State-Dependent Preferences

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June 20, 2005

State-dependent preferences. Theories of individual decision making under uncertainty pertain to situations in which a choice of a course of action, by itself, does not determine the outcome. To formulate these theories Savage (1954) introduced what has become the standard analytical framework. It consists of three sets: a set S, of *states of the world* (or *states*, for short); an arbitrary set C, of *consequences*; and the set F, of all the functions from the set of states to the set of consequences. Elements of F, referred to as *acts*, represent courses of action, consequences describe anything that may happen to a person, and states are the resolutions of uncertainty, that is, "a description of the world so complete that, if true and known, the consequences of every action would be known" (Arrow [1971], p. 45). Decision makers are characterized by preference relations, $\succcurlyeq$, on F. With few exceptions, preference relations are taken to be complete (that is, for all f and g in F, either $f \succcurlyeq g$ or $g \succcurlyeq f$) and transitive binary relations on F. The symbols $f \succcurlyeq g$ have the interpretation "the course of action f is preferred or indifferent to the course of action g." The strict preference relation, $\succ$, and the indifference relation, $\sim$, are the asymmetric and symmetric parts of $\succcurlyeq$, respectively.

Loosely speaking, a preference relation is state dependent when the prevailing state of nature is itself of direct concern to the decision maker. For example, taking out a health insurance policy is choosing an act whose consequences – the indemnities – depend on the realization of the decision maker's state of health. In this example, the state is the decision maker's state of health. It affects the decision maker's well-being directly, and indirectly, through the payoff prescribed by the health insurance policy. The preference relation may display ordinal state dependence, in which case the underlying state may affect the decision maker's preferences by altering his ordinal ranking of the consequences; or cardinal state dependence, by altering his risk attitudes; or both.

To define state dependence formally, it is convenient to adopt the model of Anscombe and Aumann (1963). In this model the state space is finite, and the consequences are lotteries, that is, probability distributions that assign strictly positive probability to a finite number of outcomes. Denote by $L(X)$ the set of lotteries on an arbitrary set, X, of possible outcomes. Given a preference relation $\succcurlyeq$ on F; a state s; and f, f', g, g' in F, define a preference relation on F conditional on s, $\succcurlyeq_s$, by $f \succcurlyeq_s f'$ if $g \succcurlyeq g'$ whenever $f(s) = g(s)$, $f'(s) = g'(s)$ and $g(s') = g'(s')$ for all $s' \in S - \{s\}$. Because acts are functions, $f(s)$ is defined uniquely. Thus $\succcurlyeq_s$ defines a preference relation on $L(X)$ conditional on s. This induced preference relation is also denoted by $\succcurlyeq_s$.

A state $s \in S$ is said to be *null* if $f \succcurlyeq_s f'$ for all $f, f' \in F$, otherwise it is *nonnull*.

Definition: A preference relation $\succcurlyeq$ on F is *state dependent* if $\succcurlyeq_s \neq \succcurlyeq_{s'}$ for some nonnull s and s' in S.

Because consequences are lotteries, if a preference relation $\succcurlyeq$ on F displays state dependence, then $\succcurlyeq_s$ and $\succcurlyeq_{s'}$ must differ on the ranking of some lotteries in $L(X)$. This may be due to distinct attitudes toward risk and/or distinct ordering of outcomes, that is, degenerate lotteries that assign the given outcomes probability one. Circumstances in which the dependence of the decision maker's preferences on the state of nature constitutes an indispensable feature of the decision problem include the choice of health insurance coverage (see Arrow, 1974; Karni, 1985); the choice of flight insurance coverage (see Eisner and Strotz, 1961); the choice of optimal consumption and life insurance plans in the face of uncertain life time (see Yaari, 1965; Karni and Zilcha 1985); and the provision of collective protection (see Cook and Graham, 1977).

Subjective expected utility representations. Preferences among acts are a matter of personal judgement, presumably combining the decision maker's valuation of the consequences and his beliefs regarding the likely realization of alternative events (that is, subsets of the state space). Subjective expected utility theory pertains to preference relations whose structures allow the decision makers' valuations of the consequences to be expressed numerically, by a utility function; his beliefs to be quantified by a (subjective) probability measure on the set of states; and the acts to be evaluated by the expectations of the utility of the corresponding consequences with respect to the subjective probability. In other words, the theory depicts the decision makers' choice among alternative acts as expected utility maximizing behavior.

By the classic von Neumann-Morgenstern expected utility theorem, $\succcurlyeq$ on F satisfies the axioms of expected utility theory (that is, $\succcurlyeq$ is a complete and transitive binary relation satisfying the Archimedean and independence axioms) if and only if there exist real-valued functions $w(\cdot, s)$ on X, $s \in S$, such that for all $f, g \in F$,

$$f \succcurlyeq g \;\Leftrightarrow\; \sum_{s \in S} \sum_{x \in X} w(x, s) f(x, s) \geq \sum_{s \in S} \sum_{x \in X} w(x, s) g(x, s). \tag{1}$$

Furthermore, the functions $\{w(\cdot, s) \mid s \in S\}$ are unique up to cardinal unit comparable transformation. (That is, if some other utility functions $\{w'(\cdot, s) \mid s \in S\}$ represent $\succcurlyeq$ on F, in the sense of equation (1), there exit $b > 0$ and real numbers $a(s)$, one for each state, such that $w'(s) = bw(s) + a(s)$ for all $s \in S$.)

The function $w(\cdot, \cdot)$ captures the decision maker's valuation of the outcomes and his beliefs about the likely realization of the states. The axioms of expected utility theory do not imply a unique decomposition of w into subjective probability distribution on S and utility on outcome-state pairs. Indeed, let $p(s)$, $s \in S$, be any list of positive numbers that sum up to 1, where $p(s) = 0$ if and only if s is null. Define $u(x, s) = w(x, s)/p(s)$ for all nonnull $s \in S$ and all $x \in X$, and $u(x, s) = \bar{u}$ if s is null. Then, by equation (1), $\succcurlyeq$ is represented by $\sum_{s \in S} p(s) \sum_{x \in X} u(x, s) f(x, s)$. The question is, are there additional conditions that would imply a *unique* decomposition of $w(x, s)$ into a product of utility representing the (possibly state-dependent) valuation of the outcomes and probabilities representing beliefs that govern the decision maker's choice among acts?

Anscombe and Aumann (1963) show that a preference relation is nontrivial (that is, $f \succ g$ for some $f, g \in F$) satisfying the axioms of expected utility theory and state independence, that is, $\succcurlyeq_s = \succcurlyeq_{s'}$, for all nonnull $s, s' \in S$ if and only if there exist a real-valued function, u, on X, and a subjective probability distribution, π, on S such that, for all $f, g \in F$,

$$f \succcurlyeq g \;\Leftrightarrow\; \sum_{s\in S} \pi(s) \sum_{x\in X} u(x) f(x,s) \geq \sum_{s\in S} \pi(s) \sum_{x\in X} u(x) g(x,s). \tag{2}$$

Moreover, u is unique up to positive linear transformation, and π is unique satisfying $\pi(s) = 0$ if and only if s is null.

The subjective expected utility representation (2) separates risk attitudes, represented by the utility function, from beliefs, represented by the subjective probabilities. However, the uniqueness of the probabilities depends crucially on the premise that constant acts are constant utility acts. This premise is not implied by the axioms. In particular, *state-independent preferences do not imply state-independent utility function.* To see why, let γ be a strictly positive real-valued function on S and $\Gamma = \sum_{s\in S} \gamma(s) \pi(s)$. Define $\hat{u}(x,s) = u(x)/\gamma(s)$, for all $x \in X$ and $s \in S$, and let $\hat{\pi}(s) = \gamma(s)\pi(s)/\Gamma$, for all $s \in S$. Then, by equation (2) and the uniqueness of u, for all $f, g \in F$,

$$f \succcurlyeq g \;\Leftrightarrow\; \sum_{s\in S} \hat{\pi}(s) \sum_{x\in X} \hat{u}(x,s) f(x,s) \geq \sum_{s\in S} \hat{\pi}(s) \sum_{x\in X} \hat{u}(x,s) g(x,s). \tag{3}$$

Thus, the utility-probability pair $(\hat{u}, \hat{\pi})$ induces a subjective expected utility representation of $\succcurlyeq$ that is equivalent to the one induced by the pair (u, π). There are infinitely many distinct utility-probability pairs that represent the same preference relation in the sense of equation (2). Moreover, because π and $\hat{\pi}$ are distinct, even if beliefs exist a priori and are coherent enough to allow their representation by probabilities, it is not evident which of the infinitely many probability distributions consistent with $\succcurlyeq$ actually represents the decision makers beliefs. But if the probabilities that figure in the representation are meaningless, there seems to be no compelling reason to prefer the expect utility representation (2) over the more general additive representation (1). On the contrary, because the additive representation does not require that the preferences be state independent, it is applicable to the analysis of problems, such as the demand for health and life insurance, in which the assumption of state-independent preferences is clearly inadequate.

Hypothetical preferences and subjective expected utility representations of state-dependent preferences. An alternative analytical framework, introduced by Karni and Schmeidler (1981) postulates the existence of a preference relation on hypothetical lotteries, whose prizes are outcome-state pairs. This preference relation is assumed to satisfy the axioms of expected utility and to be consistent with the actual preference relation on acts. Because the hypothetical lotteries imply distinct, hence incompatible, marginal distributions on the state space, preferences among such lotteries are *introspective* and may be expressed verbally only as hypothetical choices. Decision makers are supposed to be able to conceive of such hypothetical lotteries and to invoke, for the purpose of their evaluation, the same mental processes that govern their actual decisions.

To express these ideas formally, denote by $L(X \times S)$ the set of all probability distributions on $X \times S$ that assign strictly positive probabilities to a finite number of outcome-state pairs. A lottery $\ell \in L(X \times S)$ is said to be *nondegenerate* if $\sum_{y \in X} \ell(y,s) > 0$ for all $s \in S$. Denote by $\hat{\succcurlyeq}$ an introspective preference relation on $L(X \times S)$. For each $s \in S$, define the conditional introspective preferences, $\hat{\succcurlyeq}_s$, on $L(X \times S)$ analogously to the definition of $\succcurlyeq_s$ (that is, $\ell \hat{\succcurlyeq}_s \ell'$ if and only if $\ell \hat{\succcurlyeq} \ell'$, for all $\ell, \ell' \in L(X \times S)$ such that $\ell(\cdot, s') = \ell'(\cdot, s')$ for all $s' \in S - \{s\}$).

Loosely speaking, the introspective preference relation $\hat{\succcurlyeq}$ and the actual preference relation $\succcurlyeq$ are consistent if they are induced by the same utilities. Formally, define a mapping, H, from $L(X \times S)$ to F as follows: For each nondegenerate $\ell \in L(X \times S)$, let $H(\ell)(x,s) = \ell(x,s) / \sum_{y \in X} \ell(y,s)$, for all $(x,s) \in X \times S$. A state s is said to be *obviously null* if $f \succcurlyeq_s f'$ for all f and f' in F and there are $\ell, \ell' \in L(X \times S)$ such that $\ell \hat{\succ}_s \ell'$. A state s is *obviously nonnull* if $f \succ_s g$ for some f and g. A state s is *essential* if there are ℓ and ℓ' in $L(X \times S)$ such that $\ell \hat{\succ}_s \ell'$.

Strong consistency: For all $s \in S$ and nondegenerate ℓ and ℓ' in $L(X \times S)$, $H(\ell) \succ_s H(\ell')$ implies $\ell \hat{\succ}_s \ell'$, and if s is obviously nonnull, then $\ell \hat{\succ}_s \ell'$ implies $H(\ell) \succ_s H(\ell')$.

Theorem: (Karni and Schmeidler, 1981) Let $\succcurlyeq$ be a nontrivial binary relation on F and $\hat{\succcurlyeq}$ a binary relation on $L(X \times S)$. Then each of the two relations satisfies the axioms of expected utility and jointly they satisfy strong consistency if and only if there exist a real-valued function, u, on $X \times S$ and a probability distribution, π, on S such that, for all f and g in F,

$$f \succcurlyeq g \;\Leftrightarrow\; \sum_{s \in S} \pi(s) \sum_{x \in X} u(x,s) f(x,s) \geq \sum_{s \in S} \pi(s) \sum_{x \in X} u(x,s) g(x,s) \qquad (4)$$

and, for all ℓ and ℓ' in $L(X \times S)$,

$$\ell \hat{\succcurlyeq} \ell' \;\Leftrightarrow\; \sum_{s \in S} \sum_{x \in X} u(x,s) \ell(x,s) \geq \sum_{s \in S} \sum_{x \in X} u(x,s) \ell'(x,s). \qquad (5)$$

Moreover, the function u is unique up to cardinal unit comparable transformation, the probability π restricted to the event of all essential states is unique, and for s obviously null $\pi(s) = 0$ and for s obviously nonnull $\pi(s) > 0$.

The subjective expected utility representation in (4) applies whether the preference relation, $\succcurlyeq$, is state dependent or state independent. Furthermore, as Karni and Mongin (2000) observed, because the utility function is identified using hypothetical lotteries, the probability measure π in the representation theorem above quantifies the decision makers beliefs. A similar result in a somewhat different framework is proved in Karni (2003); a probabilistically sophisticated version of this approach appears in Grant and Karni (2004).

A weaker version of this result, based on restricting the consistency condition to a subset of hypothetical lotteries that have the same marginal distribution on S, due to Karni, Schmeidler, and Vind (1983), yields a subjective expected utility representation with state-dependent preferences. Wakker (1985) has extended the theory of Karni, Schmeidler and Vind (1983) to include the case in which the set of consequences is a connected topological space. However, the arbitrary choice of the subset of hypothetical lotteries renders the probabilities in these works arbitrary.

Other theories that yield subjective expected utility representations invoke preferences on conditional acts (that is, preference relations over the set of acts conditional on events). Fishburn (1973) and Karni (2005) advanced such theories assuming consequence sets that have distinct structures. Skiadas (1997) proposed a nonexpected utility model, based on hypothetical preferences, that yield a representation with state-dependent preferences. In this model, acts and states are primitive concepts, and preferences are defined on act-event pairs. For any such pair the consequences (utilities) represent the decision maker's expression of his holistic valuation of the act. The decision maker is not supposed to be aware whether the given event occurred, hence his evaluation of the act reflects, in part, his anticipated feelings, such as disappointment aversion.

Moral hazard and state-dependent preferences. Drèze (1961, 1985) and Karni (2005a) present distinct theories of individual decision-making under uncertainty with moral hazard and state-dependent preferences. Both assume that decision makers can exercise some control over the likely realization of events.

Drèze does not specify the means by which this control is exercised, relying instead on their manifestation in the decision maker's choice behavior. In particular, departing from Anscombe and Aumann's (1963) "reversal of order" assumption, Drèze assumes that decision makers strictly prefer that the uncertainty of the lottery payoff be resolved before that of the acts, presumably to allow them to exploit this information by taking action to affect the likely realization of the underlying states. Drèze obtains a unique separation of state-dependent utilities from a set of probability distributions over the set of states of nature. Choice is represented as expected utility maximizing behavior where the expected utility associated with any given act is itself the maximal expected utility with respect to the probabilities in the set.

Karni (2005a) replaces the state space with a set of effects – phenomena on which decision makers can place bets and whose realization they can influence by their actions. In Karni's theory the choice set consists of action-bet pairs. Actions affect the decision maker's well-being directly (e.g., actions may correspond to levels of effort) and indirectly (through their impact on the decision maker's beliefs); bets are functions from effects to monetary payoffs. Karni gives necessary and sufficient conditions for the existence of subjective expected utility representations with unique, action-dependent, subjective probabilities; effect-dependent utility functions representing the evaluation of wealth; and a distinct function that captures the direct impact of the choice of action on the decision maker's well-being.

Attitudes toward risk. As with state-independent preferences, the economic analysis of many decision problems involving state-dependent preferences requires measures of risk aversion. Such measures are developed in Karni (1985).

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