)$ is an upper-semi-continuous compact-valued correspondence. Therefore, by Berge (1963, p. 110, theorem 3), $F(s,a,t)$ is compact, as required.

It leads immediately to:-

Theorem 8.3.2.

X is compact in the product topology.

Proof.

By lemma 8.3.1. for all t, $G(t)$ is compact. Now X is a subset of the product $\{\bar{x}(0)\} \times \Pi_{t=1}^{\infty} G(t)$, which is a product of compact sets, and so compact in the product topology, by Tychonoff's theorem.[1] Since a closed subset of a compact set must be closed, it is therefore enough to show that X is closed, in the product topology.

Suppose that $x^n \in X$ $(n = 1,2,\ldots)$ and that $x^n \to x^*$ as $n \to \infty$. Then $x^n(0) \to x^*(0)$, and, since $x^n(0) = \bar{x}(0)$ (each n), it follows that $x^*(0) = \bar{x}(0)$. Also, for all n,t, $\left(x^n(t-1),\ x^n(t)\right) \in X(t)$, and, as $n \to \infty$, $(x^n(t-1),\ x^n(t)) \to \left(x^*(t-1),\ x^*(t)\right)$. By (A.2) $X(t)$ is closed, and so $(x^*(t-1),\ x^*(t)) \in X(t)$ $(t = 1,2,\ldots)$. Therefore $x^* \in X$.

The next assumptions is rather more restrictive:-

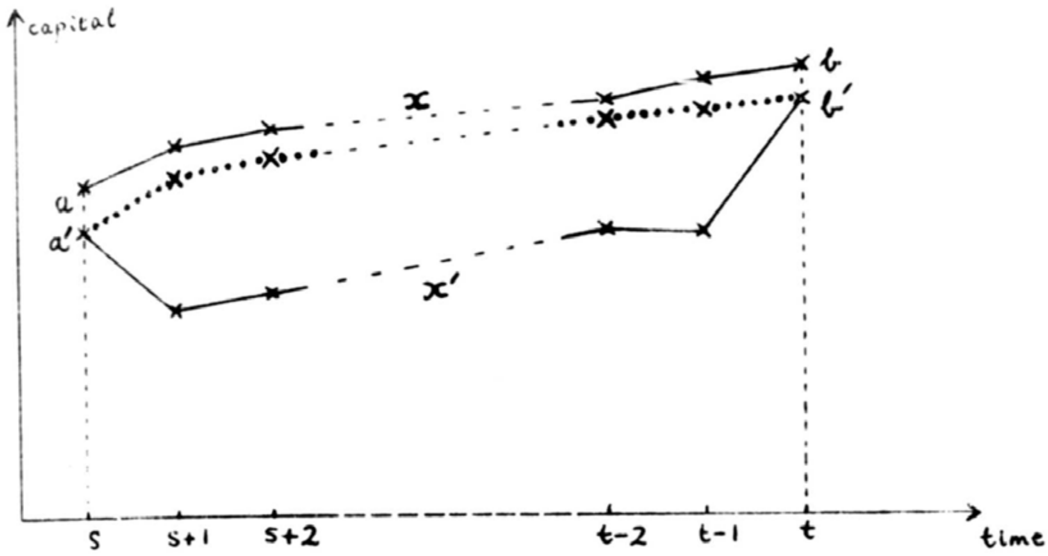
(A.4). For each pair s,t with $s < t$, $X(\cdot,\cdot,s,t)$ is a continuous correspondence.

It was seen that (A.2) alone implies that $X(\cdot,\cdot,s,t)$ is upper-semi-continuous. (A.4) adds lower semi-continuity. Roughly speaking, this means that if x is a feasible path with $x(s) = a$, $x(t) = b$, if (a',b') is near to

---

[1] See Kelley (1955), p. 143, or Berge (1963), pp. 79-80.

(a,b), and if there is a feasible path x' such that $x'(s) = a'$, $x'(t) = b'$, then there is such a feasible path which is also near to x. This is illustrated by the dotted path in the diagram below.(1)



(A.4) can probably be justified in much the same way as (A.2) was, namely as a harmless restriction on the mathematical approximations $X(t)$ to the true feasible sets, which are finite. Of course, if $X(t)$ is finite, for all t, then $X(\cdot,\cdot,s,t)$ is automatically a continuous correspondence.

---

(1) For a formal definition of lower-semi-continuity, see Debreu (1959), pp. 17 - 18, or Berge (1963), p. 109.

8.4. Assumptions Concerning the Utility Functions.

The choice function D^H for the horizon H on the underlying set $X|H$ of H-period capital streams corresponds to the H-period utility function:-

$$W^H(x) \quad (= W^H(x|H)) = \sum_{t=1}^{H} v(x(t-1),\ x(t),t).$$

It is convenient to assume that each utility function $v(\cdot,\cdot,t)$ is continuous. Then each $W^H(\cdot)$ is continuous. Since X is compact, there is then, for each horizon H, an *H-period optimum* x^H which maximizes $W^H(x)$ over X.

It may seem possible to justify continuity of the functions $v(\cdot,\cdot,t)$ by an argument similar to that used in 8.3. to justify certain topological assumptions concerning the feasible set. Nevertheless, it has been common in theory of optimal growth to admit utility functions which are not continuous at *every* point. Discontinuities occur when consumption is forced to zero - or, more generally, to a "subsistence" level. There it is common to have utility equal to $-\infty$, and so discontinuity.[1]

Now utility functions which take on the value $-\infty$ for certain options are not proper utility functions. Nevertheless, they correspond naturally to a choice function, and so we may allow such utility functions. They will be called *extended utility functions;* their formal properties are set out in section A.11 of appendix 1. Notice that an extended utility function may be continuous, in a natural sense. The only difficulty arises if $v(x(t-1),\ x(t),\ t) = -\infty$. Say that $v(\cdot,\cdot,t)$ is continuous at

---

(1) This discontinuity could be removed by redefining welfare as $W' = e^W$. But we shall assume that, because of the additive form of W, only affine transformations are allowed.

such a point if, whenever $(x^n(t-1), x^n(t))$ is a sequence of points in $X(t)$ which tends to $(x(t-1), x(t))$ as $n \to \infty$, then $v(x^h(t-1), x^h(t), t) \to -\infty$ as $n \to \infty$. Say that the extended utility function $v(\cdot,\cdot,t)$ is *continuous* if it is continuous at every point of $X(t)$.

The assumptions which will be made now are as follows:-

(A.5) For each t, $v(\cdot,\cdot,t)$ is a continuous utility function, or a continuous extended utility function, on the set $X(t)$.

An obvious consequence of (A.5) is that each of the functions, or extended functions, $W^T(\cdot)$, is continuous.

(A.6) If $b \in \text{int } G(T)$[1], then there exists $x \in X$ such that $x(T) = b$ and $W^T(x) = \sum_{t=1}^{T} v(x(t-1), x(t), t)$ is finite.

(A.6) is an assumption which restricts the points at which utility can become minus infinity. It is an assumption which is satisfied, so far as I know, by all the problems which have been considered in the optimal growth literature. Its explanation is a little involved.

Suppose that $b \in \text{int } G(T)$ and that, for all $x \in X(\bar{x}(0),b,0,T)$, $W^T(x) = -\infty$. Now there exists $b' \in \text{int } G(T)$ such that $b' > b$. To provide the extra capital stock b' presumably involves even greater sacrifices of consumption during the periods $1,2,\dots,T$ than does the provision of b. So, there presumably exists an $x \in X(\bar{x}(0),b,0,T)$ and an $x' \in X(\bar{x}(0),b',0,T)$ such that x' is definitely worse than x. But $W^T(x)$ is already $-\infty$, so to make $W^T(x') < W^T(x)$ is impossible. Choice can no longer correspond to a utility function. (A.6) rules out this possibility.

---

(1) int S denotes the topological interior of the set S.

8.5. Optimal Capital Accumulation - Existence.

In section 7.4., a feasible option x^* was defined as *optimal* if, for all feasible x:-

$$\liminf_{H \to \infty} \left[W^H(x^*) - W^H(x) \right] \geq 0.$$

Evidently this criterion can be applied to infinite-horizon capital streams. The question which then arises is whether there exists a capital stream which is optimal in this sense. There are, in fact, two powerful existence theorems. The first concerns "valuation finite economies", and does not involve finding the optimal path; the second can only be used when a potentially optimal path has been characterized - then it demonstrates sufficient conditions for the path to be optimal. Both theorems apply when the assumptions of sections 8.3. and 8.4. are satisfied.

For the first result, say that an economy - represented by the feasible set X and the utility functions $v(\cdot,\cdot,t)$ - is *valuation finite*(1) if:-

(1) For each t, there exists a finite upper bound $\bar{v}(t)$ such that, whenever $x \in X$, $v(x(t-1), x(t), t) \leq \bar{v}(t)$.

(2) There exists $\bar{x} \in X$ such that

$$\sum_{t=1}^{\infty} \left[v(\bar{x}(t-1), \bar{x}(t), t) - \bar{v}(t) \right]$$

converges.

Evidently, after redefining the utility functions $v(\cdot,\cdot,t)$ by subtracting the constants $\bar{v}(t)$, a valuation-finite economy has the following properties:-

---

(1) Such economies have been considered by Mirrlees (1968), Brock and Gale (1969), and McFadden (1973). They are a logical development of the "good" paths considered by Gale (1967).

(1) For each t and each $x \in X$, $v\big(x(t-1),\ x(t),\ t\big) \leq 0$.

(2) There exists $\bar{x} \in X$ such that:-

$$W^{\infty}(\bar{x}) = \sum_{t=1}^{\infty} v\big(\bar{x}(t-1),\ \bar{x}(t),\ t\big)$$

is well-defined.

The following existence theorem can now be proved.

Theorem 8.5.1.

Suppose that an economy is valuation-finite, and satisfies assumptions (A.1), (A.2), (A.3) and (A.5). Then there is at least one optimal capital stream.

Proof

The proof is an elementary consequence of lemma 2, p. 236 in Brock and Gale (1969). An alternative proof emerges in chapter 10.

The second result concerns a path which is "competitive". Say that $x^* \in X$ is *competitive* if there exist dual vectors $p(t)$ $(t = 0,1,2,\ldots)$ such that, for each t, $\big(x^*(t-1),\ x^*(t)\big)$ maximizes

$$v\big(x(t-1),\ x(t),\ t\big) + p(t)\, x(t) - p(t-1)\, x(t-1)$$

subject to $\big(x(t-1),\ x(t)\big) \in X(t)$.

This is, effectively, the discrete time version of Pontryagin's maximum principle. The following result is easy to prove.[(1)]

---

(1) It appears that this theorem was first proved by Weizsäcker (1965), in the one good case; on p. 91, he notes that the condition that the value of capital tends to zero, together with competitiveness, is a special case of the sufficient conditions he demonstrates.

The condition that the value of capital tends to zero - that $p(T)\, x^*(T) \to 0$ as $T \to \infty$ - is often called the "Malinvaud condition", because Malinvaud used it to establish the efficiency of a competitive path. See Malinvaud (1953), (1969), lemmas 5 and 5'.

(1) For each t and each $x \in X$, $v\big(x(t-1), x(t), t\big) \leq 0$.

(2) There exists $\bar{x} \in X$ such that:-

$$w^{\infty}(\bar{x}) = \sum_{t=1}^{\infty} v\big(\bar{x}(t-1), \bar{x}(t), t\big)$$

is well-defined.

The following existence theorem can now be proved.

Theorem 8.5.1.

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$$v\big(x(t-1), x(t), t\big) + p(t)\, x(t) - p(t-1)\, x(t-1)$$

subject to $\big(x(t-1), x(t)\big) \in X(t)$.

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---

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Theorem 8.5.2.

Suppose that an economy satisfies (A.1), and that $x^* \varepsilon X$ is a capital stream, which is competitive at non negative prices $p(t)$, and suppose that $p(T)\ x^*(T) \to 0$ as $T \to \infty$. Then x^* is optimal.

Proof

Let $x \varepsilon X$ be any other path.

Then $$\sum_{t=1}^{T} \left[v\big(x(t-1),\ x(t),\ t\big) - v\big(x^*(t-1),\ x^*(t),\ t\big)\right]$$

$$\leq \sum_{t=1}^{T} \{p(t)\ \big(x^*(t) - x(t)\big) - p(t-1)\big(x^*(t-1) - x(t-1)\big)\}$$

$$= p(T)\ \big(x^*(T) - x(T)\big) - p(0)\big(x^*(0) - x(0)\big)$$

$$\leq p(T)\ x^*(T)$$

because $p(T) \geq 0$, $x(T) \geq 0$, and $x^*(0) = x(0) = \bar{x}(0)$.

So $$\liminf_{T \to \infty} \left[W^T(x^*) - W^T(x)\right] \geq \liminf_{T \to \infty} \left[- p(T)\ x^*(T)\right] = 0$$

as required.

Even if there is no optimal path, there may be a maximal path, as defined in section 7.4. - i.e. an x^* for which there is no other path x such that:-

$$\liminf_{H \to \infty} \left[W^H(x) - W^H(x^*)\right] > 0$$

Despite the above two powerful results on the existence of an optimal path, there remains a large class of economies for which there is no optimum - and, indeed, no maximal path. The theoretical literature on optimal growth abounds with them. For present purposes, it is enough to give just one example which seems particularly disturbing, because it may well happen that there is a fairly obviously appropriate infinite-horizon choice, even though the chosen path cannot be maximal.

The example is presented in the next section.

8.6. Maximal Capital Accumulation - Non-Existence.[1]

For this example, we change from discrete to continuous time. This makes no real difference; it is just that the equations are easier to analyse.

The model is the standard one-good case. There is a single good which is used for both consumption and investment. At time t, the flow of net output is a function $f(k(t))$ of the capital stock $k(t)$. Investment is the residue of this output after consumption has been taken away. So the rate of increase of the capital stock, $\dot{k}(t)$, is equal to $f(k(t)) - c(t)$. If the horizon is H, the welfare function is assumed to be a simple integral of the form $\int_0^H u(c(t))\, dt$. It is assumed that the functions f and u are both twice differentiable, and that $f(0) = 0$, $f' > 0$, $f'' < 0$, $u' > 0$, $u'' < 0$ everywhere.

These are standard assumptions.[2] Although they concern a grossly over-aggregated economy, they are not totally implausible. In particular, if difficulties are encountered in such a simple model, it is hard to believe that they can always be avoided in more realistic, and so more complicated, models.

Shortly, we shall assume that the functions f and u take very special forms. But, for the time being, there is nothing to be gained by departing from the general case.

---

(1) This section relies heavily on the paper of Hammond and Mirrlees (1973), and some of the results for which Professor Mirrlees was primarily responsible.

(2) They were first exploited by Ramsey (1928).

Consider the planning problem which arises when the horizon is known to be H. The the objective is $\int_0^H u(c(t))\,dt$, and the constraints are:-

(1) $c + \dot{k} \leq f(k)$ $(t = 0 \text{ to } H)$

(2) $k(0) \leq \bar{k}(0)$ (given)

(3) $k(H) \geq 0$.

Proceeding non-rigorously, one can set up a Lagrangean as follows:-

$$\mathcal{L} = \int_0^H u(c)\,dt - \int_0^H \lambda(c + \dot{k} - f(k))\,dt - \mu(k(0) - \bar{k}(0)) + \nu\,k(H).$$

Integrating by parts,

$$\int_0^H \lambda\,\dot{k}\,dt = [\lambda k]_0^H - \int_0^H \dot{\lambda}\,k\,dt.$$

Therefore:-

$$\mathcal{L} = \int_0^H \{u(c) - \lambda(c - f(k)) + \dot{\lambda}k\}\,dt + \lambda(0)k(0) - \lambda(H)k(H) - \mu(k(0) - \bar{k}(0)) + \nu\,k(H).$$

Given our assumptions, the constraints $k(0) \leq \bar{k}(0)$ and $k(H) \geq 0$ are both binding, and so $\nu > 0$, $\mu > 0$.

Thus $\mu = \lambda(0)$, $\nu = \lambda(H)$ and the Lagrangean becomes simply:-

$$\mathcal{L} = \int_0^H \{u(c) - \lambda(c - f(k)) + \dot{\lambda}k\}\,dt + \mu\,\bar{k}(0).$$

It is now sufficient to find a global maximum of $\mathcal{L}$.[1] The first order conditions - which, because of concavity of the functions u and f, are sufficient - give:-

$$\left.\begin{aligned} u'(c(t)) - \lambda(t) &= 0 \\ \lambda(t)\,f'(k(t)) + \dot{\lambda}(t) &= 0 \end{aligned}\right\} \quad (t = 0 \text{ to } H)$$

[1] This, of course, is Pontryagin's maximum principle, which has been derived heuristically.

In addition, $\dot{k}(t) = f(k(t)) - c(t) \qquad (t = 0 \text{ to } H).$

It is easy to verify that these equations are all satisfied if and only if

$$u(c) + (f(k) - c)\, u'(c) = \text{constant.}$$

which is the famous Keynes-Ramsey equation.[1] Moreover, it is easy to show rigorously that any path satisfying this equation, together with $\dot{k} = f(k) - c$, *is* a solution to the finite-horizon optimization problem.

In general, the solution to the differential equations depends on $c(0) = \gamma$, the initial level of consumption. Remember that $k(0) = \bar{k}(0)$ is given, as the other initial condition. The solution path, then, is $c(t,\gamma)$, $k(t,\gamma)$, $\lambda(t,\gamma)$. Moreover, for $\gamma > 0$, c, k, λ are all continuous functions of γ, and as γ increases:-

c strictly increases

λ strictly decreases

k strictly decreases.

Moreover, if for any t_0, $c(t_0) > f(k(t_0))$, then $\dot{k}(t) < 0$ for all $t \geq t_0$.[2]

By choosing γ large enough, therefore – e.g. $\gamma > f(\bar{k}(0))$ – it is possible to ensure that $k(t,\gamma)$ steadily decreases over time, eventually. Moreover, since $\dot{k}$ is bounded away from zero, there exists a finite time T such that $k(T,\gamma) = 0$. Then the path is feasible up to time T, but no further.

Now, if the time horizon is H, the optimal path is found by choosing $\gamma = \gamma(H)$, where $k(H,\gamma(H)) = 0$. Of course, $\gamma(H)$ is unique. To show that

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(1) See Ramsey (1928), section 1, equations (4) and (5).

(2) See the appendix to Weizsäcker (1965).

this path is optimal, notice that:-

(a) If $\gamma > \gamma(H)$, then $k(t,\gamma) = 0$ for some $t < H$, and so the path is not feasible, for the horizon H.

(b) If $\gamma < \gamma(H)$, then $c(t,\gamma) < c(t,\gamma(H))$ for all t, and so the path starting with $c(0) = \gamma$ cannot possibly be optimal, because there is another feasible path on which consumption is always higher.

Notice, finally, that as the time horizon H lengthens, so $\gamma(H)$ decreases. As $H \to \infty$, there are two possibilities:-

(i) $\gamma(H) \to 0$ as $H \to \infty$

(ii) $\gamma(H) \to \gamma^* > 0$ as $H \to \infty$.

(ii) is the more interesting, because $c(t,\gamma^*)$ is a viable consumption plan, and the limit of the finite horizon choices.

Now we ask two questions:-

(a) When does $\gamma(H) \to \gamma^* > 0$ as $H \to \infty$?

(b) If $\gamma(H) \to \gamma^* > 0$, is $c(t,\gamma^*)$ a maximal consumption path, for the infinite horizon?

Consider the path given by:-

$$\left.\begin{aligned} u(c) + (f(k)-c)\, u'(c) &= A \\ \dot{k} &= f(k) - c \end{aligned}\right\} \text{ all } t$$

<u>Lemma 8.6.1.</u>

If there exists $\bar{k}$ such that $u(f(\bar{k})) \geq A$, then $\dot{k} < 0$ eventually (and so $k(t) = 0$, for some finite t).

<u>Proof</u>

Suppose $\dot{k} \geq 0$, for all time.

Now $\left(f(k)-c\right) u'(c) = A - u(c)$

$$\leq u(f(\bar{k})) - u(c)$$

$$\leq \left(f(\bar{k})-c\right) u'(c)$$

for all t, because c is concave.

Therefore $\quad f(k) \leq f(\bar{k}) \quad$ (all t),

and so $\quad k(t) \to k^* \leq \bar{k}$, as $t \to \infty$.

Then $c(t) \to c^*$ for some c^*.

But $\dfrac{u''(c)\ \dot{c}}{u'(c)} = \dfrac{d}{dt} \log u'(c) = - f'(k)$

Therefore, as $t \to \infty$, $\dot{c} \to \dfrac{- f'(k^*)\ u'(c^*)}{u''(c^*)} > 0$

This contradicts $c(t) \to c^*$.

Lemma 8.6.2.

If $A > u(f(k))$, for all k, then $\dot{k} > 0$, for all t, and $k(t) \to \infty$.

Proof

Notice that

$$u(c(0)) + \left(f(k(0)) - c(0)\right) u'(c(0)) = A > u\left(f(k(0))\right)$$

and that this is only possible if $c(0) < f(k(0))$.

Suppose that $c(t) = f(k(t))$, for some t.

Then $\quad u(c(t)) = A = u\left(f(k(t))\right)$, a contradiction.

A similar argument establishes that $k(t) \to k^*$, for finite k^*, is impossible.

Lemma 8.6.3.

Either (1) $u(f(k))$ is unbounded above, as $k \to \infty$, and then $\gamma(H) \to 0$ as $H \to \infty$.

or (2) There exists A such that

$$A = \sup \{u(f(k)) \mid k > 0\}$$

and then $\gamma(H) \to \gamma^*$ as $H \to \infty$

where $u(\gamma^*) + \left(f(\bar{k}(0))-\gamma^*\right) u'(\gamma^*) = A$.[1]

Proof

(1) If $u(f(k))$ is unbounded above, then, however $\gamma > 0$, and so A, are chosen, by lemma 8.6.1. $k(\gamma,t) = 0$ for some finite t. So for all $\gamma^* > 0$, there exists H s.t. $\gamma(H) < \gamma^*$.

(2) If $\gamma > \gamma^*$, as defined, then, by lemma 8.6.1., $k(\gamma,t) = 0$, for some finite t. But if $\gamma < \gamma^*$, then $k(t) \to \infty$, and so $\gamma < \gamma(H)$, for all H.

Therefore $\gamma^* = \inf\limits_{H} \gamma(H)$.

This answers question (a) - $u(f(k))$ must be bounded above.

There are two ways in which $u(f(k))$ can be bounded above.

(i) $u(c)$ may be bounded above - $u(c) \leq B$ for all finite c. There is strict inequality because $u'(c) > 0$ for all c. B is Ramsey's "bliss" level of utility.[2]

(ii) $f(k)$ may be bounded above - $f(k) < b$ for all finite k. Again, there is strict inequality because $f'(k) > 0$. The bound on net output may arise because of some unspecified scarce resource, in fixed supply. For example, production uses energy, and as long as we remain earthbound, we are effectively limited to the supply of energy from solar radiation, and from the stock of energy resources within the earth.

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(1) Part (2) of this result is effectively the same as theorem 3, p. 296, of Hammond and Mirrlees (1973), and is due to Mirrlees.

(2) See Ramsey (1928).

Lemma 8.6.4.

If u is bounded above – $B = \sup_c u(c)$ – then the path given by $u(t) + (f(k)-c)\, u'(c) = B$, $\dot{k} = f(k) - c$, is optimal, provided that

$$u'(c(t))\, k(t) \to 0 \text{ as } t \to \infty$$

Proof

The proof is analogous to that of theorem 8.5.2., and will be omitted.

Notice that this result cannot help if it is f which is bounded above and if, when $b = \sup_k f(k)$, $u'(b) > 0$. For then, the obvious candidate for the optimal path has the property that $k(t) \to \infty$ and $c(t) \to b$ so that $u'(c(t))\, k(t) \to \infty$ as $t \to \infty$. Nor can theorem 8.5.1. help very easily. The obvious upper bound to take is $\bar{u}(t) = u(b)$.

But $\int_0^T \{u(b) - u(c)\}\, dt = \int_0^T (f(k) - c)\, u'(c)\, dt \geq \int_0^T \dot{k}\, u'(b)\, dt$,

which tends to infinity as $T \to \infty$.

An important and disturbing non-existence result is the following:-

Lemma 8.6.5.

Define $g(k) = b - f(k)$ (> 0, for all k)

Suppose that, for some μ such that $0 < \mu < 1$,

$$\liminf_{k \to \infty} \frac{g(k)}{g(\mu k)} = 2\alpha > 0$$

Then there is no maximal path.[1]

---

[1] This result, and its proof, are based on example (3), pp. 286-287, of Hammond and Mirrlees (1973). The example is due to Mirrlees.

Proof

(1) Suppose there is a maximal path, and let c be one.

Then it must be true that, for each H, $c(t)$ $(0 \leq t \leq H)$

maximizes $\int_0^H u(\hat{c}(t))\, dt$ subject to:-

(1) $\hat{c} + \dot{\hat{k}} \leq f(k)$ $\quad (t = 0 \text{ to } H)$

(2) $\hat{k}(0) \leq \bar{k}(0)$ $\quad$ (given)

(3) $\hat{k}(H) \geq k(H)$

Otherwise there would be an alternative feasible path $\bar{c}(t)$, with $\bar{c}(t) = c(t)$ $(t > H)$, such that

$$\int_0^H \left[u(\bar{c}(t)) - u(c(t))\right] dt > 0$$

and so $\liminf_{T \to \infty} \int_0^T \left[u(\bar{c}(t)) - u(c(t))\right] dt > 0$ -

which contradicts the maximality of c.

Now, by a similar argument to our earlier one, it follows that $u(c) + (f(k)-c)\, u'(c) = \text{constant}$ (A, say) and this must be true for all time. By lemma 8.6.1., it must be true that $A \geq u(b)$. But, if $A > u(b)$, then, by reducing A to $u(b)$, we could increase consumption at all times, and so increase welfare. Therefore,

$$u(c) + (f(k)-c)\, u'(c) = u(b)$$

Also, of course, $\dot{k} = f(k) - c$.

It is enough to show that this path is inefficient, in the sense that, at a certain time t_0, we can reduce the capital stock $k(t_0)$ to $h(t_0) = \mu k(t_0)$ and yet maintain the consumption stream $c(t)$ for all $t \geq t_0$. For then it is possible to enjoy extra consumption before t_0, without sacrificing consumption after t_0.

To show this, consider the path of the capital stock, $h(t)$, if $c(t)$ is maintained. It satisfies:-

$$\dot{h} = f(h) - c \quad (\text{all } t \geq t_0)$$

It is enough to show that if $h > \mu k$, and $t \geq t_0$, then $\dot{h} - \mu\dot{k} > 0$.

$$\begin{aligned}\text{But } \dot{h} - \mu\dot{k} &= f(h) - \mu f(k) - (1-\mu)c \\ &\geq f(\mu k) - \mu f(k) - (1-\mu)c \\ &= \mu g(k) - g(\mu k) + (1-\mu)y\end{aligned}$$

where $y = b - c$.

$$\text{Also, } \frac{u(b) - u(b-y)}{y\, u'(b-y)} = \frac{(f(k)-c)}{y} = 1 - \frac{g(k)}{y}$$

But we know that $y \to 0$ as $t \to \infty$, so the left hand side $\to 1$, and

$\dfrac{g(k)}{y} \to 0$ as $t \to \infty$.

Also $k \to \infty$ as $t \to \infty$. So we can choose t_0 large enough so that, for all $t \geq t_0$,

$$g(k) > \alpha g(\mu k)$$

$$y > g(k)\,\frac{(1 - \mu\alpha)}{(1 - \mu)\alpha} \quad \text{(where we have taken } \alpha < 1) .$$

$$\begin{aligned}\text{Thus } \dot{h} - \mu\dot{k} &> \alpha\mu\, g(\mu k) - g(\mu k) + \frac{g(k)(1-\mu\alpha)}{\alpha} \\ &> 0\end{aligned}$$

as required.

This result is troubling because it is not obvious how to exclude production functions $f(k)$ with the property that $\liminf\limits_{k \to \infty} \dfrac{g(k)}{g(\mu k)} > 0$.

For example, if $f(k) = b - (a+k)^{-m}$,

then $g(k) = (a+k)^{-m}$

and $\dfrac{g(k)}{g(\mu k)} = \dfrac{(a+k)^{-m}}{(a+\mu k)^{-m}} \to \mu^m$ as $k \to \infty$.

8.7. Conclusions

This chapter has presented two useful positive results on the existence of an optimal infinite horizon plan of capital accumulation. It has also presented a disturbing negative result. An economic model of the kind considered in section 8.6. is too simple, because it is too aggregated. But unless any planning rule we are intending to use works well in such a model, it is hard to see how it will work well in other, more realistic models. The conclusion suggested by the economy of lemma 8.6.5. is that the rule of choosing an optimal path does not always work well enough. Nor does that of choosing a maximal path. In that economy there is no optimal path, nor even a maximal path. Yet there is a path satisfying the Keynes-Ramsey equation, and moreover this path is the limit of the sequence of finite horizon optima, as the horizon H tends to infinity.

Such a limit path has been called "agreeable". It is the purpose of the next chapter to see what kind of infinite horizon planning rules lead to the choice of such limit paths, and to relate these rules to the notion of agreeability.

Chapter 9.

INFINITE HORIZON OVERTURE CHOICE.

9.1. Introduction.

In chapter 7, it was suggested that it might be advantageous to consider infinite-horizon choice, as a relatively simple and clear method of allowing for uncertainty about the true time-horizon. But, in chapter 8, [illegible]ox kind of infinite horizon choice ran into difficulties. It seems that it is too easy for there to be no maximal plan, in the sense of chapter 7. In the example of section 8.6., not only was there no maximal plan; more worrying, there was no maximal plan in a situation where a sensible infinite-horizon choice seemed possible.

Since the problem of non-existence was first recognized,[1] there have been a number of suggestions for overcoming it. Chakravarty recognized that it is really a finite horizon plan which is being sought, but that the horizon is unknown.[2] Nevertheless, if the horizon is long enough, the optimal choice in the early years may not be sensitive to the horizon.[3] If it is not, there may be an "insensitive" plan - one which is "nearly optimal", regardless of the horizon.

To make this view more precise, we must clarify what is meant by "insensitive", and by "nearly optimal". Suppose that, as in chapter 8, we are trying to choose a program of capital accumulation $x = \langle x(t) \rangle_{t=0}^{\infty}$.

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(1) Notably, by Tinbergen (1960) and Chakravarty (1962a).

(2) This, of course, is the point from which chapter 7 starts.

(3) This seems first to have been realized by Modigliani and Hohn (1955). For an extended discussion, see Chakravarty (1962b), (1966) and Manesc[illegible] (1966a), (1966b).

Introduce the notation $x|T$ for the truncated sequence $\langle x(t) \rangle_{t=0}^{T}$ and call $x|T$ the *T-overture* of x.[1]

Now let us suppose that, for each finite horizon H, there is a unique optimal program x^H, which maximizes the welfare function W^H. Of course, this is a strong assumption, but it is one which could be weakened without [illegible] than to make the following argument more complicated. $x^H|T$ [illegible] T-overture of the optimum x^H.

With this notation, we can now define "insensitivity". Suppose that a plan for T-periods is to be chosen - i.e. a T-overture. Then, roughly, the optimum is not sensitive to the horizon if there is a small set N such that $x^H|T \in N$ for all H - or, at least, $x^H|T \in N$ for all large enough H. The presumption is that if any overture in N is chosen, then the mistake will not be too large. By time T, the path the economy has followed will not have deviated very far from the true optimum.

There are two difficulties with this idea of "insensitivity". The first is that it is not clear how small N has to be. But this is easily dealt with. It is more precise to say that a particular T-overture $x^*|T$ is *insensitive* if $x^H|T$ tends to $x^*|T$ as $H \to \infty$. Then, given any neighbourhood N of $x^*|T$, $x^H|T$ eventually lies in N. The limit overture $x^*|T$ is insensitive to any required degree, in effect, provided that the horizon is long enough. One can then go on to say that an infinite horizon program x^* is insensitive, provided that each of its overtures is insensitive.[2]

The second difficulty with insensitivity is more fundamental. The relationship of insensitivity to choice is far from clear. A plan may be

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(1) The term is due to Bliss (1971).

(2) See Brock (1971) for a full discussion of insensitive programs in [illegible] model of capital accumulation.

insensitive without being optimal, or even nearly optimal, in any acceptable sense. Or, at least, we need to check to see how nearly optimal an insensitive plan is. Otherwise, we cannot be sure that there is not another program which is preferable to the insensitive program.

An alternative choice procedure which does consider the optimality - or suboptimality - of its choices, is that of finding an "agreeable plan".[1] It is an alternative approach which recognizes that the horizon is finite, but unknown, and suggests a program which is "nearly optimal", regardless of the horizon. But, whereas an insensitive plan x^* is one which has its T-overture capital stream $x^*|T$ near to the optimum $x^H|T$ for large finite horizons, an agreeable plan is one which is nearly as good as the optimum, according to the welfare function W^H.

Let $\bar{W}^H$ denote the welfare $W^H(x^H)$ derived from the optimum for the finite horizon H. Let $W^H(x|T)$ denote the maximum possible value of W^H which can be achieved after following the T-overture $x|T$. In general, $W^H(x|T) < \bar{W}^H$, because $x|T$ is the "wrong start" when the horizon is H, and x^H is optimal.[2] Following $x|T$ was a mistake: the welfare loss incurred is $\bar{W}^H - W^H(x|T)$, assuming that the optimum is chosen after time T. So, a plan x^* is *agreeable* provided that for each T,

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(1) Hammond and Mirrlees (1973).

(2) Of course, this is always true if x^H is the unique optimum, but we can relax this assumption now. We only need $\bar{W}^H$ to be unique.

$\bar{W}^H - W^H(x^*|T) \to 0$ as $H \to \infty$.(1) (2) That is, a plan is agreeable provided that the welfare loss from the wrong start becomes insignificant as the horizon H tends to infinity. It is a plan which people with different views about the true distance of the horizon can agree to - provided the horizon is thought to be very distant. When a person agrees to x^*, he [illegible] presumes, certain extensions of the overtures $x^*|T$ in mind. The extensions he prefers depend on his views about the horizon. Another person has different extensions in mind. Nevertheless, there may be agreement on the appropriate T-overture, for each T, and it seems quite possible that there will be, if there is an agreeable plan.

Although they may be found appealing, there is still a possible objection to agreeable plans. Suppose I happen to believe that the true horizon is certain to be $\bar{H}$. Then, I can agree to x^* as a provisional

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(1) The units of welfare are not defined, so no meaning can be attached to $\bar{W}^H - W^H(x|T)$. But the property $\bar{W}^H - W^H(x^*|T) \to 0$ is invariant under increasing affine transformations, and so unambiguous if W^H is additive.

An interesting suggestion for measuring welfare losses is contained in Mirrlees and Stern (1972). But "fairly good plans" go beyond the scope of this study, because their justification is the saving in computational effort.

There is no necessary logical connection between agreeable plans and insensitive plans, as can be seen from example 10.2.2. below. So agreeable plans are more than a possible justification for insensitivity. Nevertheless, as will be seen in chapter 10, there are many cases in which agreeable plans are insensitive, and *vice versa*.

(2) A similar notion is Łoś' "ε-horizon", see Łoś (1967), (1971) and Keeler (1973). This work considers purely terminal objectives - e.g. where only the final capital stock $x(H)$ is of concern. Nevertheless, it easy to arrive at a similar definition for the more general problem. K is an *ε-horizon* for the plan x^* if, for all T, $\bar{W}^H - W^H(x^*|T) < \varepsilon$ whenever $H \geq T+K$. If there is an ε-horizon for x^*, however small ε is, then x^* is agreeable; but the converse is false. K is the number of periods before the mistake becomes insignificant. It is possible that $K \to \infty$ as $T \to \infty$.

choice that is almost optimal, provided that I believe it will become generally recognized, at some time T, that $\bar{H}$ is the true horizon, and provided that I also believe that T is early enough, and $\bar{H}$ late enough, for the welfare loss $\bar{W}^{\bar{H}} - W^{\bar{H}}(x^*|T)$ to be acceptably small. But can I reasonably be expected to believe all this? Is it likely that there will be such knowledge of what the horizon will turn out to be before we are right on top of it? In other words, can I really be sure that, in agreeing to an "agreeable" plan, I am not accepting a welfare loss which is too large?

These questions may not be very precise - and to attempt to meet them would involve a discussion of when a welfare loss is "too large", and of how much notice of the true horizon (if there is one) we are likely to have. But they only arise if we insist on the welfare losses being small. Now, it may be impossible to ensure that welfare losses in fact turn out to be small, even if there is an agreeable plan. Nevertheless, we can still ask what an appropriate plan is, and it turns out that we can justify agreeable plans, and generalizations of them, on other grounds.

The central point is that we shall find ourselves choosing *some* overture $x^*|T$, in the end. This is true for each T up to the true horizon H. Such a choice cannot be avoided. So, how is a T-overture $x^*|T$ to be chosen? If x^* is an agreeable plan, then $x^*|T$ may well be an appropriate choice. But, to discuss this, we should consider overture choice in general. Throughout this study, we have used the notion of a choice function to discuss any choice problem which has arisen. We will do the same here.

Section 9.2. considers T-overture choice functions, for any fixed T. Then, since our main concern is with infinite horizon choice, we are led to consider infinite horizon T-overture choice, for any fixed T. This is done in section 9.3. Section 9.4. is a discussion of consistency. There are now two kinds of consistency. The first is dynamic consistency as the time at which choice is made changes: this was the kind of consistency discussed in chapter 4. The second kind is consistency of resulting choice as the length of the overture, T, varies. An [illegible] in section 9.5. shows that this second kind of consistency may [illegible]ed. A procedure for rectifying this inconsistency is suggested in section 9.6. Section 9.7. looks a little more deeply into the difference between this new procedure and the infinite horizon choice procedures of chapter 7.

The aim of this chapter, then, is to consider the purely choice-theoretic aspects of infinite horizon overture choice. Chapter 10 will consider the problems of existence, uniqueness, and the characterization of infinite horizon overture choice, with particular reference to the capital accumulation problem of chapter 8.

9.2. T-overture Choice.

Suppose that a T-overture $x|T$ is to be chosen. This faces [illegible] with a choice situation. It is a special kind of the partial choice situation, con[illegible] in section A.10 of appendix 1. The feasible set is X. The s[illegible] possible T-overtures is:-

$$[illegible]\langle x_t \rangle_{t=0}^{T} \;\Big|\; \exists \langle \bar{x}_t \rangle_{t=0}^{\infty} \in X \text{ s.t. } x_t = \bar{x}_t (t=0 \text{ to } T)\}$$

[illegible] overture $\bar{x}|T \in X|T$, define:-

$$X(\bar{x}|T) = \{x \in X \;\Big|\; x|T = \bar{x}|T\}$$

$$= \{x \in X \;\Big|\; x_t = \bar{x}_t \quad (t = 0 \text{ to } T)\}$$

Thus, $X(\bar{x}|T)$ is the set of feasible paths with $\bar{x}|T$ as T-overture.

Now the feasible set X can be partitioned as follows:-

$$X = \bigcup\{X(x|T) \;\Big|\; x|T \in X|T\}$$

So, X is partitioned into subsets of paths which share the same T-overture.

Given any $B|T \subseteq X|T$, define:-

$$X(B|T) = \{x \in X \;\Big|\; x|T \in B|T\}$$

$$= \bigcup\{X(x|T) \;\Big|\; x|T \in B|T\}$$

Thus, $X(B|T)$ is the set of feasible paths having a T-overture in $B|T$.

Suppose that C is a coherent choice function on the underlying set X. Suppose, too that, whenever $B|T$ is a finite subset of $X|T$, $C(X(B|T))$ is non-empty. Then, by theorem A.10.1 of appendix 1, a coherent partial choice function C_T can be defined on the underlying set $X|T$, as follows:-

$$[illegible] = \{x|T \;\Big|\; C(X(B|T)) \cap X(x|T) \neq \emptyset\} \quad (\text{each } B|T \subseteq X|T).$$

[illegible] in the choice set $C_T(B|T)$ if and only if there is an extension [illegible] - i.e. a path $y \in X$ such that $y|T = x|T$ - such that $y \in C(X($[illegible] -

i.e. y is a possible choice if the feasible set is the set of all possible extensions of overtures in $B|T$.

The partial choice function C_T is called the *T-overture choice function corresponding to* C.

[illegible] in addition, C corresponds to a welfare function W, then, [illegible] A.10.2., C_T corresponds to a *T-overture welfare function* W_T [illegible] defined as follows:-

$$W_T(\bar{x}|T) = max \{W(x) \mid x \in X, x|T = \bar{x}|T\}$$

$W_T(\bar{x}|T)$ is simply the maximum possible welfare which can be achieved, once the T-overture $x|T$ has been followed. Thus the welfare associated with a T-overture depends upon the possibilities of extending it.

If x^* is an optimum - i.e. maximizes $W(x)$ over X - then $x^*|T$ must maximize $W_T(x|T)$ over $X|T$. Conversely, if $x^*|T$ maximizes $W_T(x|T)$ over $X|T$, then there is an extension $\hat{x}$ of $x^*|T$ which maximizes $W(x)$ over X. These are consequences of the results on partial choice functions derived in section A.10 of the appendix. Accordingly, if it is clear what choice from X is appropriate, then the choice of a T-overture is a trivial problem.

It should be remembered, however, that we started to consider overture choice because there are situations in which the appropriate choice from X is not clear. This was true, for example, in section 8.6. There, there is an optimal plan for each finite horizon, and these optimal [illegible] converge to a limit as the horizon tends to infinity. The limit [illegible] seemed as though it might be an appropriate choice, yet it was [illegible], and so the appropriate infinite choice was unclear. And, [illegible] an agreeable plan is one which, it is claimed, is an appropriate infinite-horizon choice, its justification rests on its overtures giving small welfare losses. This suggests that it may be easier to find an appropriate overture choice than to find an appropriate choice from X itself.

9.3. Infinite Horizon *T*-overture Choice.

Suppose that we have the kind of choice situation described in chapters 7 and 8. That is, for each finite horizon H, there is a choice function C^H on the underlying set X. But, because the horizon is [illegible], we are led to consider infinite-horizon choice.

[illegible] for each $T \leq H$, there is an overture choice function C^H_T, which is derived from C^H as in section 9.2. Then, too, there is an overture welfare function W^H_T, defined on $X|T$, which is derived from the welfare function W^H. We shall assume that W^H has the additive form:-

$$\sum_{t=1}^{H} u_t(x_t)$$

where the funtions $u_t(\cdot)$ are independent of H. This was the assumption made in section 7.4.

In chapter 7, the form of the underlying set was deliberately not spelt out very precisely. So the arguments of that chapter apply with equal force to overture choice functions C^H_T on the underlying set $X|T$. Accordingly, the *strict T-overture preference relation* $P|T$ is defined on $X|T$ as follows:-

$$a|T \;\; P|T \;\; b|T \quad \text{iff} \quad \liminf_{H \to \infty} \left[W^H_T(a|T) - W^H_T(b|T) \right] > 0$$

and the *weak T-overture preference relation* $R|T$ is defined as follows:-

$$a|T \;\; R|T \;\; b|T \quad \text{iff} \quad \liminf_{H \to \infty} \left[W^H_T(a|T) - W^H_T(b|T) \right] \geq 0$$

Of course, $P|T$ is irreflexive, asymmetric, and transitive. $R|T$ is reflexive and transitive. But, as with R in section 7.4., $R|T$ may fail to be connected.

Let $O_T(B|T) = \{x|T \in B|T \mid y|T \in B|T \text{ implies } x|T \; R|T \; y|T\}$

be the set of optimal T-overtures in $B|T$.

Let $M_T(B|T) = \{x|T \in B|T \mid y|T \in B|T \text{ implies not } y|T \; P|T \; x|T\}$

be the set of maximal T-overtures in $B|T$.

It is worth noting that, by theorem A.8.4. of appendix 1, there is a T-overture choice function C_T such that, for all $B|T \subseteq X|T$:-

$$O_T(B|T) \subseteq C_T(B|T) \subseteq M_T(B|T).$$

But, as in chapter 7, infinite horizon choice has not been uniquely specified. Instead, there are upper and lower bounds on the choice set.

Even at this stage, in such a general choice situation, we can note the relationship between agreeable plans and infinite-horizon T-overture choice. First:-

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Theorem 9.3.1.

If x^* is an agreeable plan, then, for all T, $x^*|T$ is an optimal T-overture.

Proof

Trivial.

So, optimal T-overture choice generalizes agreeable plans. The following is an example in which there is no agreeable plan, and yet, there is a plan x^* such that, for each T, $x^*|T$ is an optimal T-overture:-

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Example 9.3.2.

Suppose that $X = \{x; z^1, z^2, \ldots\}$

and that the tree structure is:-

$$[illegible](x(t)) = \{x\} \qquad (t = 1,2,\ldots)$$

$$[illegible](z^h(t)) = \{z^h\} \qquad (t = 1,2,\ldots,\; h = 1,2,\ldots)$$

Thus, choice is effectively possible only at time 0.

Suppose too that $W^H(x) = H$, $W^H(z^h) = min(2H, 2h)$.

Then, for all $T \geq 1$, $W_T^H(x) = H$, $W_T^H(z^h) = min(2H, 2h)$.

$$\text{Now,} \quad \liminf_{H \to \infty} \{W_T^H(x) - W_T^H(z^h)\} = \liminf_{H \to \infty} \{H - min(2H, 2h)\}$$

$$= +\infty, \text{ whenever } T \geq 1,$$

so x is clearly the unique optimal T-overture,

but is not agreeable, because $\bar{W}^H = 2H$,

$$\text{and so} \quad \liminf_{H \to \infty} \left[\bar{W}^H - W_T^H(x)\right] = +\infty$$

Of course, in example 9.3.2., there is no agreeable plan. When there is an agreeable plan, then any other plan, whose overtures are each optimal, is agreeable.

Theorem 9.3.3.

Suppose x^* is an agreeable plan, and that $\hat{x}$ is any other plan such that, for each T, $\hat{x}|T$ is an optimal T-overture. Then $\hat{x}$ is agreeable.

Proof

For any fixed T:-

$$\liminf_{H \to \infty} \left[\bar{W}^H - W_T^H(\hat{x}|T)\right] \leq \liminf_{H \to \infty} \left[\bar{W}^H - W_T^H(x^*|T)\right] \quad \text{(because } \hat{x}|T \text{ is optimal)}$$

$$= 0 \quad \text{(because } x^* \text{ is agreeable)}$$

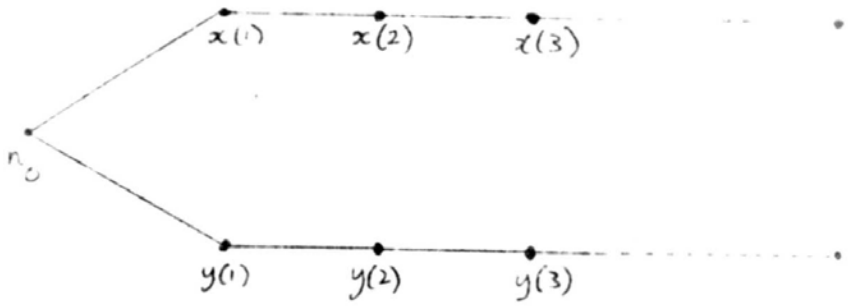
Because $\bar{W}^H \geq W_T^H(\hat{x}|T)$ (each H), it follows that

$\bar{W}^H - W_T^H(\hat{x}|T) \to 0$, as required.

Even if there is an agreeable plan, there may be another plan, whose overtures are each maximal, and yet is disagreeable, as the following example shows:-

Example 9.3.2.

[illegible] $= \{x, y\}$, and the tree structure is simply described as follows:-



The welfare functions W^H are such that:-

$$W^H(x) = 0 \qquad (\text{all } H)$$

$$W^H(y) = \begin{cases} -H & (H \text{ odd}) \\ 0 & (H \text{ even}) \end{cases}$$

Then $\overline{W}^H = 0$, so x is agreeable, but y is not. Nevertheless, each T-overture $y|T$ of y is a maximal T-overture, because:-

$$\liminf_{H \to \infty} \left[W^H_T(x|T) - W^H_T(y|T) \right] = 0$$

and so it is false that $x|T \;\; P|T \;\; y|T$.

The first property of infinite horizon T-overture choice functions to be investigated is consistency. Remember, after all, that the object of infinite horizon choice is to achieve consistency which would otherwise be lacking. It turns out that checking consistency is of crucial importance, because there is one consistency property which the choice functions C_T may violate. Infinite horizon overture choice will have to be modified to take account of this.

9.4. Consistent Overture Choice.

Let X be the feasible set, with tree structure [illegible] [illegible], as in section 4.3. Note that, in the notation of section 9.2.:-

[illegible] (s), then $X(n) = X(x|s)$ (each $x|s \in X|s$).

[illegible] the T-overture choice function indicating the choice of [illegible] at node n. The underlying set for this choice function is:-

$$[illegible]|T = \{x|T \mid \exists \bar{x} \in X(n) \text{ s.t. } x|T = \bar{x}|T\}$$

[illegible] the set of T-overtures of paths in $X(n)$. Of course, if $n = x(s)$, [illegible] $s \geq T$, then a T-overture has already been chosen and followed by [illegible] accordingly, assume that $s < T$, so that genuine choice is still possible.

There are two kinds of consistency properties which seem desirable for the overture choice functions $C_T(n)$. The first is dynamic consistency as defined in section 4.4. Suppose that n Pr n' (i.e. n precedes n' on some branch x of X), and that $n' = x'(s')$, where $s' < T$. Suppose, too, that $x|T \in C_T(n)$ $(A(n)|T)$. Now consistency demands that, if $x|T$ is followed up to node n' - as it may be - then the choice set at n' is precisely that part of the original choice set which is still feasible. Thus:-

[illegible] If n Pr n' and $x|T \in C_T(n)$ $(A(n)|T)$, then, for all $A|T \subseteq X|T$:-

$$C_T(n') \; (A(n')|T) = A(n')|T \cap C_T(n) \; (A(n)|T).$$

[illegible], $A(n)|T = \{y|T \in A|T \mid y|s = x|s, \; n = x(s) = y(s) \text{ (some } s)\}$

[illegible] second consistency property is one we have not met before, because [illegible] what happens as the length of an overture, T, changes.

To ease notation, ignore the dependence on the node n, so that the overture choice function $C_T(n)$ becomes simply C_T. For the time being, n is fixed anyway; later, when we want to see again what happens at [illegible], [illegible] shall reintroduce the full notation.

Suppose now that $T' > T$, that $A|T$ is the set of feasible T-overtures, [illegible] the set of feasible options. Write B for $X(A|T)$. Then [illegible], of course.

Suppose next that $\bar{x}|T' \in C_{T'}(B|T')$, but that $\bar{x}|T \notin C_T(B|T)$. Then, if a T-overture is chosen first, instead of a T'-overture, there is no way in which $\bar{x}|T'$ can eventually be chosen, even though it is in the T'-overture choice set. This is a clear inconsistency. It is as though a man, planning one week ahead, decided to eat an apple a day throughout the week, but, planning one day ahead, refused to eat an apple on the first day.

On the other hand, suppose that $x|T \in C_T(B|T)$, but there is no $\bar{x}|T' \in C_{T'}(B|T')$ such that $\bar{x}|T = x|T$. Then $x|T$ is a chosen T-overture; but if a T'-overture were chosen instead, $x|T$ would never be followed. Again, this is a clear inconsistency. It is as though a man, planning one day ahead, decided not to eat an apple, but, planning one week ahead, he would always eat an apple on the first day.

So, to rule out such inconsistencies, we impose the following condition:-

[illegible] $T' > T$ and $B = X(B|T)$,

[illegible] $x|T \in C_T(B|T)$

[illegible] $\exists\ \bar{x}|T' \in C_{T'}(B|T')$ such that $\bar{x}|T = x|T$.

The conditions (α) and (β) are together equivalent to two other conditions of some importance. Suppose that, at the initial node n_0, a plan for T periods is being contemplated. Taking n_0 loses no generality here. Suppose that the planning interval from 0 to T is then split into two sub-intervals, from 0 to t, and from t to T - where $0 < t < T$ of course. Suppose that, at node n_0, a plan is made based on the overture choice function $C_t(n_0)$. Suppose that, at time t, the chosen plan arrives at node n, and that there a new plan is made based on the overture choice function $C_t(n)$. Consistency demands that the resulting path should correspond to the plan which would be made at time 0, if choice were based on the overture choice function $C_T(n_0)$. In other words, the outcome of the planning procedure should not depend on the precise dates at which plans are made, nor on how long the planning interval is. This should be true for all types of planning - five-year plans, rolling plans, sequential planning revision, etc.(1) What we have is the following condition:-

(γ) If $0 < t < T$, and if $B = X(B|t)$, then:-

$x|T \in C_T(n_0)\ (B|T)$

iff (i) $x|t \in C_t(n_0)\ (B|t)$

and (ii) $n = x(t)$ implies $x|T \in C_T(n)\ (B(n)|T)$

The condition $B = X(B|t)$ is needed because, if $B = X(B|T)$, then overture choice from $B|t$ will wrongly assume that $X(B|t)$ is the set of options which are feasible, if it is known that some overture from $B|t$ is to be chosen.

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(1) For a discussion of such planning rules, see Goldman (1968), (1969).

The other condition arises as follows. Suppose that $0 < t < T$ and $B = X(B|t)$ as before. Suppose too that $x|t \in C_t(n_0)(B|t)$. Then the overture was chosen with some extension y in mind, presumably. It follows that if, at time t, a T-overture is to be chosen, this is not [illegible] soluble choice problem; $y|T$, for instance, is an acceptable [illegible] Then we have the following condition:-

[illegible] $t < T$, $B = X(B|t)$, and $x|t \in C_t(n_0)(B|t)$, then

$C_T(n)(B(n)|T)$ is non-empty, where $n = x(t)$.

We have the following result:-

Theorem 9.4.1.

The pair of conditions (α) and (β) are together equivalent to the pair of conditions (γ) and (δ).

Proof

(1) Suppose that (α) and (β) are satisfied, that $0 < t < T$, and that $B = X(B|t)$.

(a) (i) If $x|T \in C_T(n_0)(B|T)$, then $x|t \in C_t(n_0)(B|t)$ (by (β)).

(ii) Also, if $n = x(t)$, it is easy to see that $x|T \in C_T(n)(B(n)|T)$ (by (α)), because n_0 *Pr* n, and $x|T \in B(n)|T \cap C_T(n_0)(B|T)$.

(b) (i) On the other hand, if $x|T \in C_t(n_0)(B|t)$ then, by (β), there exists $y|T \in C_T(n_0)(B|T)$ such that $y|t = x|t$.

Let $n = x(t) = y(t)$.

(ii) If, in addition to (i), $x|T \in C_T(n)(B(n)|T)$, then, by (α), $x|T \in B(n)|T \cap C_T(n_0)(B|T)$. In particular, $x|T \in C_T(n_0)(B|T)$. Together with (a), this verifies (γ).

(iii) Again, because of (i), and by (α):-
$C_T(n)(B(n)|T) = B(n)|T \cap C_T(n_0)(B|T)$. Since $y|T \in B(n)|T$, it follows that $y|T \in C_T(n)(B(n)|T)$. This verifies (δ).

(2) Suppose, conversely, that (γ) and (δ) are satisfied, and that $B = X(B|t)$, where $0 < t < T$.

(a) First, (α) is verified. Suppose $x|T \in C_T(n_0)(B|T)$. Let $n = x(t)$.

(i) By (γ), $x|t \in C_t(n_0)(B|t)$. Now if, $y|T \in C_T(n)(B(n)|T)$, then $y(t) = n$. So $y|t = x|t$, and, by (γ), it follows that $y|T \in C_T(n_0)(B|T)$.

(ii) Conversely, if $y|T \in B(n)|T \cap C_T(n_0)(B|T)$, then, by ($\gamma$), $y|T \in C_T(n)(B(n)|T)$, because $y(t) = n$.

(iii) Taking (i) and (ii) together gives:-

$C_T(n)(B(n)|T) = B(n)|T \cap C_T(n_0)(B|T)$ as required.

(b) Last, (β) is verified.

(i) Suppose that $x|t \in C_t(n_0)(B|t)$. Then, by (δ), there exists $y|T \in C_T(n)(B(n)|T)$, where $n = x(t)$. Then $y(t) = n$, and $y|t = x|t \in C_t(n_0)(B|t)$. By ($\gamma$), it follows that $y|T \in C_T(n_0)(B|T)$. Just take $\bar{x}|t = y|t$.

(ii) Conversely, suppose that $x|T \in C_T(n_0)(B|T)$. Then, by (γ), $x|t \in C_t(n_0)(B|t)$, and so just take $\bar{x}|t = x|t$.

This completes the proof.

The next result shows that (γ) and (δ) - and so, of course, (α) and (β) - are both satisfied in the important special case when the overture choice functions $C_T(n)$ are derived from a consistent dynamic choice function $\{C(n)\}$.

Theorem 9.4.2.

If $\{C(n)\}$ is a consistent dynamic choice function, then its associated overture choice functions $C_T(n)$ satisfy conditions (γ) and (δ).

Before giving the proof of this theorem, we note the following result, which is useful for the proof:-

Lemma 9.4.3.

If $X(B|t) = B$, and $n = x(t)$, where $x \in B$, then $X(B(n)|T) = B(n)$, whenever $T > t$.

Proof.

[illegible] $X(B(n)|T)$

[illegible] $B(n)$ s.t. $y|T = z|T$.

[illegible] B s.t. $y|T = z|T$ and $x|t = z|t$

[illegible] $= y|t$ and $z \in B$ s.t. $y|T = z|T$.

[illegible] $x|t = y|t$ and $y \in X(B|T) = B$

iff $y \in B(n)$.

Proof of Theorem 9.4.2.

Again, without loss of generality, assume the first choice occurs at the initial node n_0. Suppose that $0 < t < T$, and $B = X(B|t)$.

(1) First, (δ) is verified.

Suppose $x|t \in C_t(n_0)(B|t)$. Let $n = x(t)$. Then there exists $y \in C(n_0)(B) \cap X(n)$, because of the definition of overture choice, and because $B = X(B|t)$. Now $y \in C(n_0)(B) \cap B(n)$, because $y \in B \cap X(n) = B(n)$. By dynamic consistency, it follows that $y \in C(n)(B(n))$. So, by lemma 9.4.3., $y \in C(n)(X(B(n)|T)) \cap X(y|T)$. Therefore $y|T \in C_T(n)(B(n)|T)$, which verifies (δ).

(2) Now (γ) is verified.

(a) If $x|T \in C_T(n_0)(B|T)$, then there exists $y \in C(n_0)(B) \cap X(x|T)$.

(i) Then $y \in C(n_0)(B) \cap X(x|t)$, and so $x|t \in C_t(n_0)(B|t)$.

(ii) Also, if $n = x(t) = y(t)$, then, by dynamic consistency, $y \in C(n)(B(n))$. Since $B(n) = X(B(n)|T)$, by lemma 9.4.3., and $y \in X(x|T)$, it follows that $x|T \in C(n)(B(n)|T)$.

(b) If $x|t \in C_t(n_0)(B|t)$, $n = x(t)$, and $x|T \in C_T(B(n)|T)$, then there exist y and z such that $y \in C(n_0)(B) \cap X(x|t)$ and $z \in C(n)(B(n)) \cap X(x|T)$. So $y \in C(n_0)(B) \cap B(n)$. By dynamic consistency, it follows that $C(n)(B(n) = C(n_0)(B) \cap B(n)$. Therefore $z \in C(n_0)(B) \cap X(x|T)$, and so $x|T \in C_T(n_0)(B|T)$, as required.

[illegible] is a generalized version of Bellman's "Principle of Optimality"[1]. Provided dynamic choice is consistent, any chosen (or optimal) plan can be achieved by choosing an appropriate (optimal) overture, and then an appropriate (optimal) completion to the plan.

We are now in a position to see that adopting finite horizon overture choice is likely to be naive. Suppose that we start with a plan indicated by $C_T^H(n_0)$ for some finite H. This is not the end of planning, of course. When period T has been reached, at the latest, a further choice has to be made. There is no guarantee that the new choice will be consistent with the old - in the sense of conditions (α) and (β) - unless the horizon H remains unchanged. On the other hand, unless H is changed, we shall in the end come up against the horizon H, and be forced to change it. So, to avoid eventual inconsistency, overture choice, like ordinary choice, has to be based on an infinite horizon.

But is infinite horizon overture choice itself consistent? Recall that (α) is the dynamic consistency condition of section 4.4. In section 7.5 we saw how the dynamic set function $\{O(n)\}$[2] is consistent, but the

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(1) Bellman (1957).

(2) $O(n)$ is a set function, rather than a choice function, because $O(n)(A)$ may be empty even when A is finite.

dynamic choice function $\{M(n)\}$ is no more than weakly consistent, in general. But there is a consistent dynamic choice function $\{C(n)\}$ with ordinal components, such that, whenever $A \subseteq X(n)$, $O(n)(A) \subseteq C(n)(A) \subseteq M(n)(A)$. The same is true of the dynamic overture choice functions $\{O_T(n)\}$ and $\{M_T(n)\}$, defined in section 9.3. The argument is effectively the same as in section 7.5. and so there is no need to repeat it here. So, as regards (α), infinite horizon overture choice has the same consistency properties as the more orthodox infinite horizon choice of chapter 7.

Condition (β), however, creates a serious new problem, as the example in the next section shows.

9.5. An Example of Inconsistent Overture Choice.

The owner of some newly cleared land is deciding whether to make an orchard or a vineyard of it. If he decides on an orchard, he cannot later convert it to a vineyard. Nor can he convert a vineyard into an orchard.

The orchard will produce £1000 worth of apples each year, starting the year after the apple-trees are planted. The vineyard will produce a single harvest of grapes, one year after the vines are planted. The harvested grapes will then be used to produce 1000 bottles of *vin mystérieux*.

There are an infinite number of different types of *vin mystérieux*. Type m matures steadily for m years, then deteriorates. Provided that its age a is not greater than m, a bottle of type m *vin mystérieux* will sell for £$2a$. But if the age a exceeds m, the price of type m *vin mystérieux* is less than £$2m$.(1)

The owner of the land wants to maximize the total present value of his sales of apples and of wine. The rate of discount is zero.

His options can be described as follows:-

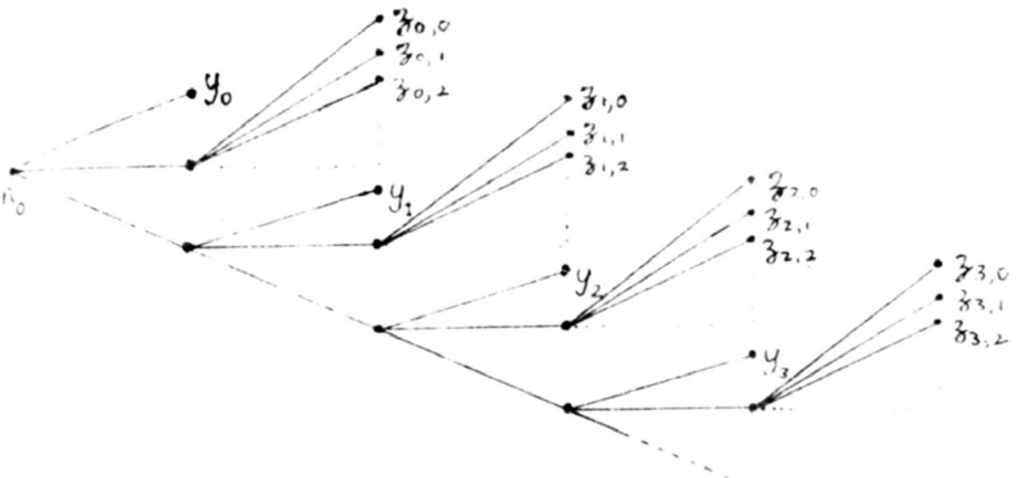
y_T - leave land idle for T years, then plant apples.

$z_{T,m}$ - leave land idle for T years, then plant vines, and produce type m *vin mystérieux*, to be sold in year $T + m$.

y_∞ - leave land idle for ever.

His decision tree can be depicted as follows:-

(1) The precise price is irrelevant, because it never pays to keep type m *vin mystérieux* after its age is m.



where the move from any node to the next takes one year.

W^H is his revenue, in thousands of pounds, when the horizon is H.

Thus $W^H(y_T) \;\; = H - T$

$$W^H(z_{T,m}) = 2m \qquad (\text{if } H \geq T + m + 1)$$

If the horizon is H, the optimum is to plant vines immediately, and then produce type $(H-1)$ *vin mystérieux*. So $\bar{W}^H = 2(H-1)$.

The values of the various possible T-overtures are as follows:-

(a) $$W^H(y_S|T) = \begin{cases} H - S & (S < T) \\ 2(H-T-1) & (S \geq T) \end{cases}$$

For, if $S < T$, then apples have already been planted in year S, and so the revenue is £$(H-S) \times 1000$. But, if $S \geq T$, and the horizon is H, it is optimal to plant vines at onee and produce *vin mystérieux* which does not fully mature until H at the earliest.

(b)

$$W^H(z_{S,m}|T) = \begin{cases} 2(H-T) & (S = T-1) \\ 2(S+m) & (S < T-1,\ H \geq S+m) \\ 2(H-T-1) & (S \geq T) \end{cases}$$

For, if $S < T-1$, the vines have already been planted, and type m *vin mystérieux* produced. If $S \geq T-1$, it is still possible to produce *vin mystérieux* which matures right up to the horizon. When $S = T-1$, there is an extra year in which the wine can mature because the grapes will already have been harvested at time T.

It follows that the optimal T-overture is to plant vines at time $T-1$. The assumption is that the horizon is known at time T, and so it will be possible to decide the appropriate type of *vin mystérieux* then. But if the horizon H is, in fact, not known at time T, the decision to plant vines turns out to be premature.

It is clear that choosing an optimal T-overture violates the consistency condition (β). The optimal policy, too, seems clear. Assuming as we do, that the horizon H will be known at some time T, and that H is large enough, it is optimal to leave the land fallow until the horizon is known, and then to grow vines and produce an appropriate type of *vin mystérieux*. Such a plan is called *overture optimal*, and is studied in section 9.6.

On the other hand, there is an infinite horizon optimal plan for this example. It consists of planting apple trees at once.

For:- $W^H(y_0) - W^H(y_T) = T$

$$W^H(y_0) - W^H(z_{T,m}) = H - 2m \quad (\text{if } H \geq T+m)$$

$$\to \infty \text{ as } H \to \infty.$$

This is an example in which there is an infinite horizon optimal plan which is not agreeable.

Indeed, $\bar{W}^H - W_T^H(y_O|T) = 2H - T$

$$\rightarrow \infty \text{ as } H \rightarrow \infty,$$

for all T, and so the optimum is infinitely disagreeable. In fact, the features of the optimum are extremely unsatisfactory, because, for the path y_∞ which involves never planting anything,

$$W_T^H(y_\infty|T) - W_T^H(y_O|T) = 2(H-T-1) - H$$

$$\rightarrow \infty \text{ as } H \rightarrow \infty$$

In section 9.6., we shall see that this means that y_O is not even "overture maximal".

Notice that, if $T' > T$, then:-

$$W_{T'}^H(z_{T'-1,m}|T') - W_{T'}^H(z_{T-1,m}|T')$$

$$= 2(H-T') - 2(T-1+m) \quad (\text{if } H \geq T-1+m)$$

$$\rightarrow \infty \text{ as } H \rightarrow \infty$$

whereas

$$W_T^H(z_{T'-1,m}|T) - W_T^H(z_{T-1,m}|T) = 0$$

So, whether the horizon is going to be revealed at time T or time T', it is no worse to set off on an optimal T'-overture than on an optimal T-overture, and it may be very much better. Given the uncertainty about the horizon and when more about it will be known, the optimal T'-overture is definitely superior. That is, the more farsighted choice is superior. This is true generally, and suggests a way of overcoming violations of (3), which is the topic to which we now turn our attention.

9.6. A Procedure for Consistent Infinite-Horizon Overture Choice.

Let us recapitulate. Suppose that, for each horizon H, there is an additive welfare function:-

$$W^H(x) = \sum_{t=1}^{H} u_t(x_t)$$

For each T, and each $H \geq T$, there is a T-overture welfare function W_T^H defined on $X|T$, the underlying set of T-overtures, as follows:-

$$W_T^H(\bar{x}|T) = max\ \{W^H(x) \mid x \in X,\ x|T = \bar{x}|T\}$$

Then, there are infinite-horizon T-overture preference relations defined as follows:-

$$x|T \quad P|T \quad y|T \quad \text{iff} \quad \liminf_{H \to \infty} \left[W_T^H(x|T) - W_T^H(y|T)\right] > 0$$

$$x|T \quad R|T \quad y|T \quad \text{iff} \quad \liminf_{H \to \infty} \left[W_T^H(x|T) - W_T^H(y|T)\right] \geq 0.$$

For each T, one can then define optimal and maximal T-overtures, as was done in section 9.3.

A simple procedure would be to select T arbitrarily, find an optimal T-overture (assuming there is one), and follow it. Unfortunately, as was seen in section 9.5., this is not always consistent, in the sense of property (β):-

If $T' > T$ and $B = X(B|T)$,

then $x|T \in C_T(B|T)$

iff $\exists\, \bar{x}|T' \in C_{T'}(B|T')$ such that $\bar{x}|T = x|T$.

That is, there may be an optimal T-overture $x|T$, such that, for any optimal T'-overture $\bar{x}|T'$, the corresponding T-overture $\bar{x}|T$ is different from $x|T$. Then, the question arises as to which should be chosen - the optimal T-overture, or the optimal T'-overture, or something else entirely?

In section 9.5., we saw that there were advantages to lengthening the overture which was being chosen. If the horizon becomes known before time T, then the choice of an optimal T-overture, to be followed until the horizon is known, is the best that can be done, it seems. But if the horizon does not become known until after time T, the choice of an optimal T-overture could turn out to be disastrous.

Suppose that x and y are two possible paths. Suppose that their T-overtures $x|T$ and $y|T$ satisfy $x|T \;\; R|T \;\; y|T$, but that, for all $T' > T$, $y|T' \;\; P|T' \;\; x|T'$. Suppose, too, that the horizon H becomes known at time $T' > T$. Then, provided that H is long enough after T', $y|T$ turns out to be a better choice of T-overture than $x|T$ does. Indeed, for all T', $W^H_{T'}(y|T') > W^H_{T'}(x|T')$ provided that H is large enough. It follows that until more is known about the horizon, it is better to follow the path y than it is to follow x.

Notice that we are comparing two *paths* again, rather than T-overtures. This is inevitable, if we wish to ensure consistency, in the sense of property (B). In principle, we must choose arbitrarily long overtures, and so, in the end, entire paths. But the choice criterion is different from any of those in chapter 7. This is because paths are being compared via their overtures, with extensions of these overtures in mind. Section 9.5. shows that there is a fundamental difference between optimal paths in the usual sense, and agreeable or overture optimal plans, as considered in this chapter. It is a difference which we shall explain further in section 9.7. Notice, however, that we may be faced with something of a dilemma: do we follow the optimal plan or the agreeable plan? The answer depends upon how likely it seems that there will be advance notice of the true horizon. Fortunately, as will be seen in chapter 10, in most cases it is not a dilemma which actually arises, because an optimal plan is also agreeable.

Once it is realised that entire paths have to be compared, through the properties of their overtures, the form of choice criterion becomes obvious. We wish to define new preference relations $\bar{R}$ and $\bar{P}$. These will be the limits, as T tends to infinity, of the original preference relations $R|T$ and $P|T$. That way, we compare overtures of infinite length - i.e. entire paths - by looking at the T-overtures. In order to ensure continuity within finite product subspaces, we shall proceed along similar lines to those of section 7.4. and define $\bar{R}$, $\bar{P}$ as follows:-

$$x \; \bar{R} \; y \quad \text{iff} \quad \liminf_{T \to \infty} \left[\liminf_{H \to \infty} \left[W_T^H(x|T) - W_T^H(y|T) \right] \right] \geq 0$$

$$x \; \bar{P} \; y \quad \text{iff} \quad \liminf_{T \to \infty} \left[\liminf_{H \to \infty} \left[W_T^H(x|T) - W_T^H(y|T) \right] \right] > 0$$

As far as consistency property (β) is concerned, there is now no problem, because an entire path is effectively being chosen anyway. As for property (α), dynamic consistency, it is easy to check that the relations $\bar{R}$ and $\bar{P}$ are consistent. The proofs are virtually identical to those of section 7.5.

Say that a plan $x^* \in X$ is *overture optimal* if, for all $x \in X$, $x^* \bar{R} x$. Say that $x^* \in X$ is *overture maximal* if there is no $x \in X$ such that $x^* \bar{P} x$.

As in section 7.5., to choose an overture optimal plan at each moment of time is dynamically consistent: to choose an overture maximal plan may not be. But, provided there is always an overture maximal plan, consistent dynamic choice is possible. Also, if there is a unique overture maximal plan at each moment of time, then to choose it is certainly dynamically consistent.

Finally, it is worth noting the following result:-

<u>Lemma 9.8.1.</u>

Let $x^* \in X$ be a plan such that, for all T, $x^*|T$ is an optimal (resp. maximal) T-overture (as defined in section 7.3.) Then x^* is overture optimal (resp. maximal).

<u>Proof.</u>

Trivial.

Of course, because of theorem 7.3.1., an agreeable plan is overture optimal.

9.7. A Note on Double Limits.

Section 9.5. brought out the difference between choosing an optimal plan, and choosing an overture optimal plan. It is worth noting that this is another instance in which the order in which one carries out limiting processes may matter.

Consider two paths, x and y, and the associated values of their measures, when the horizon is H, $W_T^H(x)$ and $W_T^H(y)$. Now, in section 9.6., we defined the overture relation $\bar{R}$ as follows:-

$$x \bar{R} y \text{ iff } \liminf_{T \to \infty} \liminf_{H \to \infty} \left[W_T^H(x|T) - W_T^H(y|T)\right] \geq 0$$

Suppose we reverse the order of the two limits, and define R^* by:-

$$x R^* y \text{ iff } \liminf_{H \to \infty} \liminf_{T \to \infty} \left[W_T^H(x|T) - W_T^H(y|T)\right] \geq 0.$$

But, for all $T \geq H$, $W_T^H(x|T)$ is simply $W^H(x)$, and $W_T^H(y|T) = W^H(y)$.

So $x R^* y$ iff $\liminf_{H \to \infty} \left[W^H(x) - W^H(y)\right] \geq 0.$

Thus R^* is the relation "catches up to" of section 7.4., and, by reversing the order of the two limits, we are back with the usual optimality criterion.

9.7. A Note on Double Limits.

Section 9.5. brought out the difference between choosing an optimal plan, and choosing an overture optimal plan. It is worth noting that this is another instance in which the order in which one carries out the limiting processes may matter.

Consider two paths, x and y, and the associated values of their overtures, when the horizon is H, $W_T^H(x)$ and $W_T^H(y)$. Now, in section 9.6., we defined the overture relation $\bar{R}$ as follows:-

$$x \bar{R} y \text{ iff } \liminf_{T \to \infty} \liminf_{H \to \infty} \left[W_T^H(x|T) - W_T^H(y|T)\right] \geq 0$$

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$$x R^* y \text{ iff } \liminf_{H \to \infty} \liminf_{T \to \infty} \left[W_T^H(x|T) - W_T^H(y|T)\right] \geq 0.$$

But, for all $T \geq H$, $W_T^H(x|T)$ is simply $W^H(x)$, and $W_T^H(y|T) = W^H(y)$.

So $x R^* y$ iff $\liminf_{H \to \infty} \left[W^H(x) - W^H(y)\right] \geq 0.$

Thus R^* is the relation "catches up to" of section 7.4., and, by reversing the order of the two limits, we are back with the usual optimality criterion.

9.8. Conclusions.

This chapter has developed an alternative infinite horizon choice procedure. It is an alternative to that of chapter 7. The point is that, if we accept that *some* T-overture has to be chosen, which one do we want to choose? This was the starting point of our search for a choice procedure, and infinite horizon T-overture choice was defined in section 9.3. But, in checking the consistency of this procedure, we received a setback in section 9.5. This showed that we cannot avoid choosing an entire path, in the end, because we have to contemplate choosing arbitrarily long overtures. Section 9.5. suggested a criterion for comparing paths in the light of their overtures, and the finite horizon extensions of these overtures. The resulting overture choices can be made dynamically consistent.

We are left with a number of important theoretical questions:-

(1) When is there a plan which is

(a) overture maximal, (b) overture optimal, (c) agreeable?

This question is important, because our search was motivated by the troublesome example 8.6., in which there is no ordinary maximal plan. Our search was fruitless unless we can now handle such problems.

(2) When is there a unique overture maximal plan?

This question is important for two reasons. First, the set of maximal plans is an upper bound on the limit choice set, as was seen in section 7.3. Thus, overture maximality is *necessary* for a plan to be an appropriate choice; we cannot be sure that it is *sufficient* unless there is one overture maximal plan only. Second, there is no dynamic consistency problem when there is a unique overture maximal plan.

(3) Is it possible to characterize an agreeable, overture optimal, or overture maximal, plan - in a way which makes it fairly easy to find in practice?

It is to these questions which chapter 10 is addressed.

Chapter 10.

AN INTRODUCTION TO THE THEORY OF OVERTURE PLANNING.

10.1. Introduction.

At the end of chapter 9, three questions were raised:-

(1) When is there an agreeable plan, an overture optimal plan, or an overture maximal plan ?

(2) When is there a unique overture maximal plan?

(3) Can we characterize an agreeable plan, an overture optimal plan, or an overture maximal plan?

One other question, suggested by example 9.5.1., and of some importance, is:-

(4) When is an optimal plan agreeable, etc.?

These questions will not be answered separately. Instead, we shall consider a number of different situations in which we can say something useful about several of these four questions. The situations can be described as follows:-

(A) The economy is described by a one-good model of capital accumulation, with a concave production function, and strictly concave utility functions $u_t(\cdot)$. This case, as described in section 8.6., has been discussed in "Agreeable Plans".[1] Here, the main results are summarized:-

(1) There is an agreeable plan if and only if there is a locally optimal path (i.e. one that maximizes $\sum_{t=0}^{T} u_t(c_t)$, given initial and terminal capital stocks, for all T) on which consumption is always positive (or, at any rate, on which $u_t(c_t)$ is always finite).

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(1) In section 8.6., and in "Agreeable Plans", time was continuous, whereas here it is discrete. This makes no real difference.

(2) If there is an overture maximal plan, it is the unique locally optimal path on which consumption at each moment of time is maximized. Moreover, it is agreeable. (This is not proved directly. But, given the assumptions, any overture maximal plan is unique, by theorem 10.6.3. Also, the "maximal" locally optimal path is agreeable.)

(3) If there is an optimal plan, it is agreeable. But an agreeable plan may not even be maximal, as was seen in section 8.6.

(B) There is a path x^* such that, for some sequence of finite horizon optima x^H, $x^H|T \to x^*|T$ as $H \to \infty$, for all T (or, there is a subsequence of finite horizon optima $x^{H(n)}$ such that $x^{H(n)}|T \to x^*|T$ as $n \to \infty$).

This case, the "insensitivity" case, is discussed in section 10.2. It will be seen that x^* is agreeable (or overture maximal) provided that it satisfies a certain flexibility property. This demonstrates existence, and also provides a characterization.

(C) There is an optimal path x^*, whose existence can be demonstrated in one of the two standard ways discussed in section 8.5. That is, either the economy is valuation-finite, or x^* is a price-competitive path, satisfying the Malinvaud condition.

This case is discussed in section 10.3., 10.4., and 10.5. We shall see that x^* is then an agreeable plan. Again, this demonstrates existence, and also provides a characterization. In addition, of course, it shows that in certain important cases, an optimal plan is agreeable.

(D) The economy has the property that X is a convex set, and each one period utility function $v(\cdot,\cdot,t)$ is strictly concave. This is shown to guarantee that, if there is an overture maximal plan, then it is unique.

Also, if there is an agreeable plan, then it must be the limit of the sequence of unique finite horizon optima x^H as $H \to \infty$. This case is considered in 10.6. Some remarks on the price decentralization of plans are also made in section 10.7.

It is assumed throughout that we are in the model of capital accumulation of chapter 8. In particular, we make the assumptions (A.1) – (A.5) of that chapter, although some of these assumptions are only needed for one or two of the results.

10.2. Insensitivity, Agreeable Plans, and Overture Maxima.

In section 9.1., a plan x^* was said to be *insensitive* if, for each T, and for the sequence of finite horizon optimal plans x^H, $x^H|T \to x^*|T$ as $H \to \infty$. An economy with an insensitive plan of capital accumulation was discussed in section 8.6. We are now in a position to show that, though there is no maximal plan even, the insensitive plan is agreeable. Indeed, we can prove that an insensitive plan is agreeable under very general conditions. But before doing so, let us consider two examples in which the insensitive plan is not agreeable, or even overture maximal.

Example 10.2.1.[1]

There is a single good, cake, of which the initial supply is one. Cake can never be produced. If x_t is the stock of cake at time t, and c_t is the consumption of cake in period t, then $x_{t+1} = x_t - c_t$. (Cake does not go mouldy.) The welfare function is $W^H \equiv \sum_{t=0}^{H} u(c_t)$, where u is strictly concave, and $u(0) = -\infty$. Clearly, the optimal finite horizon plan, if the horizon is H, is to eat equal quantities of cake, $\frac{1}{H+1}$, in each period up to the horizon. In the limit, as $H \to \infty$, this involves eating no cake at all. But this is infinitely worse, even in an overture sense, than any plan which involves eating positive quantities of cake each period. So the insensitive plan, of eating no cake, is certainly not overture maximal even.

In fact, there can be no overture maximal plan in this example. For, to have any hope of being overture maximal, a path (x_t^*, c_t^*) must have $c_t^* > 0$, for all t. Consider the alternative path with $c_t = \frac{1}{2} c_t^*$ (each t).

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(1) See Gale (1967), example 2, p. 4.

Then $x_T > x_T^*$ (each $T > 0$). So

$$W_T^H(x|T) - W_T^H(x^*|T)$$

$$= \sum_{t=0}^{T} \left[u(\tfrac{1}{2}\, c_t^*) - u(c_t^*)\right] + \sum_{t=T+1}^{H} \left[u\left(\frac{x_T}{H-T}\right) - u\left(\frac{x_T^*}{H-T}\right)\right]$$

$$= W^T(x) - W^T(x^*) + (H-T) \left[u\left(\frac{x_T}{H-T}\right) - u\left(\frac{x_T^*}{H-T}\right)\right].$$

$$(H-T) \left[u\left(\frac{x_T}{H-T}\right) - u\left(\frac{x_T^*}{H-T}\right)\right] \geq (x_T - x_T^*)\, u'\left(\frac{x_T}{H-T}\right)$$

$$\to \infty \text{ as } H \to \infty$$

$$\liminf_{H \to \infty} \left[W^H(x|T) - W^H(x^*|T)\right] = +\infty$$

whenever $T > 0$.

So there is no infinite horizon plan of any kind which one can recommend. This failure seems inevitable. The horizon is crucial; some kind of probability distribution over the possible horizons is required before *sensible choice* is possible.

It might be thought that, provided minus infinite utility is excluded, an insensitive plan would be at least overture maximal. In the cake-eating example, if $u(0)$ is finite, then it is easy to see that to consume nothing for ever is, in fact, agreeable, because u is continuous. But the following is an example in which an insensitive plan is not even overture maximal.

Example 10.2.2.

There are two capital goods, *1* and *2*. Good *2* is also a consumption good, and is like the cake of example 10.2.1. Good *1* is needed in order to consume good *2* - in fact, for each unit of good *2* consumed in period t, there must be a unit of good *1* available. Good *1* is infinitely durable. There are constant returns to scale.

Initially, there is one unit of good *1*. In period *0*, some or all of this can be converted into good *2*, unit for unit. Thereafter, no conversion either way is possible.

It follows that the consumption stream $(c(1), c(2), \ldots)$ is feasible if and only if

(i) $\sup c(t)$ units of good *1* are made available, at least

(ii) $\sum_{t=1}^{\infty} c(t) + \sup c(t) \leq 1.$

The welfare function, if the horizon is H, is

$$\sum_{t=1}^{H} \sqrt{t\, c(t)}.$$

If the horizon is H, the optimal consumption stream satisfies the equations:-

$$\sqrt{\frac{t}{c^H(t)}} = \lambda, \qquad \sum_{t=1}^{H} c^H(t) = 1 - c^H(H).$$

$$c^H(t) = t/\lambda^2, \text{ and } \frac{H(H+1)}{2\lambda^2} = 1 - \frac{H}{\lambda^2}$$

Thus $2\lambda^2 = H^2 + 3H.$

Therefore $c^H(t) = \dfrac{2t}{H(H+3)}$ $\quad (t = 1 \text{ to } H).$

Also $\quad \bar{W}^H = \sqrt{\dfrac{H}{2(H+3)}} \cdot (H+1).$

As $H \to \infty$, $c^H(t) \to c^*(t) = 0,$

and $x_1^H(t) = \dfrac{2}{H+3} \to x_1^*(t) = 0$

while $x_2^H(1) = 1 - x_1^H(1) \to x_2^*(1) = 1.$

So the insensitive path involves converting all of good 1 into good 2 initially. But, for all H and T, $W_T^H(c^*|T) = 0$, because, with zero provision of good 1, no consumption is ever possible. So the insensitive plan is clearly not agreeable - indeed, $\lim\limits_{H\to\infty} \bar{W}^H - W_T^H(c^*|T) = +\infty$ so that c^* is infinitely disagreeable.

Moreover, c^* is not overture maximal. To show this, consider the following consumption program:-

$$c(t) = 4^{-t}.$$

together with

$$x_1(t) = {}^1/_4,$$

$$x_2(t) = \tfrac{3}{4} - \frac{1 - 4^{1-t}}{3}$$

If, at time T, the horizon is discovered to be H, the best plan is to switch to a path $c'(t)$ on which

$$\sqrt{\frac{t}{c'(t)}} = \lambda \text{ and } \sum_{t=T}^{H} c'(t) = x_2(T),$$

This is feasible provided that $c'(H) \leq {}^1/_4$, which will be true if H is large enough.

So $$c'(t) = \frac{t}{\lambda^2}, \quad \lambda^2\, x_2(T) = \frac{(H+T)(H-T+1)}{2}$$

i.e. $$c'(t) = \frac{2t\, x_2(T)}{(H+T)(H-T+1)}$$

So $$W_T^H(c|T) = \sum_{t=1}^{T-1} \sqrt{t}\,.\, 2^{-t} + \sqrt{\frac{(H+T)(H-T+1)x_2(T)}{2}}$$

Since $x_2(T) \to {}^5/_{12}$ as $T \to \infty$,

it is clear that:-

$$\liminf_{T \to \infty} \liminf_{H \to \infty} \left[W_T^H(c|T) - W_T^H(c^*|T) \right] = +\infty$$

So c^* is infinitely "overture inferior" to c.

In example 10.2.2., although it is true that $x^H|T \to x^*|T$ as $H \to \infty$ the difficulty is in getting back near the path x^H after following x^*. In fact, this is impossible, because none of good 1 has been provided.

Nevertheless, it is possible to see the cases in which, although $x^H|T \to x^*|T$ as $H \to \infty$ for each T, x^* is not agreeable, as special cases. They can be ruled out by making an assumption.

First, recall the definition of $F(s,a,t)$ in section 8.3. If $s \leq t$, then:-

$$F(s,a,t) = \{b \mid \exists \langle x(r) \rangle_{r=s-1}^{t} \text{ s.t. } x(s-1) = a,\ x(t) = b$$
$$\text{and } (x(t-1), x(r)) \in X(r) \quad (r = s \text{ to } t)\}$$

Say that a feasible path x is ***flexible*** if, for all s, there exists $t \geq s$ such that $x(t) \in \operatorname{int} F(s,x(s),t)$.[1]

Notice that, for a given path x, if $s' \leq s$ and $x(t) \in \operatorname{int} F(s,x(s),t)$, then

$x(t) \in \operatorname{int} F(s',x(s'),t)$

because $F(s,x(s),t) \subseteq F(s',x(s'),t)$.

In particular, $x(t) \in \operatorname{int} G(t)$, where

$G(t) = F(0, \bar{x}(0), t)$, and $\bar{x}(0)$ is the initial capital stock. By assumption (A.6), this implies that there is a path $\hat{x}$, with $\hat{x}(0) = \bar{x}(0)$, $\hat{x}(t) = x(t)$, such that $W^t(\hat{x})$ is finite.

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[1] I owe the term "flexible" to Geoffrey Heal and his forthcoming book, Heal (1973).

Suppose that each capital good is either freely disposable, or can be disposed at some cost in terms of current resources. In other words, suppose that if $(a,b) \in X(t)$, and $a' \geq a$, then $(a',b) \in X(t)$. Then flexibility is equivalent to:-

For all s, there exists $t \geq s$ and $b > x(t)$ such that $b \in F(s,x(s),t)$.

In other words, given enough time, it is possible to produce more of every capital good.

It may be worth comparing flexibility with a number of other concepts in capital theory, such as "nontightness".(1) First, however, it should be realized that concepts such as nontightness refer to net output vectors, rather than to stocks of capital. The most closely related concept seems to be Kurz's "capable of output increase" condition.(2) A corresponding concept is the following:-

Say that a path x^* in X is *capable of stock increase* if, for *every* s and *every* t such that $0 \leq s < t$, there exists a number $k(t) > 1$ and a path $x \in X$ such that:-

$$x(t) = k(t)\, x^*(t)$$

and $\quad x(r) = x^*(r)$ for all $r < s$ and all $r > t$.

If there is free disposal, flexibility is equivalent to:-

For every s, there exists *some* $t > s$, a number $k > 1$, and a path $x \in X$ such that:-

$$x(t) = k\, x^*(t)$$

$$x(r) = x^*(r) \text{ for all } r < s \text{ and all } r > t.$$

(1) See Malinvaud (1962), (1969), Kurz (1969) and Kurz and Starrett (1970).

(2) See Kurz (1969), definition 8, p. 270.

So, for x^* to be capable of stock increase is considerably stronger than flexibility. In fact, if a plan is capable of stock increase, it may well necessarily involve inefficient storage. To see this, take $t = s + 1$, and suppose that good 1 is a four-period old machine, while good 2 is a five-period old machine. Since the only way of producing five-period old machines is to keep a four-period old machine for a whole period, $x_2^*(s+1) \leq x_1^*(s)$, if x^* is feasible. Suppose storage is efficient, so that $x_2^*(s+1) = x_1^*(s)$. Then it is impossible to find a $k(s+1) > 1$ such that the path x with:-

$$x(s+1) = k(s+1)\, x^*(s+1)$$

$$\text{and} \quad x(r) = x^*(r) \ (\text{all } r < s \text{ and all } r > s+1)$$

is feasible. So x^* is not capable of stock increase. Yet it may be flexible, because it may be possible to increase the number of machines of all ages available at time t, by producing more machines far enough in advance.

Flexibility does, however, imply that goods which can never be produced must eventually have their stocks reduced by more than is necessary. But, as example 10.2.2. showed, capital goods which cannot be produced may well cause trouble.

It is worth remarking that, in the standard one good model of section 8.6., a path is flexible if and only if there is an infinite set of periods in which the capital stock is reduced either by consumption, or by disposal. For, with just one capital good, and ignoring depreciation, x^* is flexible if an only if, for every s, there exists $t > s$ and a feasible path x such that:-

$$x(t) > x^*(t)$$

$$\text{and} \quad x(r) = x^*(r) \text{ for all } r < s \text{ and all } r > t.$$

So, x^* is flexible if and only if, for every s, there exists $t' > s$ and a feasible path x such that:-

$$x(t') \leq x^*(t'),\ x(t'+1) > x^*(t'+1)$$

Such a path x exists if and only if the capital stock is reduced in period t' on the path x^*.

We can now prove that flexibility is sufficient to ensure that an competitive path is agreeable.

Theorem 10.2.3.

Suppose that x^* is flexible, and that, for all T, $x^H|T \to x^*|T$ as $H \to \infty$. Then x^* is agreeable.

Proof

For any T, there exists $S > T$ such that

$$x^*(S) \in \text{int } F(T, x^*(T), S).$$

For this S, consider the correspondence $X(\cdot,\cdot,T,S)$ defined in section 8.3. as follows:-

$$X(a,b,T,S) = \{x \in X \mid x(T) = a,\ x(S) = b\}$$

By (A.4), it is lower semi-continuous. Since $x^H(S) \to x^*(S)$, and $x^* \in X(x^*(T), x^*(S), T, S)$, it follows that there exists a sequence of paths y^H and a number H_0 such that, whenever $H \geq H_0$, $y^H \in X(x^*(T), x^H(S), T, S)$, and $y^H|S \to x^*|S$. Moreover, we can take $y^H|T = x^*|T$, for all $H \geq H_0$.

Define the path z^H for all $H \geq H_0$ by:-

$$z^H(t) = \begin{cases} y^H(t) & (t \leq S) \\ x^H(t) & (t > S) \end{cases}$$

Evidently, $W_T^H(x^*|T) \geq W^H(z^H)$. Therefore, it remains only to show that $\overline{W}^H - W^H(z^H) \to 0$. But $\overline{W}^H - W^H(z^H) = W^S(x^H) - W^S(z^H)$, and $x^H|S \to x^*|S$, $z^H|S \to x^*|S$. Also, since x^* is flexible, $W^S(\cdot)$ is finite and so continuous at x^*. Therefore $\bar{W}^H - W^H(z^H) \to 0$, as required.

These conditions are powerful. The only cases in which they are not satisfied are:-

(1) There is no insensitive path. This means that there exists T_0 such that, for all $T \geq T_0$, the sequence $x^H|T$ fails to converge as $H \to \infty$.

(2) Although there is an insensitive path x^*, x^* is inflexible.

There is little to be done in the second case, as examples 10.2.1. and 10.2.2. showed. But the first case is more easily dealt with. Because X is compact, there is at least one path x^*, and a sequence of horizons $H(n)$, such that $x^{H(n)}|T \to x^*|T$ as $n \to \infty$, for all T. Should x^* be inflexible, we are back with case (2) again. But, as will shortly be seen, if there is such a limit plan x^* which is flexible, then x^* is at least overture maximal.

First, however, let us show that the problem is not vacuous, by producing an example with no insensitive plan.

Example 10.2.4.

There are two types of capital good - blighted blackberry bushes and cleared land. Initially, there is one unit of land uniformly covered by blighted blackberry bushes. Each unit of land left to blackberry bushes provides 2^{-t} units of food in period t. On any piece of land, at any time, the blackberry bushes can be burnt down costlessly, and thereafter the land can be used to produce a stream of food output $(0,1,0,1,0,1,\ldots)$ per unit area (i.e., 0 just after initial planting, 1 at harvest time, and so on). Assume that food can be stored for one period costlessly, and that initially $\frac{1}{2}$ unit of food is in store. Then, given a strictly concave utility function $u(c(t))$ of food consumption in period t, it is enough to maximize total food output.

Feasible paths of food output are convex combinations of:-

$y^0 = (0,1,0,1,0,1,\ldots)$ (blackberry bushes removed at once

$y^1 = (1,0,1,0,1,0,\ldots)$ (" " " after one period)

$y^2 = (1,\frac{1}{2},0,1,0,1,\ldots)$ (" " " after two periods)

$y^3 = (1,\frac{1}{2},\frac{1}{4},0,1,0,\ldots)$ (" " " after three periods)

etc.

Evidently, of these, only y^1 and y^2 are undominated. If the horizon is H, then the optimum is y^1 if H is odd, and y^2 if H is even.

Indeed, $$W^H(y^1) = \begin{cases} \frac{1}{2}(H+1) & (H \text{ odd}) \\ \frac{1}{2} H & (H \text{ even}) \end{cases}$$

$$W^H(y^2) = \begin{cases} \frac{1}{2}(H+1) & (H \text{ even}) \\ \frac{1}{2} H & (H \text{ odd}, \; H \geq 3) \end{cases}$$

So, for any undominated path $y = \lambda y^1 + \mu y^2$ (where $\lambda + \mu = 1$, and $\lambda,\mu \geq 0$),

$$W^H(y) = \begin{cases} \frac{1}{2}(H+\lambda) & (H \text{ odd}) \\ \frac{1}{2}(H+\mu) & (H \text{ even}) \end{cases}$$

Also, $$\bar{W}^H = \frac{1}{2}(H+1)$$

There is no insensitive path. Also, because:-

$$\bar{W}^H - W^H(y) = \begin{cases} \frac{1}{2}\mu & (H \text{ odd}) \\ \frac{1}{2}\lambda & (H \text{ even}) \end{cases}$$

there is no agreeable path either. But any path $y = \lambda y^1 + \mu y^2$ is overtaking maximal, because if $y^* = \lambda^* y^1 + \mu^* y^2$, then

$W^H(y^*) - W^H(y) = (-1)^H \frac{1}{2}(\lambda - \lambda^*)$, so that the *lim inf* as $H \to \infty$ is certainly not positive.

Now for the result:-

Theorem 10.2.5.

Let $H(n)$ be any sequence of horizons. Suppose that, for each T, $x^{H(n)}|T \to x^*|T$ as $n \to \infty$. Suppose too that x^* is flexible. Then x^* is overture maximal.

Proof

It is more than sufficient to prove that, for all T,

$$\liminf_{H \to \infty} \left[\bar{W}^H - W_T^H(x^*|T)\right] = 0 \qquad \text{(A)}$$

because then, for all T and for all paths $x \in X$,

$$\liminf_{H \to \infty} \left[W_T^H(x|T) - W_T^H(x^*|T)\right] \leq 0.$$

But to prove that (A) is true, it is only necessary to show that

$$\lim_{n \to \infty} \left[\bar{W}^{H(n)} - W_T^{H(n)}(x^*|T)\right] = 0.$$

The proof of this is so close to that of theorem 10.2.3. that it is not worth setting out here.

We can now see that a *necessary* condition for the non-existence of an overture maximal plan is that *every* limit path (there is at least one, because X is compact) of the sequence x^H must be inflexible.

There is one case of some importance in which there is certain to be an overture maximum. Suppose that, for all t, $X(t)$ is a finite set. Give it the discrete topology, i.e. the topology in which every subset is open. Then all the assumptions are satisfied, and every path is flexible, for all time. Hence there is at least one limit path of the sequence x^H, and that limit path is certainly overture maximal.

These results suggest ways of finding agreeable or overture maximal plans, even when there is no ordinary infinite-horizon optimum. For example, one could find x^H, for all horizons H, find a limit x^*, and see whether x^* was flexible. This would perhaps not be very easy in practice, but it may suggest better methods.

10.3. Disagreeable Infinite-Horizon Optima.

The example in 9.5. was an instance of an infinite-horizon optimum which is not agreeable. But, in that example, there were an infinite number of capital goods - *vin mystérieux* of different ages and types. Also, there was no agreeable plan. In the following example, there are a finite number of capital goods, and there is both an infinite horizon optimal plan and an agreeable plan, yet these two plans are different.

Example 10.3.1.

There are two capital goods, good *0* and good *1*. Initially, there is just one unit of good *0*. This can either be held for ever, producing one unit of consumption each period, or converted into an equal quantity of good *1* immediately. If it is converted into good *1*, the stock of good *1* doubles each period, as long as no consumption takes place. Good *1* can be consumed, and each unit yields one unit of consumption. But, if *any* of good *1* is consumed, it *all* disappears.

The possible consumption streams are therefore:-

$$c^0 = (1,1,1,\dots)$$

and

$$c^1 = (0,1,0,0,0,\dots)$$
$$c^2 = (0,0,2,0,0,\dots)$$
$$c^3 = (0,0,0,4,0,\dots)$$

etc.

So c^t is the consumption stream which results from converting good *0* into good *1* initially, and then holding good *1* until period $t+1$, whereupon it is exhausted and yields 2^{t-1} units of consumption.

Assume that the welfare function, for the horizon H, is simply:-

$$W^H(c) = \sum_{t=1}^{H} c(t)$$

i.e. total consumption.

Now, the infinite-horizon optimal plan for the economy is c^o, because for any other c^t,

$$\lim_{H\to\infty} \left[W^H(c^o) - W^H(c^t) \right] = \lim_{H\to\infty} (H - 2^t) = \infty .$$

But, if the horizon is H, the optimal plan is to convert to good 1, and consume 2^{H-1} units in period H. So $\bar{W}^H = 2^{H-1}$. There is an agreeable plan, c^*, which consists of converting to good 1 and never consuming anything. This is agreeable, because $W_T^H(c^*|T) = 2^{H-1} = \bar{W}^H$ whenever $H > T$.

It is also possible that there is no agreeable plan at all, even though there is an infinite-horizon optimum, as the following example shows:-

<u>Example 10.3.2.</u>

There are three capital goods, 0, 1, and 2. Initially, there is just one unit of good 0. This can be held forever, producing one unit of consumption each period. Alternatively, it can be converted immediately into an equal quantity of good 1, or into an equal quantity of good 2. As long as no consumption takes place, each unit of good 1 becomes two units of good 2 one period later; each unit of good 2 becomes two units of good 1 one period later. Good 1 can be consumed; good 2 cannot. Each unit of good 1 yields one unit of consumption, but if *any* of good 1 is consumed, it *all* disappears.

The possible consumption streams are precisely the same as for example 10.3.1. If t is odd, c^t is achieved by converting initially into good 1; if t is even, c^t is achieved by converting initially into good 2.

Taking the same welfare function:-

$$W^H(c) = \sum_{t=1}^{H} c(t)$$

it follows that c^o is still the unique infinite-horizon optimum. But now there is no agreeable plan. There are *two* paths on which consumption is zero for all time - on one, the initial conversion is into good 1, and on the other, the initial conversion is into good 2. Call these paths d^1 and d^2 respectively.

Then $$W_T^H(d^1|T) = \begin{cases} 2^{H-1} & (H \text{ even}) \\ 2^{H-2} & (H \text{ odd}) \end{cases}$$

If H is even, then, in period H, there will be 2^{H-1} units of good 1 available for consumption. But, if H is odd, period $H-1$ is the last period in which good 1 is available, and so the best that can be done is to consume the 2^{H-2} units which are available then.

Similarly, $$W_T^H(d^2|T) = \begin{cases} 2^{H-1} & (H \text{ odd}) \\ 2^{H-2} & (H \text{ even}) \end{cases}$$

Since $\bar{W}^H = 2^{H-1}$, it follows that neither $d^1|T$ nor $d^2|T$ is agreeable. Of course, both are overture maximal.

So it is only special types of infinite horizon optima which are agreeable. Which special types?

There are, in fact, two special types, at least. In section 8.5. it was observed that there are two standard ways of proving the existence of an infinite horizon optimum. What can now be shown is that, if either of these ways of proving existence works, then any infinite horizon optimum is agreeable. The two ways of proving existence are:-

(1) Show that the economy is valuation finite

(2) Find a competitive path, for which one can show that the value of the capital stock tends to zero as time tends to infinite (the Malinvaud condition).

Each of these cases is now discussed in turn.

10.4. Valuation-finite Economies.

The economy, in which we are trying to choose a plan of capital accumulation, is said to be *valuation finite* if:-

(1) There exist upper bounds $\bar{v}(t)$ $(t = 1,2,\ldots)$ such that, whenever $x \in X$,

$$v(x(t-1),\ x(t),\ t) \leq \bar{v}(t).$$

(2) There is a path $\hat{x} \in X$ such that the sum

$$\sum_{t=1}^{\infty}\left[v\left(\hat{x}(t-1),\ \hat{x}(t),\ t\right) - \bar{v}(t)\right]$$

(of nonpositive terms) converges.[1]

In such an economy, we can prove that an infinite horizon optimum exists, and that it is agreeable. In fact, we can prove rather more - there is a subsequence of horizons $H(n)$ such that (1) the corresponding finite-horizon optima $x^{H(n)}$ tend to a limit path x^* as $n \to \infty$, (2) the limit x^* is optimal. (3) $W^{H(n)}\left(x^{H(n)}\right) - W^{H(n)}(x^*) \to 0$ as $n \to \infty$, (4) x^* is agreeable.

Because of the valuation-finiteness assumption, we can subtract the constants $\bar{v}(t)$ from each one period indirect utility function $v(\cdot,\cdot,t)$, to get a new one period indirect utility function which is nonpositive for all plans in the feasible set. Since

$W^T(x) = \sum_{t=1}^{T} v\left(x(t-1),\ x(t),\ t\right)$, it follows that, for the correspondingly redefined welfare sums:-

$$W^{T+1}(x) \leq W^T(x) \leq W^{T-1}(x) \leq \ldots \leq W^1(x) \leq 0$$

for all $x \in X$, and all T.

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(1) See Mirrlees (1968), Brock and Gale (1969), McFadden (1973).

So the sequence $W^T(x)$ either converges, or else diverges to $-\infty$. In either case, we can define

$$W^\infty(x) = \lim_{T\to\infty} W^T(x)$$

where it is understood that $W^\infty(x) = -\infty$ if the sequence $W^T(x)$ diverges. But, in fact, for some $\hat{x} \in X$, $W^\infty(\hat{x})$ is finite. Let $\hat{W}$ denote $W^\infty(\hat{x})$.

Now we can prove the results mentioned above, via a series of lemmas.

Lemma 10.4.1.

Let x^H denote an optimum if the horizon is H. Let $\bar{W}^H = W^H(x^H)$ Then, in a valuation finite economy, $\bar{W}^H$ is a nonincreasing sequence which converges to a finite limit W^*.

Proof

$$\begin{aligned}
\bar{W}^{H+1} &= W^{H+1}(x^{H+1}) \\
&\leq W^H(x^{H+1}) \text{ (because } W^H \text{ is a sum of nonpositive terms)} \\
&\leq W^H(x^H) \quad \text{(by definition of } x^H\text{)} \\
&= \bar{W}^H
\end{aligned}$$

Moreover, for each H,

$$\bar{W}^H \geq W^H(\hat{x}) \geq \hat{W}$$

where $\hat{W}$ is finite. Therefore $\bar{W}^H$ converges to a limit $W^* \geq \hat{W}$.

Lemma 10.4.2.

Any limit point x^* of a sequence of finite horizon optima x^H has the property that

$$W^\infty(x^*) = W^*.$$

Proof

Suppose $x^{H(n)} \to x^*$.

Now, whenever $H(n) \geq T$, $\bar{W}^{H(n)} \leq W^T(x^{H(n)})$

But $\bar{W}^{H(n)} \to W^*$ as $n \to \infty$, by lemma 10.4.1. So, as $n \to \infty$, $W^* \leq W^T(x^*)$, by continuity of $W^T(\cdot)$.

Suppose that there exists $\varepsilon > 0$ such that

$$W^T(x^*) > W^* + \varepsilon \text{ (all } T\text{)}.$$

Now there exists n_0 such that, whenever $n \geq n_0$,

$$\bar{W}^{H(n)} < W^* + \tfrac{1}{2}\varepsilon$$

Also, $\quad W^{H(n)}(x^*) > W^* + \varepsilon$

So $\quad W^{H(n)}(x^*) > \bar{W}^{H(n)} + \tfrac{1}{2}\varepsilon$

contradicting the definition of $\bar{W}^{H(n)}$

Therefore $W^T(x^*) \to W^*$ as $T \to \infty$.

Lemma 10.4.3.

In a valuation finite economy, a path x^* is optimal if and only if $W^\infty(x^*) = W^*$ (where $W^* = \lim\limits_{H\to\infty} \bar{W}^H$).

Proof

(1) Suppose $W^\infty(x^*) = W^*$.

Then, for any path $x \in X$,

$$\liminf_{H \to \infty} \left[W^H(x^*) - W^H(x)\right]$$

$$= \liminf_{H \to \infty} \left[W^* - W^H(x)\right] \quad \text{(because } W^H(x^*) \to W^*\text{)}$$

$$= \liminf_{H \to \infty} \left[\bar{W}^H - W^H(x)\right] \quad \text{(because } \bar{W}^H \to W^*\text{)}$$

≥ 0, by definition of $\bar{W}^H$

So x^* is optimal.

(2) Notice that there is a path x^* such that $W^\infty(x^*) = W^*$, because X is compact, so the sequence x^H has a limit point, and this point serves as x^* by lemma 10.4.2.

By part (1), there can be no path x such that $W^\infty(x) > W^*$.

On the other hand, if $W^\infty(x) < W^*$ then x cannotbe optimal, because:-

$$\liminf_{H \to \infty} \left[W^H(x^*) - W^H(x)\right] = W^* - W^\infty(x) > 0.$$

So, if x is optimal, then $W^\infty(x)$ (which is defined for every $x \in X$) must be equal to W^*.

Lemma 10.4.4.

In a valuation-finite economy, any infinite-horizon optimum is agreeable.

Proof

Suppose x^* is an optimum. By lemma 10.4.3., $W^\infty(x^*) = W^*$.

So $$\lim_{H\to\infty} \left[\bar{W}^H - W^H(x^*)\right] = W^* - W^* = 0$$

which shows that x^* is certainly agreeable.

It is worth noticing that, if one is satisfied with arbitrarily small welfare losses as the horizon tends to infinity, there is no need ever to deviate from an agreeable plan x^* which is optimal in a valuation finite economy. By remaining on the optimal path x^*, the welfare losses tend to zero as the horizon tends to infinity in any case.

These lemmas might be summarized as follows. Let $W^* = \lim_{H\to\infty} \bar{W}^H$. Then W^* is finite (lemma 10.4.1). Now define the following sets:-

S_1 : the set of limit points of some sequence of finite horizon optima x^H

S_2 : the set of infinite-horizon optima

S_3 : the set of paths with $W^\infty(x) = W^*$

S_4 : the set of agreeable plans.

Then $S_1 \subseteq S_2 = S_3 \subseteq S_4$.

Notice that, in a valuation finite economy, any limit point of the sequence x^H is agreeable. This is a stronger result than those obtained in section 10.2., in a number of respects:-

(1) There is no need to assume flexibility.

(2) Any limit point is agreeable, rather than merely overture maximal.

(3) The sufficient condition for agreeability, $x^H \to x^*$, has been weakened to $x^{H(n)} \to x^*$.

10.5. Competitive Paths.

It is usual to investigate competitive paths in an economy with a convex feasible set, and quasi-concave preferences. However, competitiveness as a *sufficient* condition for optimality does not require any such assumptions, as will be seen.

Let $\langle x(t)\rangle_{t=0}^{\infty}$ be any path of capital accumulation. Say that it is *competitive* if, for each t, there is a nonnegative price vector $p(t)$, dual to the capital stock vector $x(t)$, such that, for $t = 1, 2, \ldots$

$(a^*, b^*) = (x(t-1),\ x(t))$ maximizes

$$v(a,b,t) + p(t)b - p(t-1)\,a$$

subject to $(a,b) \in X(t)$.

This is just the discrete-time version of Pontryagin's maximum principle.

Now we have the following simple result:-

Theorem 10.5.1.

If $x^* \in X$ is competitive at prices $\langle p(t)\rangle_{t=0}^{\infty}$ such that $p(t)\, x^*(t) \to 0$ as $t \to \infty$, then x^* is both infinite horizon optimal and agreeable.

Proof

If x is any feasible path, then:-

$$W^H(x) - W^H(x^*)$$

$$= \sum_{t=1}^{H} \left[v\left(x(t-1), x(t), t\right) - v\left(x^*(t-1), x^*(t), t\right)\right]$$

$$\leqq \sum_{t=1}^{H} \left[p(t)x^*(t) - p(t-1)x^*(t-1) - p(t)x(t) + p(t-1)x(t-1)\right]$$

$$= p(H)\left(x^*(H) - x(H)\right) - p(0)\left((x^*(0) - x(0)\right)$$

But $x^*(0) = x(0) = \bar{x}(0)$, the historically given initial capital stock.

So $\liminf_{H \to \infty} \left[W^H(x^*) - W^H(x) \right] \geq p(H)\, x(H) \geq 0$

Thus x^* is optimal.

Also, if $x = x^H$, then:-

$$0 \leq \bar{W}^H - W^H(x^*) \leq p(H) \left(x^*(H) - x(H) \right)$$

Taking the limit as $H \to \infty$, it follows that

$$\bar{W}^H - W^H(x^*) \to 0.$$

So x^* is agreeable.

Once again, there is no need to depart from the agreeable plan ever, if one is content to have welfare losses which tend to zero as $H \to \infty$.

The condition that the value of capital - $p(t)\, x^*(t)$ - tends to zero, is familiar from the work of Malinvaud.

10.6. The Convex Economy.

As was mentioned at the end of chapter 9, uniqueness is an important property for infinite horizon choice, of all the various kinds. In general, however, there is no reason to expect a choice set to consist of a single option. Uniqueness is a property which only holds under certain conditions. One particularly important set of conditions is the following. Suppose that A is a convex subset of a linear space. Suppose that the choice function C corresponds to a utility function u which is strictly quasi-concave. Then, as is well known, $C(A)$ consists of a single utility-maximizing option - assuming it is non-empty.

The result is so frequently seen that it might be called the "fundamental uniqueness theorem". In this section, we shall present extensions of it to the type of infinite horizon choice function considered in chapter 8 and in this chapter, in connectiors with planning capital accumulation.

The convexity assumptions we shall make are:-

(A.7) X is convex

(A.8) For each t, $v(\cdot,\cdot,t)$ is strictly concave.

Assumption (A.8) deserves some comment. For it is usual to assume that utility is a strictly concave function of *consumption bundles*. $v(\cdot,\cdot,t)$, however, is an indirect utility function, defined on pairs of capital stock vectors as follows:-

$$v(a,b,t) = \max \{u(a,b,c,t) \mid (a,b,c) \in Y(t),\ c \in Q(t,a)\}$$

where $Y(t)$ is the set of triples (a,b,c) which are technologically possible in period t, and $Q(t,a)$ is the consumption set in period t.[1]

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[1] See sections 8.1. and 8.2.

Now, the usual assumptions would be:-

(1) $Y(t)$ and $Q(t,a)$ are convex (for each t and $a \geq 0$)

(2) $u(a,b,\cdot,t)$ is strictly concave, as a function of c, and $u(\cdot,\cdot,\cdot,t)$ is concave, as a function of (a,b,c).

To assume that u is strictly concave in all of (a,b,c) together involves a serious loss of generality, because u may well be independent of (a,b) - or, at least, of several components of (a,b). From these assumptions, it only follows that $v(\cdot,\cdot,t)$ is concave - it does *not* follow that $v(\cdot,\cdot,t)$ is strictly concave. Nevertheless, even with these more usual assumptions, it is possible to prove uniqueness of the chosen consumption stream, and so to derive results similar to those below. It is only to simplify the analysis that the more rigorous assumption is made.

Another point is that the customary assumption would be:-

For each H, $W^H(\cdot)$ is strictly quasi-concave.

This is clearly weaker than strict concavity of the functions $v(\cdot,\cdot,t)$ Nevertheless, the difference does not seem too important in practice.

Given assumptions (A.7) and A.8), we have the following simple and well-known results:-

Theorem 10.6.1.

For each H, the optimum x^H for this finite horizon is unique.

Proof

If x^H, $\bar{x}^H$ are different paths, then

$$W^H(\tfrac{1}{2}x^H + \tfrac{1}{2}\bar{x}^H) > \tfrac{1}{2} W^H(x^H) + \tfrac{1}{2} W^H(\bar{x}^H),$$

and so x^H and $\bar{x}^H$ cannot both be optima.

Theorem 10.6.2.

If there is an infinite-horizon maximal path, then it is unique (and so, *a fortiori*, if there is an optimal path, it must be unique).

Proof

Suppose that x_1^* and x_2^* are two different infinite horizon maximal paths. Then $x_0^* = \frac{1}{2}(x_1^* + x_2^*)$ is feasible. Also, for some finite T, and some $\varepsilon > 0$

$$W^T(x_0^*) - \tfrac{1}{2} W^T(x_1^*) - \tfrac{1}{2} W^T(x_2^*) \geq \varepsilon$$

It follows that, whenever $H \geq T$,

$$W^H(x_0^*) - \tfrac{1}{2} W^H(x_1^*) - \tfrac{1}{2} W^H(x_2^*) \geq \varepsilon$$

and so it is not possible for both x_1^* and x_2^* to be maxima after all.

The following shows comparable uniqueness of overture choice:-

Theorem 10.6.3.

If there is an overture maximal path, then it is unique (and so, *a fortiori*, if there is an overture optimal path, then it is unique; and if there is an agreeable plan, it is unique).

Proof

Suppose that x_1^* and x_2^* are two different overture maximal paths. Then $x_0^* = \frac{1}{2}(x_1^* + x_2^*)$ is feasible. Also, there exists T_0 and $\varepsilon > 0$ such that, whenever $T \geq T_0$

$$W^T(x_0^*) \geq \tfrac{1}{2} \{W^T(x_1^*) + W^T(x_2^*)\} + \varepsilon$$

Let x_{Ti}^H denote a path which maximizes $W^H(x)$ subject to

$$x|T = x_i^*|T \text{ and } x \in X \quad (i = 0,1,2;\ H \geq T).$$

Write ${}^TW^H(x)$ for $\sum_{t=T+1}^{H} v\big(x(t-1), c(t), t\big)$.

Then $x_{T_0}^H$ maximizes ${}^T W^H(x)$ subject to $x|T = x_0^*|T$ and $x \in X$.

But $\frac{1}{2}\left(x_{T_1}^H + x_{T_2}^H\right)$ is feasible, given these constraints. Therefore:-

$$\begin{aligned}
{}^T W^H\left(x_{T_0}^H\right) &\geq {}^T W^H\left(\tfrac{1}{2}\left(x_{T_1}^H + x_{T_2}^H\right)\right) \\
&\geq \tfrac{1}{2}\left({}^T W^H\left(x_{T_1}^H\right) + {}^T W^H\left(x_{T_2}^H\right)\right)
\end{aligned}$$

$$\begin{aligned}
\therefore \quad W_T^H(x_0^*|T) &= W^T(x_0^*) + {}^T W^H\left(x_{T_0}^H\right) \\
&\geq \tfrac{1}{2}\left[W^T(x_1^*) + W^T(x_2^*)\right] + \varepsilon + \tfrac{1}{2}\left[{}^T W^H\left(x_{T_1}^H\right) + {}^T W^H\left(x_{T_2}^H\right)\right] \\
&= \tfrac{1}{2}\left[W_T^H(x_1^*|T) + W_T^H(x_2^*|T)\right] + \varepsilon
\end{aligned}$$

whenever $T \geq T_0$.

This contradicts:-

$$\liminf_{T \to \infty} \; \liminf_{H \to \infty} \left[W_T^H(x_0^*|T) - W_T^H(x_i^*|T)\right] \leq 0 \qquad (i = 1,2)$$

and so it is impossible for both x_1^* and x_2^* to be overture maximal.

Under assumptions (A.7) and (A.8), it is also possible to say something more about the relationship between insensitive plans and agreeable or overture maximal plans, provided an extra minor assumptions is made:-

(A.9) If either $v(a_1,b_1,t)$ or $v(a_2,b_2,t)$ is finite, and if

$\lambda > 0$, $\mu > 0$, $\lambda + \mu = 1$, then $v(\lambda a_1 + \mu a_2, \lambda b_1 + \mu b_2, t)$ is finite.

This is very close to (A.8) but, given the possibility that $v(a,b,t)$ may be $-\infty$, is not implied by it.

Lemma 10.6.4.

Suppose that x_1^* is a path such that, for each T, $x^{H(n)}|T \to x_1^*|T$ $n \to \infty$. Let x_2^* be any feasible path, other than x_1^*, for which $W^T(x_2^*)$ is finite. Let $x_0^* = \frac{1}{2}(x_1^* + x_2^*)$.

Then $\liminf_{T \to \infty} \liminf_{n \to \infty} \left[W_T^{H(n)}(x_0^*|T) - W_T^{H(n)}(x_2^*|T)\right] > 0$

Proof

Let T_0 be such that $x_1^*|T_0 \neq x_2^*|T_0$.

(1) Note that there exists $\varepsilon > 0$ such that, whenever $T \geq T_0$:-

$$\liminf_{n \to \infty} \left[W^T(x_0^*) - \tfrac{1}{2} W^T(x^{H(n)}) - \tfrac{1}{2} W^T(x_2^*)\right] \geq \varepsilon$$

For if $W^T(x^{H(n)}) \to -\infty$ as $n \to \infty$, this is true for arbitrary $\varepsilon > 0$ because $W^T(x_2^*)$ is finite by hypothesis, and so $W^T(x_0^*)$ is finite by (A.9). On the other hand, if $W^T(x^{H(n)}) \not\to -\infty$, then $W^T(x_1^*)$ must be finite, and so $W^T(x^{H(n)}) \to W^T(x_1^*)$. Then just take $\varepsilon = W^{T}O(x_0^*) - \frac{1}{2} W^{T}O(x_1^*) - \frac{1}{2} W^{T}O(x_2^*)$.

(2) Introduce the notation x_{Ti}^H and ${}^TW^H(x)$ as in the proof of theorem 10.6.3. Then, for each $T \geq T_0$, there exists $n(T)$ such that whenever $n \geq n(T)$:-

$$\begin{aligned}
W_T^{H(n)}(x_0^*|T) &= W^T(x_0^*) + {}^TW^{H(n)}(x_{T0}^{H(n)}) \\
&\geq \tfrac{1}{2} W^T(x^{H(n)}) + \tfrac{1}{2} W^T(x_2^*) + \tfrac{1}{2}\varepsilon + \tfrac{1}{2}{}^TW^{H(n)}(x_{T1}^{H(n)}) + \tfrac{1}{2}{}^TW^{H(n)}(x_{T2}^{H(n)}) \\
&= \tfrac{1}{2} \bar{W}^{H(n)} + \tfrac{1}{2} W_T^{H(n)}(x_2^*|T) + \tfrac{1}{2}\varepsilon \\
&\geq W_T^{H(n)}(x_2^*|T) + \tfrac{1}{2}\varepsilon, \text{as required.}
\end{aligned}$$

Theorem 10.6.5.

x^* is agreeable only if, for all T, $x^H|T \to x^*|T$ as $H \to \infty$.

Proof

Suppose that, for some T_0, $x^H|T_0 \not\to x^*|T_0$. Since X, the set of feasible paths, is compact, there is a sequence $x^{H(n)}$ and a path $\hat{x}$ such that $x^{H(n)} \to \hat{x}$ as $n \to \infty$, and $\hat{x}|T_0 \neq x^*|T_0$. Then $x_0^* = \frac{1}{2}(\hat{x} + x^*)$ is feasible, and by lemma 10.6.4:-

$$\liminf_{T \to \infty} \; \liminf_{n \to \infty} \left[W_T^{H(n)}(x_0^*|T) - W_T^{H(n)}(x^*|T)\right] > 0$$

But $$\lim_{n \to \infty} \left[W_T^{H(n)}(x^*|T) - \bar{W}^{H(n)}\right] = 0$$

Together, these contradict $W_T^{H(n)}(x_0^*|T) \leq \bar{W}^{H(n)}$.

Theorem 10.6.6.

If, for all T, $x^H|T \to x^*|T$ as $H \to \infty$ then $\hat{x}$ is overture maximal only if $\hat{x} = x^*$.

Proof

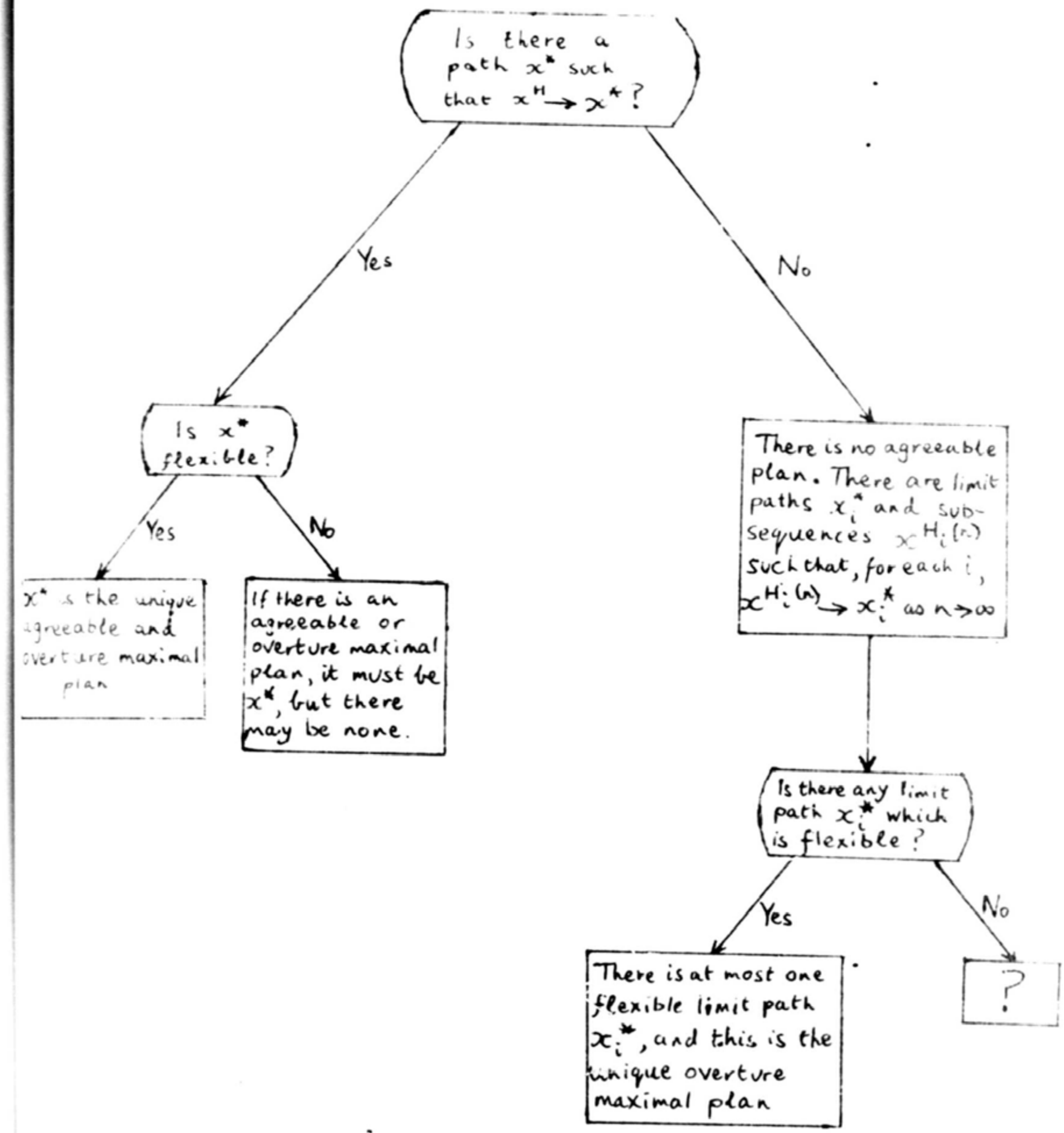
Suppose $\hat{x}$ is overture maximal, but $\hat{x} \neq x^*$. Then $W^T(\hat{x})$ is certainly finite, for all T. So, by lemma 10.6.4., there is a path $x_0^* = \frac{1}{2}(\hat{x} + x^*)$ such that:-

$$\liminf_{T \to \infty} \; \liminf_{H \to \infty} \left[W_T^H(x_0^*|T) - W_T^H(\hat{x}|T)\right] > 0$$

i.e. $\hat{x}|T$ is not overture maximal after all.

We can now summarize the results of this section. First, recall theorems 10.2.4. and 10.2.6. - if $x^H \to x^*$ and x^* is flexible, then x^* is agreeable; if $x^{H(n)} \to x^*$ is x^* is flexible, then x^* is overture maximal. With this remark, the diagram on the following page becomes self-explanatory.

Only in the case when *every* one of several limit paths x_i^* is inflexible is there really anything left to say. Such a case seems somewhat farfetched.


Is there a path x^* such that $x^H \to x^*$?
Yes
No
Is x^* flexible?
Yes
No
x^* is the unique agreeable and overture maximal plan
If there is an agreeable or overture maximal plan, it must be x^*, but there may be none.
There is no agreeable plan. There are limit paths x_i^* and subsequences $x^{H_i(n)}$ such that, for each i, $x^{H_i(n)} \to x_i^*$ as $n \to \infty$
Is there any limit path x_i^* which is flexible?
Yes
No
There is at most one flexible limit path x_i^*, and this is the unique overture maximal plan
?

10.7. Local Optimality and Price Decentralization.

A path $x^* \in X$ is said to be *locally optimal* if, for all T, and for all $x \in X$, $x(T) = x^*(T)$ implies $W^T(x) \leq W^T(x^*)$.

In other words, x^* is locally optimal unless there is an alternative path x which is preferable, but which has the same capital stock at all sufficiently distant times. If a path is not locally optimal, it can be improved simply by altering its T-overture, for some finite T. It seems clearly undesirable to follow such a path.

In fact, it is very easy to show that none of the suggested infinite horizon choice procedures allows one to choose a path which is not locally optimal. Local optimality *is* a necessary condition for a path to be chosen.

In the convex economy of section 10.6., this may well allow price decentralization of a chosen path. There are fairly general conditions under which a locally optimal path in a convex economy is competitive,[1] in the sense that there exist nonnegative price vectors $p(t)$ at each time t ($t = 0,1,2,\ldots$), such that, if $(a^*,b^*) = (x(t-1),x(t))$, then

(a^*,b^*) max $v(a,b,t) + p(t)\, b - p(t-1)\, a$

subject to $(a,b) \in X(t)$,

for each t ($t = 1,2,\ldots$).

Moreover, because $X(t)$ is convex, and $v(\cdot,\cdot,t)$ is strictly concave, the maximum is unique. So it would be possible to announce prices so that the decisions taken separately, one period at a time, led to the chosen path being followed. Of course, finding such prices is far from easy, but the theoretical possibility may be useful in suggesting ways of getting near to the ideal choice.

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[1] See, for example, Gale (1967).

10.8. Conclusions.

In the special case considered in section 10.6., when X is convex and each $v(\cdot,\cdot,t)$ is strictly concave, we have been able to answer fairly comprehensively the first three questions raised at the start of the chapter. The aim of the chapter was to see how sensible overture choice might be in practice. In the special case, it is almost equivalent to selecting an insensitive path. But it works more generally, and some optimality properties are guaranteed.

Once we allow nonconvex production sets, or non strictly quasi-concave welfare functions, there are plenty of problems even when there is a finite horizon. So the failure to find similarly comprehensive answers to our original questions is perhaps excusable. Nevertheless, we still know that a flexible insensitive path is agreeable, and sections 10.2., 10.3., 10.4., and 10.5. all contain partial answers to some of the original questions.

Finally, it should be emphasized that we have *not* shown that overture choice generalizes the earlier infinite horizon choice procedures. While, in many cases, the two approaches yield identical results, they still remain essentially different, as sections 9.7. and 10.3. emphasize. If there is an advantage to overture choice, it is that the cases in which the choice set is empty seem more farfetched and less paradoxical.

CONCLUSIONS

11.1. A Procedure for Consistent Dynamic Social Choice.

Warnock has suggested that

> "... any account of morality must be mistaken which does *not* yield the consequence that moral judgment is sometimes, and may be often, exceedingly difficult."[1]

This was to support his argument that Utilitarianism cannot really be attacked on the grounds that there is so much which it must take into account. The same applies to welfare economics, which is virtually utilitarianism applied to questions of economic policy.

This thesis has attempted to suggest general methods for surmounting some of the difficulties which arise in a welfare analysis, particularly one in which time is important. The particular difficulties which have received most attention here are endogenous tastes, endogenous population, and the time horizon. All these difficulties have previously been discussed. But much of the earlier discussion, quite naturally, has concentrated on special cases. Here I have attempted to devise a procedure which can handle all these difficulties together, in a more systematic fashion.

Economists, on the whole, have apparently been reluctant to suggest procedures which lead to a social choice function, even in outline. But it seems that most economists, when they discuss policy, have at the back of their minds a certain procedure, which I shall call the "orthodox procedure". It runs through two stages:-

---

(1) Warnock (1971), p. 29.

(1) Individuals' own tastes are estimated, on the basis of their own choices, or their own expressions of preference.

(2) These tastes are amalgamated in some way into a Bergson social welfare function.

Of course, it has been generally realized that such a procedure rests on special assumptions. Nevertheless, suggestions for modifying the procedure, when the assumptions are violated, are hard to find.

First, let us note some rather obvious objections to the two parts of the orthodox procedure:-

(1) (a) Individuals' tastes are ill-defined, because they change over time and are subject to influence.

(b) Even where tastes are well-defined, actual choice and expressions of preference may fail to correspond to tastes. The individual may indulge in naive or sophisticated dynamic choice, which corresponds to tastes only in some complicated manner. The individual may deliberately act out of accord with his tastes, because he has a higher set of ethical values, or for other reasons. Or he may simply be irrational.

(2) (a) Certain tastes should be ignored or overruled, on ethical grounds.

(b) Tastes are not the only factor affecting the desirability of economic policies. Indeed, for some policies which involve life and death, they may not be relevant at all.

To meet such objections, less emphasis must be placed on tastes and individuals' own choices. This is not to suggest that they should be overruled lightly; rather, it is to recognise that sometimes they are uninformative. Accordingly, an alternative to the orthodox procedure will now be suggested. It is one which is designed specifically to handle intertemporal problems.

(1) Estimate, by whatever means seems most appropriate, the interests of the individuals who are, will be, or may be, members of the society.

(2) For each large time horizon, H, aggregate these interests, via a dynamic constitution, into a consistent dynamic social welfare choice function $\{C^H(n)\}$.

(3) To allow for uncertainty about the true horizon H, choose an agreeable plan, if one exists; if not, choose an infinite horizon overture maximal plan.

This procedure needs explaining and amplifying.

(1) At each moment of time, the individual has certain tastes. It is then assumed that there is a "temporary" welfare choice function which gives those choices one would make with the individual's interests in mind. This temporary WCF may or may not correspond to the individual's tastes. It is certainly coherent (as defined in 2.4.).

(2) Because both tastes and the set of living individuals are changing, it is not enough to have a single constitution to determine a social WCF. Instead, one needs a dynamic constitution, which aggregates the temporary WCF's of relevant individuals (section 5.3.). The social WCF is constructed to be both dynamically consistent and coherent. This involves a form of "intertemporal liberalism" (section 4.10.).

Notice that it is *not*, in general, appropriate to find a single WCF for each individual, representing his intertemporal interests, and to aggregate individuals' intertemporal WCF's into a social WCF. This "metastatic" approach does not allow the weighting of different individuals' interests to depend upon the inconsistent temporary WCF's underlying the intertemporal WCF's. Nor is it appropriate, in general, to aggregate individuals' temporary WCF's at one time into a temporary social WCF at the same time, and then to construct a consistent dynamic social WCF from these temporary social WCF's. This was illustrated by examples in sections 5.2. and 5.3.

This procedure appears to meet the objections (1)(a) and (1)(b) - in principle, at least. (2)(a) is also met, to some extent. (2)(b) remains, but although individuals' interests are not all that matter in justifying economic policies, and although it is hard to think about the interests of those about to be born or to die, the force of the objection has been somewhat blunted.

(3) This part of the procedure deals with the time-horizon. Because it involves technical problems, and because there had previously been much discussion of infinite horizons, there was a lengthy discussion in chapters 7,8,9 and 10. The proposal is to choose an agreeable or overture maximal plan, rather than the usual infinite-horizon optimum, for two reasons.

(i) An agreeable or overture maximal plan is likely to exist even though an infinite horizon optimum does not. See, for instance, the example of section 8.6., and compare the relative strengths of the general conditions under which existence has been proved, as set out in sections 8.5., 10.4., 10.5., and 10.6.

(ii) Provided, at least, that there is sufficient notice of horizon for significant changes of policy to be possible before the horizon is upon us, a policy which starts off with an agreeable or overture maximal plan, is, by definition, no worse than any other (chapter 9).

This thesis has been directed towards explaining and justifying this procedure. In the process, some results, which may be of interest in themselves, have emerged.

11.2. Some Incidental Conclusions.

(1) The unanimity principles acquire a somewhat stronger justification, when formulated in terms of interests. Also, when choice is not ordinal, they can be strengthened (sections 3.2., 3.3., 3.4., 3.5.).

(2) Unless some ethical values not directly related to individuals' interests - such as egalitarianism - are relevant, an additive form of Bergson social welfare function can be justified quite generally (section 3.6.).

(3) Naive and sophisticated dynamic choice are coherent if and only if they coincide, and this occurs if and only if there is no "essential inconsistency" of the kind noted in the potential addict example (section 4.9.).

(4) Weizsäcker's suggestions for making welfare judgments when tastes are changing endogenously are valid only for some special individual intertemporal WCF's. In effect, then, Weizsäcker suggests what may be a useful way of ruling out some of the many intertemporal WCF's which can be derived from a given set of inconsistent temporary WCF's (section 6.1.).

(5) The Average Utility Principle, for evaluating population policy, is dynamically inconsistent. Nevertheless, one can maintain the spirit of the principle, because there is a dynamically consistent welfare function which is homogeneous of degree zero in the population stream (section 6.3.).

(6) Choosing infinite horizon maxima, rather than optima, may be dynamically inconsistent. There is, nonetheless, a consistent infinite horizon dynamic choice function (section 7.5.).

(7) Bellman's "Principle of Optimality" applies to *any* coherent consistent dynamic choice function, in the sense that choosing first on

overture, and then the remainder of a path, is equivalent to choosing an entire path (section 9.4.).

(8) For the two general cases in which it is possible to show that there is an infinite horizon optimum, such an optimum is agreeable (sections 10.4., 10.5.).

(9) Providing that an insensitive plan - i.e. the limit of a sequence of finite horizon optima, as the horizon tends to infinity - is flexible, in a certain sense, it is agreeable (section 10.2.).

(10) When utility each period is strictly concave, and the feasible set is convex, any of the infinite-horizon choice procedures considered gives a unique choice. Also, if there is an agreeable plan, it is insensitive. If there is an insensitive plan, no other plan can be overture maximal (section 10.6.).

11.3. A Final Note on Uncertainty.

On the whole, we have chosen to ignore the problem of uncertainty, so that we could concentrate on intertemporal and dynamic problems without this additional complication. It is, however, worth noting that the most common approach to welfare economics when there is uncertainty can be subsumed within the theory developed in the earlier chapters.

The most usual approach to uncertainty is that developed by Arrow and Debreu. There is a set of consequences, K, and of states of the world, S. The underlying set X is the Cartesian product $\Pi_{s \in S} K_s$ of gambles involving the possible consequences, K. As before, welfare choice functions are defined on X. It is usual to assume that they correspond to von Neumann Morgenstern expected utilities:-

$$\sum_{s \in S} p_s \, u_s \, (k_s)$$

where $\sum_{s \in S} p_s = 1$ (by normalization), and p_s is the subjective probability of state s.

This approach to welfare economics is based upon ex-ante expected utilities, and so it will be called the *ex-ante approach*. At first, it seems appropriate when there is a single, once-and-for-all, choice of plan to be made. But as soon as we admit dynamic features of choice, it becomes rather more questionable. For example, individuals adjust their estimates of the relative probabilities of different events in the light of new information, and modify their policy accordingly. Is this not like a change of tastes? However, both changes of taste and adaptation of subjective probabilities in the light of new evidence can be handled in the welfare theory of chapter 5 - at least, in principle - and formally both problems are identical. Each $x \in X$ can include a description of random influences on subjective probabilities - such as the outcome of certain experiments, or other sources of information. Dynamic consistency involves anticipating receiving such new information - one rapidly becomes involved in adaptive control problems. The *ex-ante* approach can still survive, although it is much more complicated.

It must also be recognized that future tastes are uncertain. But again, this can be allowed for. Each $x \in X$ must also include a description of the way tastes depend upon the state of the world, and the number of states of the world must be expanded to include different "psychological states".

Population may also be uncertain. But, just as endogenous population was no more difficult, formally, than endogenous tastes, so is random population no more difficult than random tastes.

So, formally at least, uncertainty is no problem, it seems at first. It is always possible to define consequences and states of the world to allow for it, whatever form the uncertainty takes. On the other hand, there are very real problems in thinking about appropriate welfare choice functions in such complicated environments. Again, we have a powerful approach which sweeps the important problems out of sight. While it is helpful to know that the problems can be solved in principle, such knowledge is no more than a spur for detailed investigation of tractable special cases. This is particularly true of uncertainty, but it is also true of everything which has been done in this work.

However, the above discussion of uncertainty has made an implicit assumption, which obscures all the real difficulties. The assumption is that each individual in the economy, and the person contemplating economic policy, will find out what the true state of the world is in the end.. Without this assumption, many new problems arise, even in a static framework.(1) Individuals may be able to mislead one another, and also any planner carrying out a welfare analysis. The planner may be able to mislead individuals in a way which is of benefit to them. It follows that individuals and planner find themselves in a kind of game situation - using "game" in the von Neumann-Morgenstern sense. Indeed, it is not even strictly a game, because the players do not know one another's tastes. To analyse such problems would be a vast extension of the existing study, and must be left for later work.

---

(1) As has been shown in a number of papers, notably by Arrow (1970), chs. 5, 6, 8 and 9; Akerlof (1970), Mirrlees (1972b), etc.

APPENDIX 1

CHOICE FUNCTIONS AND THEIR PROPERTIES.

A.1. Definition.

In section 2.2., a *choice function* was defined as a mapping $C: \mathcal{P}(X) \to \mathcal{P}(X)$, where:-

a) X is the *underlying set.*

b) $\mathcal{P}(X)$ is the set of subsets of X.

c) For all $A \subseteq X$, $C(A) \subseteq A$.

d) Whenever A is a finite subset of X, $C(A)$ is non-empty.

Then C is said to be *defined on* (the underlying set) X.

We shall also have cause to consider functions which are similar to choice functions, but violate (d) above. So, a *set function* on X is a mapping $C: \mathcal{P}(X) \to \mathcal{P}(X)$, where (a), (b) and (c) above are satisfied.

A.2. Coherence.

In section 2.4., the following definition was made:-

A choice function C, defined on X, is *coherent* if, whenever $B \subseteq A \subseteq X$, $B - C(B) \subseteq A - C(A)$.

The following lemma concerns the existence, for a given choice function C, of a coherent choice function C' which is "finer" or "more selective" than C:-

Lemma A.2.

Let C be any choice function defined on X. For each $A \subseteq X$, define:-

$$C'(A) = \{x \in A \mid \forall B \quad x \in B \subseteq A \text{ implies } x \in C(B)\}$$

Then:-

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$$C'(A) = \{x \in A \mid \forall B \quad x \in B \subseteq A \text{ implies } x \in C(B)\}$$

Then:-

(a) $C'(A) \subseteq C(A)$ (each $A \subseteq X$).

(b) If, whenever A is finite, $C'(A)$ is non-empty, then C': $\mathcal{P}(X) \to \mathcal{P}(X)$ is a coherent choice function.

(c) If there exists a coherent choice function C'' defined on X, such that, whenever $A \subseteq X$, $C''(A) \subseteq C(A)$, then, whenever $A \subseteq X$, $C''(A) \subseteq C'(A)$.

Proof

a) Take $B = A$ in the definition of $C'(A)$.

b) Evidently, under this hypothesis, C' is a choice function. To verify coherence, suppose that $B \subseteq A$, and $x \in B - C'(B)$. Then there exists a set S such that $x \in S \subseteq B$, but $x \notin C(S)$. It follows that $x \in A - C'(A)$, because $x \in S \subseteq A$ as well.

c) Suppose that $x \in A - C'(A)$, for some $A \subseteq X$. Then there exists a set B such that $x \in B \subseteq A$ and $x \notin C(B)$. It follows that $x \in B - C''(B)$. Since C'' is coherent, $x \in A - C''(A)$. So $x \notin C''(A)$, as required.

A.3. Preferences.

The preference relations R, P, I, on X are preferences revealed by choice on pair-sets, and are defined as follows:-

Let A denote the pair $\{x,y\}$

Then $x\ R\ y$ iff $x \in C(A)$

$x\ P\ y$ iff $y \notin C(A)$

$x\ I\ y$ iff $C(A) = A$.

Notice that, because $C(A)$ is non-empty,

(1) Either $x\ R\ y$, or $y\ R\ x$, or both.

The other usual properties of preference relations are easily verified:-

(2) $x\ R\ x$ (take $A = \{x,x\}$)

(3) $x\ R\ y$ iff not $y\ P\ x$.

(4) $x\ P\ y$ iff ($x\ R\ y$ and not $y\ R\ x$)

(5) $x\ I\ y$ iff ($x\ R\ y$ and $y\ R\ x$)

etc.

A.4. Binary Choice.

For the (weak) preference relation R on the set X - where R is reflexive and connected - define

$$C_R(A) = \{x \in A \mid y \in A \text{ implies } x\ R\ y\}$$

Say that the choice function C, defined on X, is *binary*, and that it ***corresponds to the weak preference relation*** R, if $C(A) = C_R(A)$, for all $A \subsetneq X$. Then $C_R(A)$ must be non-empty whenever A is finite. Then R is the weak preference relation revealed by choice on pair sets. Also, if P is the corresponding strict preference relation, then P must be acyclic, otherwise $C(A)$ would be empty for some finite A.

Lemma A.4.

If the choice function C, defined on X, is binary, then it is coherent.

Proof

If $B \subsetneq A \subsetneq X$ and $x \in B - C(B)$, then there exists $y \in B$ such that $y\ P\ x$. Since $y \in A$ and $x \in A$, it is evident that $x \in A - C(A)$, as required.

A.5. Ordinal Choice.

Say that the choice function C, defined on X, is *ordinal*, if:-

a) C is binary.

b) The weak preference relation R which corresponds to C is transitive.

So, ordinal choice corresponds to the usual kind of weak preference relation - one that is reflexive, connected, and transitive.

Say that the choice function C on X is *effectively ordinal* if there exists a transitive weak preference relation R such that, for each $A \subsetneq X$:-

Either (1) $C(A) = C_R(A) = \{x \in A | y \in A \text{ implies } x \, R \, y\}$

or (2) $C(A)$ is empty.

Of course, since C is a choice function, $C(A)$ is non-empty whenever A is finite. Therefore, if C is also effectively ordinal, $C(A) = C_R(A)$ whenever A is finite.

A useful result is the following:-

Lemma A.5.1.

Suppose that C is a choice function on X with the property that, whenever $B \subsetneq A \subseteq X$ and $C(A) \cap B$ is non-empty, then $C(B) = B \cap C(A)$. Then C is effectively ordinal.

Proof

Define the preference relations R, P, I by choices on pair sets, as in section A.3.

(1) $C(A) \subseteq C_R(A)$.

Suppose $x \in C(A)$ and $y \in A$. Take $B = \{x,y\}$. Then $x \in B \cap C(A)$, and $B \subseteq A$. So, by hypothesis, $C(B) = B \cap C(A)$. Thus $x \in C(B)$, and $x \, R \, y$.

(2) R is transitive.

Suppose not. Then there exists a triple $S = \{x,y,z\}$ such that $x \, R \, y$, $y \, R \, z$, and $z \, P \, x$. So $x \notin C(S)$, by (1).

(a) Suppose $y \in C(S)$. Take $B = \{x,y\}$. Then $y \in B \cap C(S)$, and $B \subsetneq S$. By hypothesis, $C(B) = B \cap C(S)$. Thus $x \notin C(B)$, contradicting $x \, R \, y$.

(b) Suppose $y \notin C(S)$. Since $C(S)$ cannot be empty, $z \in C(S)$. Therefore, $z\ R\ y$, by (1). Take $B = \{y,z\}$. Then $z \in B \cap C(S)$, and $B \subseteq S$. By hypothesis, $C(B) = B \cap C(S)$. Since $y \in C(B)$, this is obviously a contradiction.

So, in either case, we have a contradiction, and R must therefore be transitive.

(3) If $C(A)$ is non-empty, then $C_R(A) \subseteq C(A)$.

Suppose that $x \in C_R(A)$ and $y \in C(A)$. Then $x\ R\ y$, and, because of (1), $y\ R\ x$. Take $B = \{x,y\}$. Then $y \in B \cap C(A)$, and $B \subseteq A$. By hypothesis, then, $C(B) = B \cap C(A)$. But $x \in C(B)$. Therefore $x \in C(A)$, as required.

An immediate corollary is the following:-

Lemma A.5.2.

If C is a coherent choice function on X, and if, for all $A \subseteq X$, $C(A)$ has no more than one member, then C is effectively ordinal.

Proof

It is enough to verify the property in the statement of lemma A.5.1. Suppose, then, that $B \subseteq A \subseteq X$ and $x \in C(A) \cap B$. Then $C(A) = \{x\}$, by hypothesis. Also $C(B) = \{x\}$, because, by coherence, $x \in C(B)$. Therefore $\{x\} = C(B) = C(A) \cap B$, as required.

In this case, too, the preference relation R corresponding to C is a strong ordering - i.e. for each $x, y \in X$, either $x\ P\ y$, or $y\ P\ x$, or $x = y$.

A.6. Choice Indicator Functions.

A *choice indicator function* (or CIF) is a mapping $u:X \to \mathbb{R}$. It *corresponds* to the choice function C defined on X as follows:-

$$C(A) = \{x \in A \mid y \in A \text{ implies } u(x) \geq u(y)\} \text{ (each } A \subseteq X).$$

If u corresponds to the choice function C, then C must be ordinal, because C corresponds to the transitive weak preference relation R defined by:-

$$x\ R\ y \quad \text{iff } u(x) \geq u(y).$$

Utility functions and welfare functions are, of course, special kinds of choice indicator functions.

A.7. Quasi-Ordinal Choice.

The choice function C, defined on X, is said to be *quasi-ordinal* if it is binary, and the associated strict preference relation P is transitive, i.e. $x\ P\ y$ and $y\ P\ z$ together imply that $x\ P\ z$. Then the weak preference relation R is said to be *quasi-transitive*.

A.8. Proper Weak Preferences, Optima, and Maxima.

When preferences are merely quasi-transitive, rather than transitive, it is often convenient to consider a restricted form $\tilde{R}$ of the weak preference relation R, such that $\tilde{R}$ is transitive. Such a relation $\tilde{R}$ can always be found; one way is the following:-

Define the indifference relation $\tilde{I}$ as follows:-[1]

$$x\ \tilde{I}\ y \quad \text{iff } [(x\ P\ z \ \text{ iff }\ y\ P\ z) \text{ and } (z\ P\ x \ \text{ iff }\ z\ P\ y)]$$

---

[1] This definition is suggested by Walsh (1970), p.79.

So $x \tilde{I} y$ if and only if interchanging x and y makes no difference to the strict preference relation P. Now define $\tilde{R}$ as follows:-

$x \tilde{R} y$ iff ($x P y$ or $x \tilde{I} y$).

Obviously, if $x \tilde{R} y$, then $x R y$ because if $y P x$, then neither $x P y$ nor $x \tilde{I} y$ is possible.

Notice that R is transitive if and only if $\tilde{I} = I$; then $\tilde{R} = R$, and $\tilde{R}$ is connected. In general, however, $\tilde{R}$ is not connected, as example A.8.2. below will demonstrate.

Of course, $\tilde{R}$ is reflexive, and the following lemma proves that $\tilde{R}$ is transitive:-

---

<u>Lemma A.8.1.</u>

If $\tilde{R} = P \cup \tilde{I}$, where

$x \tilde{I} y$ iff $\left[(x P z \text{ iff } y P z) \text{ and } (z P x \text{ iff } z P y)\right]$

then $\tilde{R}$ is transitive.

<u>Proof</u>

Suppose that $x \tilde{R} y$ and $y \tilde{R} z$

(a) If $x P y$ and $y P z$, then $x P z$, because P is transitive.

(b) If $x P y$ and $y \tilde{I} z$ or if $x \tilde{I} y$ and $y P z$, then, by definition of $\tilde{I}$, it follows that $x P z$.

(c) Finally, if $x \tilde{I} y$ and $y \tilde{I} z$, it is clear that $x \tilde{I} z$.

This particular weak preference relation $\tilde{R}$ is only one of several with the following essential properties:-

(1) $\tilde{R}$ is reflexive and transitive

(2) $x P y$ iff ($x \tilde{R} y$ and not $y \tilde{R} x$)

Given any transitive asymmetric[1] strict preference relation P, say that $\tilde{R}$ is a *proper weak preference relation corresponding to* P if

---

[1] P is asymmetric iff ($x P y$ implies not $y P x$).

$\hat{R}$ satisfies properties (1) and (2) above.

Define for each $A \subseteq X$ the following two sets:-

$O(A) = \{x \in A \mid y \in A \text{ implies } x \tilde{R} y\}$

$M(A) = \{x \in A \mid y \in A \text{ implies not } y P x\}$.

Evidently $O(A) \subseteq M(A)$, with inequality in general. Say that each member of $O(A)$ is *optimal* in A, and that each member of $M(A)$ is *maximal* in A, given these preference relations. As A varies over the subsets of X, $M(\cdot)$ defines a binary choice function. But this is not true of $O(\cdot)$, because it is possible to find a finite set A such that $O(A)$ is empty. For example:-

---

Example A.8.2.

Suppose that $X = \{a,b,c\}$ and that $a P b$, $b I c$, and $c I a$. Suppose that $\tilde{R}$ is a proper weak preference relation corresponding to P. Then $a \tilde{R} b$, but not $b \tilde{R} a$. Now:-

(1) Suppose $c \tilde{R} a$. Then, since $\tilde{R}$ is transitive, $c \tilde{R} b$. Since $c P b$ is false, $b \tilde{R} c$. Since $\tilde{R}$ is transitive, it follows that $b \tilde{R} a$ - a contradiction.

(2) Suppose $a \tilde{R} c$. Then, since $a P c$ is false, $c \tilde{R} a$. But this contradicts (not $b \tilde{R} a$), by (1).

It follow that $\tilde{R}$ cannot be a connected relation.

Next, we show that $O(X)$ is empty. Evidently, $b \notin O(X)$, so:-

(3) Suppose $a \in O(X)$. Then $a \tilde{R} c$. This gives a contradiction, because of (2).

(4) Suppose $c \in O(X)$. Then $c \tilde{R} a$. This gives a contradiction, because of (1).

It follows that O is not a choice function, in this case.

Example A.8.2. suggests the following result:-

Theorem A.8.3.

$O(\cdot)$ is a choice function on X if and only if R is transitive.

Proof

(1) Suppose that R is not transitive. Then there exists a set $A = \{x,y,z\}$ of three options in X such that $x\ R\ y$, $y\ R\ z$, and $z\ P\ x$. (Remember that $z\ P\ x$ iff not $x\ R\ z$.) Because P is transitive, it must be true that $x\ I\ y$ and $y\ I\ z$. Then, by analogy to example A.8.2., $O(A)$ is empty, and so $O(\cdot)$ is not a choice function.

(2) If R is transitive, then $R = \tilde{R}$, and $O(A) = M(A)$ for each $A \subseteq X$, so that $O(\cdot)$ is a choice function.

Finally, the following result is useful in demonstrating the existence of a choice function having certain properties. It is true even if the correspondence between R and P is weaker than that considered previously in this section.

Theorem A.8.4.[1]

Suppose that $\tilde{R}$ is a transitive weak preference relation, and $\tilde{P}$ is a transitive strict preference relation. Suppose too that if $x\ \tilde{P}\ y$, then $x\ \tilde{R}\ y$ and not $y\ \tilde{R}\ x$.

Define $O(A) = \{x \in A \mid y \in A \text{ implies } x\ \tilde{R}\ y\}$

$M(A) = \{x \in A \mid y \in A \text{ implies not } y\ \tilde{P}\ x\}$.

Then there exists an ordinal choice function $C(\cdot)$ such that, for each $A \subseteq X$, $O(A) \subseteq C(A) \subseteq M(A)$.

Proof

Define $\tilde{I}$ by $x\ \tilde{I}\ y$ iff ($x\ \tilde{R}\ y$ and $y\ \tilde{R}\ x$) - then $\tilde{I}$ is an equivalence relation. Denote the equivalence classes in $X/\tilde{I}$ by $[x]$, where $x \in [x]$.

---

[1] While this theorem is close to Arrow (1963), lemma 4, pp. 64-68, it is not identical; also, the present proof is shorter than Arrow's.

Define the relation P^* on $X/\breve{I}$ as follows:-

$[x]\ P^*[y]$ iff $\exists x \in [x],\ y \in [y]$ s.t. $x\ \breve{P}\ y$.

Then P^* is asymmetric and acyclic, because if

$[x_1]\ P^*\ [x_2]\ P^* \ldots P^*[x_n]\ P^*[x_1]$, for any finite n.

then there exist $x_i,\ x_i' \in |x_i|$ $(i = 1$ to $n)$ such that

$x_1'\ \tilde{P}\ x_2\ \breve{I}\ x_2'\ \tilde{P}\ x_3\ \breve{I}\ x_3'\ \tilde{P} \ldots \tilde{P}\ x_n\ \breve{I}\ x_n'\ \breve{P}\ x_1\ \breve{I}\ x_1'$

Because $\tilde{R}$ is transitive, it follows that $x_2\ \tilde{R}\ x_1'$, which contradicts $x_1'\ \tilde{P}\ x_2$. So there can be no such cycle.

Let $\bar{P}$ be the transitive completion of P^*, i.e. the relation defined by:-

$[x]\ \bar{P}\ [y]$ iff $\exists [z_1], [z_2], \ldots, [z_n] \in X/\breve{I}$ s.t.

$[x]\ P^*\ [z_1]\ P^*\ [z_2]\ P^* \ldots P^*[z_n]P^*[y]$

Then $\bar{P}$ is transitive and asymmetric. By a slightly modified version of Szpilrajn's theorem,[1] there exists a connected reflexive transitive relation $\bar{R}$ such that:-

$[x]\ \bar{P}\ [y]$ implies $\Big([x]\ \bar{R}\ [y]$ and not $[y]\ \bar{R}\ [x]\Big)$.

Now define R on X as follows:-

$x\ R\ y$ iff $[x]\ \bar{R}\ [y]$

Clearly R is a connected reflexive transitive relation.

Also if $x\ P\ y$, then not $y\ \tilde{R}\ x$ (for if $x\ P\ y$, and $y\ \tilde{R}\ x$, then there exists $z_1,\ldots,z_n$ s.t. $x\ \breve{R}\ z_1\ \breve{R}\ z_2\ \tilde{R} \ldots \tilde{R}\ z_n\ \breve{R}\ y\ \tilde{R}\ x$ with at least one $\breve{R}$ a $\breve{P}$ - a contradiction).

Finally, if $x\ \tilde{P}\ y$, then $[x]\ \bar{P}\ [y]$ and so ($x\ R\ y$ and not $y\ R\ x$). This confirms that, if C is the ordinal preference relation corresponding to R, then $O(A) \subseteq C(A) \subseteq M(A)$ (each $A \subseteq X$).

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(1) Szpilrajn (1930).

A.9. Dynamic Consistency

In section 4.3., a dynamic choice function $\{C(n)\}$ was defined on an underlying tree X, with tree structure $\{X(n)\}$. Moreover, in section 4.4., a dynamic choice function $\{C(n)\}$ was defined to be *consistent* if and only if, for each $A \subseteq X$, and whenever $n \ Pr \ n'$ (n is a node which precedes n') and

$A(n') \cap C(n) \ (A(n))$ is non-empty:-

$$C(n') \ (A(n')) = A(n') \cap C(n)(A(n)).$$

In this section, consistency of dynamic choice functions will be related, as far as possible, to consistency of corresponding preference relations.

First, define a *dynamic weak preference relation* $\{R(n)\}$ on the underlying tree X as a collection of weak preference relations on $X(n)$, for each $n \in N(X)$. It will be assumed that each component $R(n)$ is reflexive and transitive, but not necessarily connected, even on $X(n)$.

Similarly, define a *dynamic strict preference relation* $\{P(n)\}$ on the underlying X as a collection of strict preference relations on $X(n)$, for each $n \in N(X)$. It will be assumed that each $P(n)$ is asymmetric and transitive.

The definition of consistency for these dynamic preference relations seems fairly obvious:-

$\{R(n)\}$ is *consistent* if and only if, whenever $n \ Pr \ n'$ and $a, b \in X(n')$,

$$a \ R(n') \ b \quad \text{iff} \quad a \ R(n) \ b$$

$\{P(n)\}$ is *consistent* if and only if, whenever $n \ Pr \ n'$ and $a, b \in X(n')$

$$a \ P(n') \ b \quad \text{iff} \quad a \ P(n) \ b.$$

Given the dynamic weak preference relation $\{R(n)\}$, define a dynamic set function (i.e. a dynamic choice function, with the requirement that

$C(n)(A(n))$ must be non-empty whenever $A(n)$ is finite removed) $\{O(n)\}$ on the underlying tree X as follows:-

$O(n)(A) = \{x \in A \mid y \in A \text{ implies } x\, R(n)\, y\}$ (each $A \subset X(n)$).

So $O(n)(A)$ is just the set of optima in A, given the transitive weak preference relation $R(n)$. $\{O(n)\}$ may not be a dynamic choice function, because it is possible that $O(n)(A)$ is empty even though A is finite.

Although $\{O(n)\}$ may not be a dynamic choice function, consistency can still be defined as it was for dynamic choice functions, and it retains much of its appeal. Say that $\{O(n)\}$ is *consistent* if:-

(1) For each $A \subseteq X$, whenever $n\ Pr\ n'$ and $A(n') \cap O(n)(A(n))$ is non-empty,

then $O(n')(A(n') = A(n') \cap O(n)(A(n))$.

(2) For each $A \subseteq X(n)$, $O(n)(A) = O(n_0)(A)$,

where n_0 is the initial node of X.

Usually, part (2) of this definition is unnecessary, as was seen in section 4.4. But here it is needed, because set functions have weaker properties than choice functions. In particular, it is too easy for $O(n')(A(n'))$ to be empty. Part (2) plays a crucial role in the following useful result:-

Theorem A.9.1.

$\{O(n)\}$ is dynamically consistent if and only if $\{R(n)\}$ is a consistent dynamic weak preference relation.

Proof

(a) Suppose that $\{O(n)\}$ is consistent. Suppose that $n\ Pr\ n'$, and that $x, y \in X(n')$. Write $A = \{x,y\}$. Now:-

$x\, R(n)\, y$ iff $x \in O(n)(A)$ iff $x \in O(n_0)(A)$ iff $x \in O(n')(A)$ iff $x\, R(n')\, y$

as required.

(b) Conversely, suppose that $\{R(n)\}$ is consistent.

(1) If $A \subseteq X$, $n \; Pr \; n'$, and $x \in A(n') \cap O(n)(A(n))$, then:-

(i) If $y \in O(n')(A(n'))$, then $y \; R(n') \; x$, and so, by consistency, $y \; R(n) \; x$. But, given any $z \in A(n)$, $x \; R(n) \; z$. Since $R(n)$ is transitive, $y \; R(n) \; z$, and so $y \in O(n)(A(n))$.

(ii) If $y \in A(n') \cap O(n)(A(n))$, then, whenever $z \in A(n')$, because $z \in A(n) \supseteq A(n')$, it follows that $y \; R(n) \; z$. Since $\{R(n)\}$ is consistent, $y \; R(n') \; z$. Therefore $y \in O(n')(A(n'))$.

Thus $O(n')(A(n')) = A(n') \cap C(n)(A(n))$.

(2) If $A \subseteq X(n)$, then:-

$x \in O(n)(A)$ iff $x \; R(n) \; y$ whenever $y \in A$

iff $x \; R(n_0) \; y$ whenever $y \in A$ (because of consistency)

iff $x \in O(n_0)(A)$.

Thus $O(n)(A) = O(n_0)(A)$.

So, to choose the set of optima wherever possible is dynamically consistent if and only if the (transitive) dynamic weak preference relation is consistent. This result is used in chapters 4, 7 and 9.

Given the dynamic strict preference relation $\{P(n)\}$, define a dynamic choice function $\{M(n)\}$ on the underlying tree X as follows:-

$$M(n)(A) = \{x \in A \mid y \in A \text{ implies not } y \; P(n) \; x\} \quad (\text{each } A \subset X(n)).$$

$M(n)$ is the set of maximal options in A, given the transitive strict preference relation $P(n)$

Even if $\{P(n)\}$ is consistent, $\{M(n)\}$ may not be, as the following example shows:-

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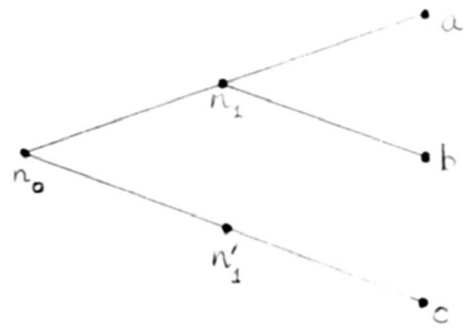
Example A.9.2.

Suppose that the underlying tree X has branches a, b, c and the following structure:-



Suppose that $c\ P(n_0)\ b$, but otherwise there are no strict preferences. Then $\{P(n)\}$ is dynamically consistent. Also:-

$$M(n_0)(X) = \{a,c\}$$

but $$M(n_1)(X(n_1)) = \{a,b\}$$

So even though $a \in X(n_1) \cap M(n_0)(X)$, it is false that $M(n_1)(X(n_1)) = X(n_1) \cap M(n_0)(X)$ and so $\{M(n)\}$ is not consistent.

However, as will shortly be shown, $\{M(n)\}$ does have a weak consistency property. Say that a dynamic choice function $\{C(n)\}$ is *weakly consistent* if:-

(1) Whenever $A \subseteq X$ and $n\ Pr\ n'$,

$$C(n')(A(n')) \supseteq A(n') \cap C(n)(A(n)).$$

(2) $C(n)(A) = C(n_0)(A)$ (each $A \subseteq X(n)$).

Part (2) is as for full consistency. Part (1) has been weakened so that some options in $A(n')$ may be chosen at the node n', even though they were not chosen at the preceding node n.

Theorem A.9.3.

$\{M(n)\}$ is weakly consistent if and only if $\{P(n)\}$ is consistent.

Proof

(a) Suppose that $\{M(n)\}$ is weakly consistent. Suppose that $n\ Pr\ n'$ and that $x, y \in X(n')$. Write $A = \{x,y\}$. Now:-

$x\ P(n)\ y$ iff $y \notin M(n)(A)$ iff $y \notin M(n_0)(A)$ iff $y \notin M(n')(A)$ iff $x\ P(n')\ y$

(b) Conversely, suppose that $\{P(n)\}$ is consistent.

(1) Suppose that $A \subseteq X$, $n\ Pr\ n'$, and $x \in A(n') \cap M(n)(A(n))$.

Suppose too that, for some $y \in A(n')$, $y\ P(n')\ x$.

Then $y \in A(n)$, and, by consistency, $y\ P(n)\ x$.

This contradicts $x \in M(n)(A(n))$. Therefore $x \in M(n')(A(n'))$.

(2) If $A \subseteq X(n)$, then

$x \in M(n)(A)$ iff not $y\ P(n)\ x$, whenever $y \in A$

iff not $y\ P(n_0)\ x$, whenever $y \in A$

iff $x \in M(n_0)(A)$

as required.

An especially important case arises when $M(n)(A(n))$ happens to contain a single option. In other words, there is a unique maximal option. Then, if $\{P(n)\}$ is dynamically consistent, it must be consistent to choose the single option in $M(n)(A(n))$ at node n.

More generally, we have the following result:-

Theorem A.9.4.

Suppose that $\{R(n)\}$ and $\{P(n)\}$ are consistent dynamic preference relations - weak and strong respectively - on the underlying tree X. Suppose that, whenever $a\ R(n)\ b$, $b\ P(n)\ a$ is false. Then there is a consistent dynamic choice function $\{C(n)\}$ such that, whenever $n \in N(X)$ and $A \subseteq X(n)$:-

$$O(n)(A) \subseteq C(n)(A) \subseteq M(n)(A).$$

Proof

By theorem A.8.4., there exists an ordinal choice function $C(n_0)$ on X such that, for each $A \subseteq X$:-

$$O(n_0)(A) \subseteq C(n_0)(A) \subseteq M(n_0)(A).$$

Because of consistency, and theorems A.9.1. and A.9.3., for any $n \in N(X)$ and $A \subseteq X(n)$:-

$$O(n)(A) = O(n_0)(A), \text{ and } M(n)(A) = M(n_0)(A).$$

Define, for each $n \in N(X)$, $A \subseteq X(n)$:-

$$C(n)(A) = C(n_0)(A).$$

Then $\{C(n)\}$ is obviously a dynamic choice function. Moreover, it is consistent, because if $n \; Pr \; n'$ and $A(n') \cap C(n)(A(n))$ is non-empty, then $A(n') \cap C(n_0)(A(n)$ is non-empty, and so because $A(n') \subseteq A(n)$ and $C(n_0)$ is ordinal:-

$$C(n_0)(A(n')) = A(n') \cap C(n_0)(A(n)).$$

Therefore $C(n')(A(n')) = A(n') \cap C(n)(A(n))$, as required.

A.10. Partial Choice Functions.

Suppose that C is a choice function defined on the underlying set X. Now, in dynamic choice, where X is an underlying tree, the agent need not make his choice in one single step. Instead, he can choose a subtree of the original underlying tree - by choosing an overture, say, as in chapter 9. The final branch only emerges after a sequence of choices.

In chapter 8, we saw how it was enough first to choose a plan of capital accumulation, and to leave the planning of consumption, labour supply, etc., until a second stage. Of course, the second stage influences the first, but one can draw a useful distinction between the two.

Both these are examples of multi-stage choice. What concerns us here is the early stages of a multi-stage choice procedure.[1] In these, instead of choosing an option in X, the agent chooses one of a collection of subsets of X. The choice from this subset is left till later. This type of choice will be called *partial choice* . It is the subject of this section. The definitions and elementary properties should come as no surprise to those familiar with dynamic programming or with indirect utility functions. Nevertheless, a formal statement is needed for parts of chapters 8 and 9.

Let X be partitioned as follows:-

$$X = \bigcup_{y \in Y} X(y)$$

The partial choice function will be defined on Y, or on a subset of Y. But in order to know which y to choose, an agent must know what is feasible when he has chosen y. For instance, in choosing between £100 and $250, an agent in Britain wishes to know the terms under which he can change the dollars into pounds, or into goods he desires.

Let the feasible set be fixed, then. It will ease notation, without losing any generality, if we assume that X is the feasible set.

If the agent chooses $y \in Y$, his feasible options afterwards will be $X(y)$. This is what we mean when we say that the agent chooses y. Assume - again, without loss of generality - that each $X(y)$ is non-empty. Then the feasible set of "partial options" is Y.

The agent's choice set is $C(X)$. He is, therefore, presumably willing to choose a partial option y if and only if, after choosing y, he can still

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(1) The last stage is an entirely orthodox choice problem, because the final choice will be specified.

proceed eventually to choose at least one option in $C(X)$. Thus, the agent's partial choice set is:-

$$\{y \in Y \mid C(X) \cap X(y) \neq \emptyset\}$$

If the agent's choice set $C(X)$ is clearly known, there is little more to be said and, of course, there is no point really in looking at partial choice anyway. The interest arises when $C(X)$ is not known, as with infinite-horizon choice in chapter 9 - or when it is easier if we do not have to specify $C(X)$, as with the choice of economic plan in chapter 8, when it is easier to consider just capital accumulation. To avoid having to consider $C(X)$itself, we consider partial choice functions.

Let $B \subseteq Y$. Suppose that the feasible set of options is $X(B) = \bigcup_{y \in B} X(y)$, instead of X. Then the partial choice y must lie in B. We can regard B as the *feasible set of partial options.*

If the feasible set of options is $X(B)$, the agent's choice set is $C(X(B))$. Presumably, then, he is willing to choose the partial option y if and only if he can still choose at least one option in $C(X(B))$ afterwards. Thus the agent's partial choice set is:-

$$D(B) = \{y \in B \mid C(X(B)) \cap X(y) \neq \emptyset\}$$

We shall now make an assumption , which concerns the nature of the partition $X = \bigcup_{y \in Y} X(y)$, and is certainly satisfied if each $X(y)$ is a finite set. The assumption is that whenever B is finite, $C(X(B))$is non-empty. Given this assumption , we have the following two results:-

Theorem A.10.1.

If C is coherent on X, then D is a coherent choice function on Y.

Proof

It is enough to show that:-

(i) *If B is finite, $D(B)$ is non-empty.*

This is true because, by assumption, there exists $x \in C(X(B))$ Then there exists $y \in B$ such that $x \in X(y)$, and so $y \in D(B)$.

(ii) *D is coherent.*

Suppose $B \subsetneq B'$ and $y \in B \cap D(B')$. Then there exists $x \in C(X(B')) \cap X(y)$. As $y \in B$ and $x \in X(y)$, $x \in X(B)$. Because C is coherent, $X(B) \subseteq X(B')$, and $x \in C(X(B'))$, it follows that $x \in C(X(B))$. Since $x \in X(y)$, it follows that $y \in D(B)$, as required.

Theorem A.10.2.

Suppose that C coresponds to the utility function $u(\cdot)$ defined on X. Then D corresponds to the utility function $v(\cdot)$, defined on Y as follows:-

$$v(y) = max\ \{u(x) \mid x \in X(y)\}$$

Proof.

(i) Notice that v is well-defined, because $u(\cdot)$ corresponds to C, and $C(X(y))$is non-empty, by assumption.

(ii) Suppose that $y^* \in D(B)$, for some $B \subseteq Y$. Then there exists $x^* \in C(X(B)) \cap X(y^*)$. Now, whenever $x \in X(B)$, $u(x^*) \geq u(x)$. Because $x^* \in X(y^*)$, $u(x^*) = v(y^*)$. For any other $y \in B$, there exists some $x \in X(y)$, such that $u(x) = v(y)$. Then $x \in X(B)$, and so $v(y^*) = u(x^*) \geq u(x) = v(y)$.

(iii) Conversely, suppose that $y^* \in B$ and that, whenever $y \in B$, $v(y^*) \geq v(y)$. Then there exists $x^* \in X(y^*)$ such that $v(y^*) = u(x^*)$.

Now, for all $x \in X(B)$, there exists $y \in B$ such that $x \in X(y)$. By definition of v, $u(x) \leq v(y)$. Therefore $u(x) \leq v(y) \leq v(y^*) = u(x^*)$.

It follows that $x^* \in C(X(B))$. Since $x^* \in X(y^*)$ also, $y^* \in D(B)$.

This completes the proof.

Two other properties of partial choice functions seem desirable:-

(P.1) If $x \in C(X(B))$ and $x \in X(y)$, then $x \in C(X(y))$. So, if we start with the partial choice $y \in D(B)$, and then choose from $C(X(y))$, the original choice $x \in C(X(B))$ may emerge. It is easy to see that (P.1) is satisfied provided that C is coherent. Also, (P.1) is satisfied for all partitions of X, and for all $x \in X$ and all sets B, only if C is coherent.

(P.2) If $y \in D(B)$ and $x \in C(X(y))$, then $x \in C(X(B))$. So, if we start with the partial choice $y \in D(B)$, and then choose $x \in C(X(y))$, x is a possible choice from the original feasible set $X(B)$. By lemma A.5.1., it is fairly easy to see that both (P.1) and (P.2) are satisfied, for all partitions of X, and for all $x \in X$ and all sets B, if and only if C is effectively ordinal - i.e. ordinal provided we disregard feasible sets A for which $C(A)$ is empty.

A.11. Extended Utility Functions.

In section 8.4., we introduced the notion of an *extended utility function*. An extended utility function $u(\cdot)$ corresponds to a choice function $C(\cdot)$ if

(1) for all $x \in X$, either $u(x)$ is a real number

or $u(x) = -\infty$

(2) for all $A \subsetneq X$,

$$C(A) = \{x \in A \mid y \in A \text{ implies } u(y) \leq u(x)\}$$

(where $-\infty < \lambda$, for all real λ, naturally).

Say that the extended utility function $u(\cdot)$ is *continuous* if, whenever $\langle x^n \rangle$ is a sequence of points of X such that $x^n \to x$ as $n \to \infty$, where $x \in X$, then:-

(1) if $u(x)$ is real, then $u(x^n) \to u(x)$ as $n \to \infty$.

(2) if $u(x) = -\infty$, then $u(x^n) \to -\infty$ as $n \to \infty$.

A useful result is the following:-

Theorem A.11.1.

Suppose that $u(\cdot)$ is a continuous extended utility function corresponding to the choice function C. Suppose that A is any non-empty compact set. Then $C(A)$ is non-empty.

Proof

(1) If $u(x) = -\infty$ for all $x \in A$, then $C(A) = A$.

(2) If there exists $x^o \in A$ such that $u(x^o)$ is finite, define $A^o = \{x \in A \mid u(x) \geq u(x^o)\}$. Then A^o is a closed subset of A, because u is continuous, and so A^o is compact. So $C(A^o)$ is non-empty. But $C(A) = C(A^o)$ and so $C(A)$ is non-empty.

APPENDIX 2

ALLAIS, 'GENERALIZED SOCIAL PRODUCT', AND CONSISTENT PREFERENCES

In Allais' "Economie et Interêt", there is an extension of the notion of Pareto efficiency to take account of changes over time in individuals' intertemporal utility functions.(1) This appendix is an assessment of Allais' contribution, in the light of the analysis of intertemporal welfare in chapter 5 of the present work.

Following Allais, assume that there are only two periods.

In chapter 6, ("Interest and Social Product"(2)), he defines "social product" so that it is maximized by an allocation which is Pareto efficient, given the tastes of the individuals in the economy (§54). He states the usual results linking competitive equilibrium and Pareto efficiency, for a given single instant of time (§55), and also over time, when each individual has tastes given at the moment when he first participates in the economy (§58). Then he extends the notion of "social product" to take account of the possibility that the same individual may have different utility functions at different moments of time, and so Pareto efficiency is no longer well-defined (§60). These utility functions, or "satisfactions" are denoted by S_0 and S_1, and apply to periods T_0, T_1, whose starting points are t_0, t_1, respectively. In addition, he emphasizes that he is discussing a world of perfect foresight.

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(1) I am grateful to E. Malinvaud and to T. de Montbrial for drawing my attention to this part of Allais' work. In addition, they have each published some comments on it - see Malinvaud (1972), ch. 10., A. §5, and de Montbrial (1971), p. 156.

(2) In the original, "Interêt et Rèndement Social".

Then, on pp. 163-164, we find the following:-(1)

"...*a priori, one perceives no logical reason for wanting to maximize utility* S_0 *rather than utility* S_1.

"Even in the world of perfect foresight we are considering, one certainly cannot say that the utility S_0 takes account of the perfectly known utility S_1 of the individual, as we consider him during his youth (T_0). Rather, what the individual takes into account is the utility he experiences at time t_0 at the prospect of the utility he will experience at time t_1, and not this utility itself.

"There is really as much difference between the utilities S_0 and S_1 of the same individual considered at two different moments, as there is between the utilities of two different individuals considered at the same moment.

"We shall say that the 'generalized social product' of an economic state, given perfect foresight, is maximized when any potential modification of that state satisfying the constraints, which augments the utility of certain individuals considered in certain periods, necessarily diminishes the utility of other individuals, or of the same individuals considered at other moments of time."

So Allais is suggesting an extension of unanimity principles to "dated individuals". An option x is strictly superior to an option y if and only if every individual, at every moment of time - i.e. every dated individual - prefers x to y. So, to apply this "Allais principle" to individuals' own choices, there is no need to assume that individuals have *consistent* dynamic choice functions.

But it seems that Allais was not intending to apply his principle to individuals' own choice functions. The second paragraph quoted above

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(1) In my translation. The italicized parts correspond to parts in bold type in the original.

suggests a form of anticipation of future utility which is akin to intertemporal liberalism. This is borne out in appendix III, pp. 757-758.

Here, A and B are the only two goods in the economy; $(A_0, B_0; A_1, B_1)$ is the individual's two-period consumption stream. In the following, I have simplified and changed the notation in order to maintain consistency with that above.

> "It is easy to show, *supposing foresight is perfect*, that the functions S_0 and S_1 are not independent ...
>
> "First, for given levels of consumption A_0 and B_0, it can be seen that if a combination (A_1, B_1) is preferred to a combination (A_1', B_1') at time t, (utility S_1), then allowing for perfect foresight, it is equally preferred at time t_0 (utility S_0). That is if, for given consumption levels A_0 and B_0, the utility S_1 remains constant, then the utility S_0 does the same, and consequently the utility S_0 must be of the form:-
>
> $$S_0 = \textstyle\sum_0 (A_0, B_0, S_1)."$$

This form of utility function is, of course, precisely that which results from intertemporal liberalism. So now the question arises: do the utility functions S_0 and S_1 represent different choices, and if so, why does this difference matter?

In fact, Allais goes on to argue that S_1 must have the form:-

$$S_1 = \textstyle\sum_1 (A_1, B_1, S_0)$$

and to make it clear that S_1 represents the choices which the individual, at time t, would like to have made at time t_0. But because S_0 has the form above, this decision is consistent with his choice at time t_0.

Now preferences concerning the past, such as those embodied in S_1, do not correspond to real choices. They are a form of regret. So Allais wishes to allow for such regrets in his generalized notion of "efficiency" or "social product". It is not enough for superiority of x over y that each individual prefers x to y at time t_0, because an individual may be impatient, for example, and express preferences which he later regrets. As Allais himself says (footnote (4), p. 758):-

"Perfect foresight is not the same as perfect forethought."

So, strictly speaking, the Allais principle, as Allais himself saw it, is not generally applicable to dynamic social choice, as considered in chapter 5. Of course, in principle, an individual's intertemporal welfare function could be constructed to take account of such regrets, and to embody the Allais principle. But something other than mere intertemporal liberalism is involved.

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