A Reformulation of the Maxmin Expected Utility Model with Application to Agency Theory

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Abstract

Invoking the parametrized distribution formulation of agency theory, the paper develops axiomatic foundations of the principal’s and agent’s choice behaviors that are representable by the maximization of the minimum expected utility over action-dependent sets of priors. In the context of this model, the paper also discusses some implications of uncertainty aversion for the design of optimal incentive schemes.

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1 Introduction

In recent years, ambiguity aversion, displayed by a pattern of choice first noted by Ellsberg (1961), has become a focal issue in the theory of decision making under uncertainty. This phenomenon is incompatible with representations of decision makers’ beliefs by additive (subjective) probability measures. Axiomatic models of decision making under uncertainty designed to accommodate ambiguity aversion include the Choquet expected utility model of Schmeidler (1989), and the maxmin expected utility model of Gilboa and Schmeidler (1989).

If ambiguity aversion is an important aspect of individual decision-making under uncertainty then, like risk aversion, it should manifest itself in shaping economic institutions and practices. In this paper, I take a step toward incorporating ambiguity aversion into the theory of incentive contracts and the study of its implications.

By and large, the analysis of the principal-agent relationship in the presence of moral hazard is based on the assumption that the two parties are expected-utility maximizers. In particular, the parametrized distribution formulation, pioneered by Mirrlees (1974, 1976), envisions a principal designing an incentive contract intended to induce an agent to choose an action that would maximize the principal’s expected utility, over a class of action-dependent probability distributions on outcomes. Given the contract, the agent is supposed to choose the action that would maximize his own expected utility over the same class of action-dependent probability distributions on outcomes.
In view of the main concern of this work, I begin by developing axiomatic models of ambiguity-averse principal and agent, whose preferences are represented by maxmin expected utility over action-dependent subjective sets of priors. These models constitute a synthesis of the maxmin expected utility model of Gilboa and Schmeidler (1989) and the analytical framework of Karni (2006). The analytical framework dispenses with the notion of states of nature which, for reasons explained in Karni (2006), I regard as unsatisfactory, introducing instead a choice space whose elements are action-bet pairs. Actions correspond to means by which agents may influence the likely realizations of different outcomes; bets represent outcome-contingent payoffs. The resulting representations are subjective versions of the parametrized distribution formulation, lending themselves naturally to the investigation of incentive contracts in the presence of moral hazard.

Despite obvious similarities, the approach taken here is different from existing models in several important respects. First, unlike Mirrlees (1974, 1976), who assumes that the family of action-dependent distributions is given, I derive the relevant family of action-dependent (sets of) distributions from the principal’s and agent’s preferences. Second, the representations are different from Gilboa and Schmeidler (1989) in that, through their selection of actions, decision makers can choose among sets of distributions on outcomes. Moreover, the preference relation on the contingent payoff schemes, representing the contracts, may be outcome dependent.

To render this discussion more precise, I denote the set of actions by $A$ and the set of contracts by $W$. Elements of $W$ are real-valued functions on a set, $\Theta$, of outcomes.
Assume that the principal and the agent have preference relations on the set, $\mathcal{A} \times \mathcal{W}$, of action-contract pairs that are denoted by $\succ^P$ and $\succ^A$, respectively.

An action-contract pair $(a^*, w^*)$ is said to be a maximal element of a subset $F \subset \mathcal{A} \times \mathcal{W}$ according to $\succ$ if $(a^*, w^*) \in F$ and $(a^*, w^*) \succ (a, w)$ for all $(a, w) \in F$. The principal’s problem may be stated as follows: Choose the maximal element according to $\succ^P$ of the “feasible” set $F \subset \mathcal{A} \times \mathcal{W}$, defined by: $(a', w') \in F$ if and only if
\[(a', w') \succ^A (a, w') \quad \text{for all } a \in \mathcal{A} \tag{1}\]
and
\[(a', w') \succ^A (a^0, w^0), \tag{2}\]
where $(a^0, w^0)$ where $w^0 \in \mathcal{W}$ denote the agent’s “outside option,” and $a^0$ satisfies $(a^0, w^0) \succ^A (a, w^0)$, for all $a \in \mathcal{A}$.

If the principal and the agent are maxmin expected utility maximizers, and the agent’s utility function is additively separable in actions and payoffs, then the problem may be stated as follows: Choose an action-contract pair $(a', w')$,
\[\begin{align*}
(a', w') &\in \arg \max_{\mathcal{A} \times \mathcal{W}} \left\{ \min_{\pi \in \mathcal{C}^P(a)} \sum_{\theta \in \Theta} u^P \left( w(\theta); \pi(\theta) \right) \right\} \\
&\text{subject to the incentive compatibility constraints, for all } a \in \mathcal{A}, \\
&\min_{\pi \in \mathcal{C}^A(a')} \sum_{\theta \in \Theta} u^A \left( w'(\theta), \pi(\theta) \right) + v(a') \geq \min_{\pi \in \mathcal{C}^A(a)} \sum_{\theta \in \Theta} u^A \left( w(\theta), \pi(\theta) \right) + v(a), \tag{4}\end{align*}\]
and the participation constraint
\[\begin{align*}
&\min_{\pi \in \mathcal{C}^A(a')} \sum_{\theta \in \Theta} u^A \left( w'(\theta), \pi(\theta) \right) + v(a') \geq \min_{\pi \in \mathcal{C}^A(a^0)} \sum_{\theta \in \Theta} u^A \left( w^0(\theta), \pi(\theta) \right) + v(a^0), \tag{5}\end{align*}\]
where \( \{u^P(\cdot; \theta) \mid \theta \in \Theta\} \) and \( \{u^A(\cdot; \theta) \mid \theta \in \Theta\} \) are outcome-dependent utility functions of the principal and the agent, respectively; \( \{C^P(a) \mid a \in A\} \) and \( \{C^A(a) \mid a \in A\} \) are their respective sets of priors on \( \Theta \); and \( v \) denotes the agent’s disutility of actions.

Assume that the payoffs of the bets are roulette lotteries (or lotteries, for short) in the sense of Anscombe and Aumann (1963). The key notion invoked in the axiomatization of the principal’s preferences is that of constant-valuation bets. Loosely speaking, constant-valuation bets are defined by the behavioral implication that, given such a bet, the decision-maker is indifferent among all actions. Because the choice of action has a direct effect on the agent’s well-being, but not necessarily on that of the principal, constant valuation bets mean different things for the two parties. To handle this difference, the axiomatic structure of the agent’s preferences permits the separation of the direct effect of the actions from the indirect effect (that is, the impact of the actions on the distributions of outcomes).

The literature dealing with agency theory with ambiguity-averse players is rather meager. Ghirardato (1994) examines the validity of some properties of optimal incentive schemes, in the context of Schmeidler’s (1989) Choquet expected utility model. While the motivation is similar, the present work differs from that of Ghirardato both in terms of the analytical framework and the representations of the players’ preferences. Like Schmeidler’s Choquet expected utility model, on which it is based, Ghirardato’s analysis does not allow for outcome-dependent preferences over monetary payoffs. This aspect of the preference relations, which is relevant in many applications, is accommodated by the approach taken here.
Mukerji (2004) studies the effect of ambiguity aversion on the contractual form of procurement contracts. To model ambiguity and the principal and agent ambiguity aversion, Mukerji invokes a model, due to Ghirardato, Maccheroni and Marinacci (2004), in which acts are evaluated by a convex combination of their respective minimum and maximum expected payoff on a set of priors. Bypassing the effects of attitude towards risk, Mukerji shows that the more ambiguity averse the agent relative to the principal (that is, the higher the weight placed by the agent on the minimum of the expected payoff, relative to that of the principal) the smaller is the power of the incentives built into the procurement contract.

Drèze (1961, 1987) was the first to axiomatize subjective expected utility theory with state-dependent preferences and moral hazard. He argues that the “reversal of order” axiom of Anscombe and Aumann (1963) embodies the decision-maker’s belief that he cannot influence the likely realization of alternative states. In situations in which a decision maker may take actions that would tilt the probabilities of the states in his favor, he would prefer knowing the outcome of a lottery (and, consequently, his payoff contingent on the state that obtains) before rather than after the state of nature is revealed. In Drèze’s theory the decision maker’s actions are tacit.1

Like the maxmin expected utility model, the representation in Drèze’s theory entails the maximization of subjective expected utility over a convex set of subjective probability measures. Yet, Drèze’s theory is different from the maxmin expected utility model in its motivation and interpretation. Drèze’s motivation is to model decision making in the presence

1The same comment applies to the extension of Drèze’s work in Drèze and Rustichini (1999).
of moral hazard, not ambiguity aversion. Consequently, the set of probability measures in
his theory represents the leverage a decision maker believes he has over the likely realization
of the states, rather than his ambiguity aversion.

The present work differs from that of Drèze in several respects, the most important of
which is the analytical framework. In this paper, the agent’s actions and the relation that the
principal and the agent believe exist between these actions and probability distributions of
outcomes, are explicit aspects of the representation. This difference in the treatment of the
actions, emanates from distinct methodological outlooks. Drèze is looking at the problem of
modeling the agent’s behavior from the perspective of an observer (or the principal), who
may “see” only the decision-makers’ preferences over acts and over the timing of resolution
of risk. In contrast, here the principal and the agent have preferences over the set of action-
contract pairs that are known to them and, in principle, observable.

Also related to the present work is Karni (2006a), where I explore axiomatic foundations
of principal and agent’s behavior based on an analytical framework similar to the one used
here. In that paper I was concerned only with principal and agent who are subjective
expected utility maximizers.

In the following section I introduce the analytical framework. The principal’s preference
relation and its representation appear in Section 3. The agent’s preference relation and its
representations is the subject matter of Section 4. In Section 5, I discuss issues pertaining
to the application of the maxmin expected utility model to agency theory and, within a
simple model, examine some welfare implications. Section 6 contains a brief summary and a discussion of related literature. The proofs appear in Section 7.

2 The Analytical Framework

2.1 Preliminaries

Let $\Theta$ be a finite set of outcomes, and let $\mathcal{A}$ be a set whose elements, called actions, describe activities by which a decision maker, the agent in this instance, may influence the likely realization of different outcomes. Let $I(\theta)$ be an interval in $\mathbb{R}$ representing monetary prizes that are feasible if the outcome $\theta \in \Theta$ obtains. Denote by $\Delta(I(\theta))$ the set of all simple probability distributions (that is, distributions that have finite support) on $I(\theta)$. Elements of $\Delta(I(\theta))$ are referred to as lotteries. A bet, $b$, is a function on $\Theta$ such that $b(\theta) \in \Delta(I(\theta))$. Denote by $B$ the set of all bets (that is, $B := \Pi_{\theta \in \Theta} \Delta(I(\theta))$). The choice set is the product set $C := \mathcal{A} \times B$, whose generic element, $(a, b)$, is an action-bet pair.

A decision maker is characterized by a preference relation, $\succ$, on $C$ that have the usual interpretation.$^2$ In other words, decision makers are supposed to be able to choose among, or express preferences, over action-bet pairs, presumably taking into account the influence the choice of action may have on the likely realization of alternative outcomes and, consequently,

$^2$A preference relation, $\succ$, is a binary relation on $C$, and $(a, b) \succ (a', b')$ means that $(a, b)$ is at least as desirable as $(a', b')$.  

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on the desirability of the corresponding bets. The strict preference relation, $\succ$, and the indifference relation, $\sim$, are defined as usual.

For all $b, b' \in B$ and $\alpha \in [0, 1]$, define $\alpha b + (1 - \alpha) b' \in B$ by: 

$$(\alpha b + (1 - \alpha) b')(\theta) = \alpha b(\theta) + (1 - \alpha) b'(\theta),$$

for all $\theta \in \Theta$. I use the notation $b_{θ} p$ to denote the bet that is obtained from the bet $b$ by replacing its $θ$-coordinate with the lottery $p$.

An outcome $θ \in \Theta$ is said to be null given the action $a$ and the preference relation $\succ$, if $(a, b_{θ} p) \sim (a, b_{θ} q)$ for all $p, q \in \Delta(I(θ))$, and $b \in B$, otherwise it is nonnull given the action $a$ and the preference relation $\succ$. In general, given $\succ$, an outcome may be null under some actions and nonnull under others. However, it is customary, in agency theory, to suppose that the set of outcomes constituting the support of the distributions is invariable with respect to the actions. Formally,

The full support assumption: For all $a \in A$, the set of outcomes that are nonnull under $a$ is $\Theta$.

### 2.2 Constant valuation bets

Given a preference relation $\succ$, the idea of constant valuation bets presumes the existence of a subset, $\hat{A}$, of actions and a subset $B^{cv}$ of bets such that, given $\hat{b} \in B^{cv}$, the variations in the decision-maker's well-being that is due to the direct impact of the actions in $\hat{A}$ are compensated by variations due to impact of these actions on the likely realization of the
different effects.\textsuperscript{3} A constant valuation bet is manifested if, given this bet, the decision maker is indifferent among all the actions in $\hat{A}$. For the constant valuation bets to be well-defined, there must exist sufficiently many distinct actions in $\hat{A}$ so as to render the coordinates of each constant valuation bet unique, in the sense of belonging to the same equivalence class of lotteries. Formally, let $\mathcal{I}(b; a) = \{b' \in B \mid (a, b') \sim (a, b)\}$ and $\mathcal{I}(p; \theta, b, a) = \{q \in \Delta(I(\theta)) \mid (a, b_{-\theta}q) \sim (a, b_{-\theta}p)\}$. $\mathcal{I}(b; a)$ denotes the indifference class of $b$ given $a$ and, similarly, $\mathcal{I}(p; \theta, b, a)$ denotes the indifference class of $p$ given $\theta$, $b$ and $a$.

**Definition 1:** A bet $\bar{b} \in B$ is a constant valuation bet according to $\succ$ if that exist $\hat{A} \subseteq \mathcal{A}$ such that $(a, \bar{b}) \sim (a', \bar{b})$ for all $a, a' \in \hat{A}$ and, for all $b \in \cap_{a \in \hat{A}}\mathcal{I}(\bar{b}; a)$, $b(\theta) \in \mathcal{I}(\bar{b}(\theta); \theta, \bar{b}, a)$ for all $\theta \in \Theta$ and $a \in \hat{A}$.

The uniqueness of the constant valuation bets, implied by Definition 1, requires that $|\hat{A}| \geq |\Theta|$. To simplify the exposition, without essential loss of generality, I assume that $\hat{A} = \mathcal{A}$.\textsuperscript{4} Let $B^{cv}(\succeq)$ denote the set of all constant valuation bets according to $\succeq$. In view of the definition of constant valuation bets, if $\bar{b}', \bar{b} \in B^{cv}(\succeq)$ I write $\bar{b}' \succeq \bar{b}$ instead of $(a, \bar{b}') \succeq (a, \bar{b})$. Also, henceforth, to simplify the notations, I shall suppress the preference relation symbol and write $B^{cv}$ instead of $B^{cv}(\succeq)$.

Assume that the choice set has a maximal and minimal elements whose bet components are constant valuation. Formally,

\textsuperscript{3} The concept of constant valuation bets first appeared in Karni (2006).

\textsuperscript{4} Karni (2006) contains a discussion on how to extend the analysis to the case in which $\hat{A}$ is a proper subset of $\mathcal{A}$.
There are $\bar{b}^{**}, \bar{b}^* \in B^c$ such that $\bar{b}^{**} \succ (a, b) \succ \bar{b}^*$ for all $(a, b) \in C$, and $\bar{b}^{**} \succ \bar{b}^*$.

### 2.3 Axioms

In general, the principal and the agent are motivated by distinct considerations. In some applications the agent’s well-being may not be affected directly by the outcome, and in most applications the principal’ well-being is not directly affected by the agent’s actions. For example, given an employment contract, the employer’s well-being is directly affected by the firm’s revenues; whereas revenues affect the employee’s well-being only to the extent that they determines his payoff under the employment contract. Moreover, the employer’s sole concern with the employee’s actions is their effect on the likely realization of profits, while the employee’s well-being is directly affected by these actions (e.g., effort). Despite their different concerns, certain principles, expressed by the following axioms, govern the choice behavior of the principal and the agent alike.

(A.1) **(Weak order)** $\succ$ is complete and transitive.

(A.2) **(Archimedean)** For all $a \in A$ and $b, b', b'' \in B$, such that $(a, b) \succ (a, b') \succ (a, b'')$, there exist $\alpha, \beta \in (0, 1)$ satisfying $(a, \alpha b + (1 - \alpha) b') \succ (a, b') \succ (a, \beta b + (1 - \beta) b'')$.

(A.3) **(Uncertainty Aversion)** For all $a \in A$, $b, b' \in B$, and $\alpha \in (0, 1)$, if $(a, b) \sim (a, b')$ then $(a, \alpha b + (1 - \alpha) b') \succ (a, b)$.
3 The Principal’s Preferences and Their Representation

3.1 Axioms

In addition to the axioms (A.1) - (A.3), the principal’s preference structure is assumed to satisfy the following addition conditions.

The statement of the first additional axiom uses the following notation and definitions. Given \( p \in \Delta(I(\theta)) \) denote by \( b_{-\theta}p \) the constant valuation bet whose \( \theta - th \) coordinate is \( p \). Define a preference relation, \( \succ_{\theta} \), on \( \Delta(I(\theta)) \) by \( p \succ_{\theta} q \) if and only if \( b_{-\theta}p \succ b_{-\theta}q \), \( p,q \in \Delta(I(\theta)) \). The symmetric and asymmetric parts of \( \succ_{\theta} \) are denoted by \( \succsim_{\theta} \) and \( \sim_{\theta} \), respectively.

(A.4) (Monotonicity) For all \( a \in \mathcal{A}, b,b' \in B \), if \( b(\theta) \succ_{\theta} b'(\theta) \) for all \( \theta \in \Theta \) then \((a,b) \succ (a,b')\).

The following axiom is analogous to the certainty-independence axiom of Gilboa and Schmeidler (1989). It requires that the independence axiom of expected utility theory applies when mixing bets with constant valuation bets.

(A.5) (Constant-Valuation Independence) For all \( a \in \mathcal{A}, b,b' \in B, \bar{b} \in B^{cv}, \) and \( \alpha \in (0,1) \), \((a,b) \succ (a,b')\) if and only if \((a,\alpha b+(1-\alpha)\bar{b}) \succ (a,\alpha b' + (1-\alpha)\bar{b})\).
The next axiom requires the set of constant valuation bets be convex (that is, \( \bar{b}, \bar{b}' \in B^{cv} \) implies \( (a, \alpha \bar{b} + (1 - \alpha) \bar{b}') \sim (a', \alpha \bar{b} + (1 - \alpha) \bar{b}') \) for all \( \alpha \in (0, 1) \)). The intuition underlying this condition is, that what the principal ultimately cares about are the consequences and not the actions. Specifically, he cares about the outcome, \( \theta \), that obtains and the associated payoff, \( b(\theta) \). Thus, facing a choice between \( (a, \alpha \bar{b} + (1 - \alpha) \bar{b}') \) and \( (a', \alpha \bar{b} + (1 - \alpha) \bar{b}') \), the principal recognizes that the consequences are identical under the two actions. Regardless of the action that is implemented, he is awarded a composite lottery \( \alpha \bar{b} + (1 - \alpha) \bar{b}' \), whose consequences are \( (\alpha \bar{b}(\theta) + (1 - \alpha) \bar{b}'(\theta), \theta) \), \( \theta \in \Theta \). Because \( \bar{b} \) and \( \bar{b}' \) are constant-valuation bets, these consequences are equally preferred regardless of the outcome that obtains. Formally,

\[(A.6) \textbf{(Convexity)} \text{ For all } a, a' \in A, \bar{b}, \bar{b}' \in B^{cv}, \text{ and } \alpha \in (0, 1), (a, \alpha \bar{b} + (1 - \alpha) \bar{b}') \sim (a', \alpha \bar{b} + (1 - \alpha) \bar{b}').\]

Because what constitutes a constant valuation bet is a personal (subjective) matter, the convexity of the set \( B^{cv} \) is not implied by the structure of the choice space.\(^5\)

### 3.2 Representation

The representation of the principal’s preferences is analogous to the maxmin expected utility representation of Gilboa and Schmeidler (1989). The statement of the representation theorem below involves the following terminology. A family \( \{f_j\}_{j=1}^n \) of real-valued functions is said

\(^5\)Gilboa and Schmeidler (1989) implicitly assume that constant acts are constant-valuation acts. Because the set of constant functions is convex, the restriction imposed by Axiom (A.6) below has no bite.
to be cardinally measurable and fully comparable if, for the purpose of representation, it is equivalent to another set of functions \( \{g_j\} \), where \( g_j = a + cf_j, \ c > 0 \) for all \( j = 1, \ldots, n \).

**Theorem 1** Let \( \succ \) be a binary relation on \( \mathbb{C} \). Assume that (A.0) and the full support assumption hold. Then the following conditions are equivalent:

(i) \( \succ \) satisfies (A.1) – (A.6).

(ii) There exist affine functions \( \{u(\cdot, \theta) : \Delta(I(\theta)) \to \mathbb{R}\}_{\theta \in \Theta} \), and a family of closed and convex sets, \( \{C(a)\}_{a \in A} \), of probability measures on \( \Theta \) such that, for all \( (a, b), (a', b') \in \mathbb{C} \),

\[
(a, b) \succ (a', b') \iff \min_{\pi \in C(a)} \sum_{\theta \in \Theta} u(b(\theta), \theta) \pi(\theta) \geq \min_{\pi \in C(a')} \sum_{\theta \in \Theta} u(b'(\theta), \theta) \pi(\theta).
\]

Moreover, the functions \( \{u(\cdot, \theta)\}_{\theta \in \Theta} \) are cardinally measurable and fully comparable, for every \( a \in A \) the set \( C(a) \) is unique, and each \( \pi \in C(a) \) has full support.

The proof of Theorem 1 appears in Section 7.1. Note that the affinity of \( u \) implies that \( u(b(\theta), \theta) = \sum_{z \in I(\theta)} u(z, \theta) b(z, \theta) \).

### 3.3 Interpretation

It is noteworthy that, while I interpret Theorem 1 as a representation of the principal’s preference relation, there is nothing in the formalism that prevents its application to the
agent’s preferences as well. However, the choice of action affects the well-being of the agent and the principal in different way. In particular, it is essential, in the analysis of principal-agent problems, that the direct costs of actions to the agent not be constant while, in so far as the principal is concerned, the direct cost of the agent’s action is typically assumed to be negligible. Yet, the “disutility” of actions does not appear explicitly in the representation in Theorem 1. This is because, as the proof of the theorem makes clear, the utilities that appear in the representation are such that

\[ u(p, \theta) := \lambda v(b_{-\theta}p) + \xi, \lambda > 0 \]  

(6)

where \( v \) is an affine, real-valued, function on \( B^{cv} \) satisfying \( v(\bar{b}) \geq v(\bar{b}') \) if and only if \( \bar{b} \succeq \bar{b}' \). Thus, for each lottery \( p \), the utility of the associated constant valuation bet, \( b_{-\theta}p \), is independent of the “cost of action”. However, the constant valuation bets themselves depend, implicitly, on the costs of actions. Put differently, distinct costs of actions induce distinct sets of constant valuation bets and, consequently, distinct utility functions. Thus, in general, the utility \( u(p, \theta) \) does not necessarily represent the value the decision maker attributes to the payoff \( p \) conditional on \( \theta \), rather it reflects the value of the constant valuation bet whose \( \theta \)-coordinate is \( p \), incorporating the cost of actions.

If the effort exerted by the agent is not, in itself, be a direct source of disutility for the principal then, even if their valuations of effect-contingent monetary payoffs are the same, the principal’s and the agent’s utility functions that figure in representations such as in Theorem 1 are distinct.
The literature, dealing with the analysis of principal-agent relationship, supposes that the agent’s choice of action has no direct effect on the principal’s utility. Hence, in so far as the principal’s preferences are concerned, constant valuation bets are constant utility bets (that is, bets whose implicit valuation of the consequences \((b(\theta), \theta)\) is the same for all \(\theta \in \Theta\)).

It is natural, therefore, to interpret the representation in Theorem 1 as pertaining to the principal’s preference relation. In so far as the agent’s preference relation is concerned, it is common practice to separate the utility of payoffs from the direct cost of effort. In the next section, I explore an alternative approach which allows for such separation.

### 4 Additive Separability over Actions and Bets

#### 4.1 Action-bet separability and constant utility bets

Assume that \( \mathcal{A} \) is a connected, separable, topological space and let \( \mathcal{A} \times \mathcal{B} \) be endowed with the product topology. Let \( \tilde{\mathcal{B}} (\succ) \) be a convex subset of \( \mathcal{B} \) on which the following, well-known, axioms hold. (Anticipating the representation below and its interpretation, henceforth I refer to elements of \( \tilde{\mathcal{B}} (\succ) \) as constant-utility bets from the point of view of the agent whose preference relation is \( \succ \)).

The first axiom requires that both actions and constant utility bets are essential in the sense that, for some \( \tilde{b} \in \tilde{\mathcal{B}} \), the agent is not indifferent among all the actions and that, for some \( a \in \mathcal{A} \), he is not indifferent among all elements of \( \tilde{\mathcal{B}} \).
(A.7) (Essentiality) There are $a, a' \in A$ and $\tilde{b}, \tilde{b}' \in \tilde{B}$ such that $(a, \tilde{b}) \succ (a', \tilde{b})$ and $(a, \tilde{b}) \succ (a, \tilde{b}')$.

The second axiom requires, that the preference ranking of constant utility bets be independent of the actions, and that of actions be independent of constant utility bets.

(A.8) (Coordinate independence) For all $a, a' \in A$ and $\tilde{b}, \tilde{b}' \in \tilde{B}$, $(a, \tilde{b}) \succ (a', \tilde{b})$ if and only if $(a', \tilde{b}) \succ (a', \tilde{b}')$ and $(a, \tilde{b}) \succ (a', \tilde{b})$ if and only if $(a, \tilde{b}') \succ (a', \tilde{b}')$.

The third axiom is the familiar condition that is necessary for the existence of separately additive representations when there are two factors (actions and bets in the this case). Using coordinate independence, and with slight abuse of notations, if $\tilde{b}, \tilde{b}' \in \tilde{B}$ I shall write $\tilde{b} \succ \tilde{b}'$ instead of $(a, \tilde{b}) \succ (a, \tilde{b}')$.

(A.9) (Hexagon condition) For all $a, a', a'' \in A$ and $\tilde{b}, \tilde{b}', \tilde{b}'' \in \tilde{B}$ if $(a, \tilde{b}') \sim (a', \tilde{b})$ and $(a, \tilde{b}'') \sim (a'', \tilde{b})$ then $(a', \tilde{b'}) \sim (a'', \tilde{b'})$.

Henceforth, where there is no risk of confusion, I write $\tilde{B}$ instead of $\tilde{B}(\succ)$. The reader should keep in mind, however, that what the agent regards as constant utility bets, is entirely a subjective matter.

The restriction of $\succ$ to $A \times \tilde{B}$ is a continuous weak order satisfying axioms (A.7) – (A.9) if and only if there exist continuous real-valued functions $V$ on $\tilde{B}$ and $\varsigma$ on $A$ such that

\[6\text{See Wakker (1989).}\]
\(\succ\) on \(\mathcal{A} \times \tilde{B}\) is represented by \(\left( a, \tilde{b} \right) \mapsto V(\tilde{b}) + \varsigma(a).\) Moreover, the functions \((V, \varsigma)\) are jointly cardinal (that is, if \(\left( \tilde{V}, \tilde{\varsigma} \right)\) is another additive representation of \(\succ\) on \(\mathcal{A} \times \tilde{B}\) then \(\tilde{V} = \gamma V + \delta_1\) and \(\tilde{\varsigma} = \gamma \varsigma + \delta_2, \gamma > 0).\)

Suppose that \(\tilde{B}\) is unique in the following sense:

**Uniqueness of** \(\tilde{B}\) - For every given \(\tilde{b} \in \tilde{B}\) and all \(b \in B, b(\theta) \in \mathcal{I}(\tilde{b}(\theta); \theta, \tilde{b})\) for all \(\theta \in \Theta\) if and only if \(b \in \tilde{B}\) and \(\tilde{b} \sim b.\)

Next I restate some of the axioms of the preceding section, replacing constant valuation bets with elements of \(\tilde{B}\).

Stating the modified monotonicity axiom, requires additional notation and definitions. Given \(p \in \Delta(I(\theta))\) denote by \(\tilde{b}_{-\theta}p\) be the bet in \(\tilde{B}\) whose \(\theta - th\) coordinate is \(p.\)

\((A.4')\) **(Monotonicity)** For all \(a \in \mathcal{A}, b, b' \in B,\) if \(\left( a, b_{-\theta}b(\theta) \right) \succ \left( a, b_{-\theta}b'(\theta) \right)\) for all \(\theta \in \Theta\) then \((a, b) \succ (a, b').\)

Constant-valuation independence is replaced by constant-utility independence.

\((A.5')\) **(Constant-Utility Independence)** For all \(a \in \mathcal{A}, b, b' \in B,\) \(\tilde{b} \in \tilde{B},\) and \(\alpha \in (0, 1),\)

\[ (a, b) \succ (a, b') \text{ if and only if } \left( a, \alpha b + (1 - \alpha) \tilde{b} \right) \succ \left( a, \alpha b' + (1 - \alpha) \tilde{b} \right). \]

\(^7\)The relation \(\succ\) is continuous (in the topological sense) if the sets \(\{(a, b) \in \mathcal{A} \times B \mid (a, b) \succ (a', b')\}\) and \(\{(a, b) \in \mathcal{A} \times B \mid (a', b') \succ (a, b)\}\) are closed.

\(^8\)See Wakker (1989) Theorem III.4.1.
Analogously to (A.0), assume that there are maximal and minimal bets. Formally,

\[(A.0')\] There are \(\tilde{b}^{**}, \tilde{b}^* \in \tilde{B}\) such that \(\tilde{b}^{**} \succ (a, b) \succ \tilde{b}^*,\) for all \((a, b) \in C,\) and \(\tilde{b}^{**} \succ \tilde{b}^*.\)

### 4.2 Representation

The next theorem is a representation of the agent’s choice behavior, separating the his attitude towards risk from his actions.

**Theorem 2** Let \(\succ\) be a binary relation on \(C.\) Assume that \((A.0')\) and the full support assumption hold. Then the following conditions are equivalent:

1. \(\succ\) is a continuous weak order satisfying \((A.3), (A.4'), (A.5')\) and, on some convex subset, \(\tilde{B},\) of bets it satisfies \((A.7)-(A.9).\)

2. There exist affine functions \(\{v(\cdot; \theta) : \Delta(I(\theta)) \to \mathbb{R}\},\) a function \(\varsigma : A \to \mathbb{R},\) and a family of closed and convex sets, \(\{C(a)\}_{a \in A},\) of full support probability measures on \(\Theta\) such that, for all \((a, b), (a', b') \in C,\)

\[(a, b) \succ (a', b') \iff \min_{\pi \in C(a)} \sum_{\theta \in \Theta} v(b(\theta), \theta) \pi(\theta) + \varsigma(a) \geq \min_{\pi \in C(a')} \sum_{\theta \in \Theta} v(b(\theta), \theta) \pi(\theta) + \varsigma(a').\]

Moreover, the functions \(\{v(\cdot, \theta)\}_{\theta \in \Theta}\) are cardinally measurable and fully comparable, \(V := \sum_{\theta \in \Theta} v(\tilde{b}(\theta), \theta)\) on \(\tilde{B}\) and \(\varsigma\) are jointly cardinal, and, for each \(a \in A,\) the set \(C(a)\) is unique.

The proof is similar to that of Theorem 1 and is outlined in Section 7.2.
The function \( \varsigma \) represents the (dis)utility of the actions. For reasons described earlier, the representation in Theorem 2 pertains to the preference relation of the agent.

### 4.3 Effect-independent preferences

In many instances, it is natural to suppose that the agent’s attitude towards risk is effect-independent. To formalize the revealed preference implications of effect independent risk attitudes, it is necessary to take account of the “rank-dependent” nature of the maxmin preference relation. Without essential loss of generality, assume that \( I(\theta) = [\bar{x}, \bar{x}] \) for all \( \theta \in \Theta \), and, consequently, \( \Delta (I(\theta)) = \Delta \) for all \( \theta \in \Theta \). The following axiom lends a choice-based meaning to the idea of effect independent.

\[ (A.10) \ (\text{Effect independence}) \text{ For all } \theta, \theta' \in \Theta \text{ and } p, q \in \Delta, \left( a, \tilde{b}_{-\theta}^{*}p \right) \succ \left( a, \tilde{b}_{-\theta}^{*}q \right) \text{ if and only if } \left( a, \tilde{b}_{-\theta'}^{*}p \right) \succ \left( a, \tilde{b}_{-\theta'}^{*}q \right). \]

Note that the effect of “rank dependence” has been neutralized by the choice of \( \tilde{b}^{**} \) as the reference bet. To grasp this, observe that \( \tilde{b}^{**} = (\delta_{\bar{x}}, ..., \delta_{\bar{x}}) \) and recall that, by definition, \( \tilde{b}^{**} \succ \tilde{b}_{-\theta}p \) for all \( p \in \Delta \) and \( \theta \in \Theta \). Hence, replacing \( \tilde{b}^{**}(\theta) \) by either \( p \) or \( q \), does not entail a change in the preference ordering of the coordinates. (If it did, then it would have implied that \( \tilde{b}_{-\theta'}p \succ \tilde{b}^{**} \) for some \( \theta' \in \Theta \), leading to a contradiction).

Adding effect-independence to the axiomatic structure of Theorem 2, yields the following representation.
Theorem 3  Let \( \succ \) be a binary relation on \( C \). Assume that (A.0’) and the full support assumption hold. Then the following conditions are equivalent:

(i) \( \succ \) is a continuous weak order satisfying (A.3), (A.4’), (A.5’), (A.10) and, on some convex subset, \( \tilde{B} \), of bets it satisfies (A.7)–(A.9).

(ii) There exist an affine function \( v : \Delta \to \mathbb{R} \), a function \( \varsigma : A \to \mathbb{R} \), and a family of closed and convex sets, \( \{ C(a) \}_{a \in A} \), of probability measures on \( \Theta \) such that, for all \( (a, b), (a’, b’) \in C \),

\[
(a, b) \succ (a’, b’) \iff \min_{\pi \in C(a)} \sum_{\theta \in \Theta} v(b(\theta)) \pi(\theta) + \varsigma(a) \geq \min_{\pi \in C(a')} \sum_{\theta \in \Theta} v(b(\theta)) \pi(\theta) + \varsigma(a').
\]

Moreover, the functions \( v \) and \( \varsigma \) are jointly cardinal and, for each \( a \in A \), the set \( C(a) \) is unique, and each \( \pi \in C(a) \) has full support.

The proof appears in section 7.3.

5 A Simple Application

To explore some rudimentary agency-theory implications of ambiguity aversion, I consider a principal who engages an agent to perform a task. The task must be performed in one of two ways, involving distinct actions, \( a_0 \) and \( a_1 \); either way may result in success or failure. Assume that the agent’s choice of action is his private information and the outcome is publicly observable. If the agent performs the task successfully, the principal stands to gain \( x_s \) dollars;
if the agent fails, the principal stands to gain $x_f$ dollars, where $x_s > x_f$. Assume that $a_1$ is more costly to the agent and yields a more favorable prospect of success. Assume also that the principal is risk neutral and the agent is risk averse. To explore the implications of ambiguity aversion for the design of incentive contracts, I consider next alternative attitudes toward ambiguity.

5.1 The benchmark case: Expected utility-maximizing behavior

Consider the standard problem in which both the agent and the principal are ambiguity neutral (that is, they are expected utility maximizers). As is customary in agency theory, assume that the principal and the agent agree on the likelihood of success, conditional on the alternative actions. Denote by $p_s(a_i)$ the probability of success under $a_i$, $i = 0, 1$, and assume that $p_s(a_1) > p_s(a_0)$. Suppose that the principal wants to implement $a_1$. His problem is to design a contract, $w = (w_s, w_f) \in \mathbb{R}^2$, so as to maximize

$$p_s(a_1)(x_s - w_s) + (1 - p_s(a_1))(x_f - w_f)$$

subject to the incentive compatibility (IC) constraint

$$(p_s(a_1) - p_s(a_0))(u(w_s) - u(w_f)) \geq v(a_1) - v(a_0)$$

and the individual rationality (IR) constraint

$$p_s(a_1)u(w_s) + (1 - p_s(a_1))u(w_f) - v(a_1) \geq u.$$
where \( u \), the agent’s von Neumann-Morgenstern utility function, is increasing and strictly concave, \( v(a_1) > v(a_0) \), and \( u \) is the agent’s utility of his, uncertainty free, “outside option.”

In this case, both constraints are binding and the solution to the problem is given by\(^9\)

\[
\begin{align*}
    u(w_s^*) &= u + v(a_0) + \frac{1 - p_s(a_0)}{p_s(a_1) - p_s(a_0)}(v(a_1) - v(a_0)) \\
    u(w_f^*) &= u + v(a_0) - \frac{p_s(a_0)}{p_s(a_1) - p_s(a_0)}(v(a_1) - v(a_0))
\end{align*}
\]

(7)

Clearly, \( w_s^* > w_f^* \).

5.2 Ambiguity-neutral principal and ambiguity-averse agent

Analogous to the assumption that the principal is risk neutral and the agent is risk averse, I assume next that the principal is ambiguity neutral and the agent ambiguity averse. The principal’s objective function is as in the previous case. To describe the constraints he faces, let \( C(a_i) = [\alpha_i, \beta_i], i = 0, 1 \), be the agent’s set of probabilities of success under \( a_i \).

It is customary to suppose that costlier actions yield a higher probability of success. In the present context, the analogous assumption requires that \( \alpha_1 > \alpha_0 \) and \( \beta_1 > \beta_0 \). Furthermore, because the agent’s preferences are represented by maxmin expected utility functional with action-dependent sets of priors, while the principal is an expected utility maximizer, the “common priors” assumption must be modified. Assumed that the principal is aware of the agent’s ambiguity aversion, that he knows \( C(a_i), i = 0, 1 \), and that his own conditional probabilities of success satisfy \( p_s(a_i) \in C(a_i), i = 0, 1 \).

As in the benchmark case, suppose that the principal would like to implement the action \( a_1 \). Note that under these assumptions, if \( w = (w_s, w_f) \) is the optimal incentive contract, \( w_s > w_f \). To see this, observe that because the IC constraint is binding, if \( w_s \leq w_f \), then the IC constraint implies that \((\beta_1 - \beta_0)(u(w_s) - u(w_f)) = v(a_1) - v(a_0) > 0\). If \( \beta_1 > \beta_0 \) then \( u(w_s) - u(w_f) > 0 \) and, since \( u \) is strictly monotonic increasing, \( w_s > w_f \), a contradiction. If \( w_s \leq w_f \) and \( \beta_1 = \beta_0 \), then the agent cannot be induced to choose \( a_1 \).

Let \( \hat{w} = (\hat{w}_s, \hat{w}_f) \) be the optimal incentive contract designed to induce the agent to choose the action \( a_1 \). Suppose that \( \alpha_1 > \alpha_0 \) (otherwise, it is impossible to implement \( a_1 \)). Because \( \hat{w}_s > \hat{w}_f \) the IC constraint is

\[
(\alpha_1 - \alpha_0)(u(\hat{w}_s) - u(\hat{w}_f)) \geq v(a_1) - v(a_0),
\]

and the corresponding IR constraint is

\[
\alpha_1 u(\hat{w}_s) + (1 - \alpha_1) u(\hat{w}_f) - v(a_1) \geq u.
\]

It is easy to verify that, since the two constraints are binding, \( \hat{w} \) is given by

\[
\begin{align*}
u(\hat{w}_s) &= u + v(a_0) + \frac{1 - \alpha_1}{\alpha_1 - \alpha_0}(v(a_1) - v(a_0)) \\
u(\hat{w}_f) &= u + v(a_0) - \frac{\alpha_1}{\alpha_1 - \alpha_0}(v(a_1) - v(a_0)).
\end{align*}
\]

Comparing this solution to the benchmark case, if \( p_s(a_1) - p_s(a_0) \geq \alpha_1 - \alpha_0 \) and \( p_s(a_1) > \alpha_1 \) then

\[
\frac{1 - p_s(a_1)}{p_s(a_1) - p_s(a_0)} < \frac{1 - \alpha_1}{\alpha_1 - \alpha_0}
\]
and
\[
\frac{p_s(a_1)}{p_s(a_1) - p_s(a_0)} < \frac{\alpha_1}{\alpha_1 - \alpha_0}
\]  
(12)

Thus \( \hat{w}_s > w_s^* \) and \( \hat{w}_f \leq w_f^* \).

(a) If \( \hat{w}_f > w_f^* \) then the cost, to the principal, of implementing \( a_1 \) when the agent is ambiguity averse is higher than in the benchmark case.

(b) If \( \hat{w}_f \leq w_f^* \) then implementing \( a_1 \), requires exposing the agent to greater risk. In addition, since \( \alpha_1 u(w_s) + (1 - \alpha_1) u(w_f) < p_s(a_1) u(w_s) + (1 - p_s(a_1)) u(w_f) \) for all \( w \) such that \( w_s > w_f \), the cost to the principal of meeting the individual rationality constraint, necessary to implement \( a_1 \), is higher.

Thus compared with the benchmark case, when the principal is ambiguity neutral and the agent is ambiguity averse, the implementation of the costly action, \( a_1 \), results in a Pareto inferior allocation.

A different question concerns the welfare effects of increasing the agent’s ambiguity aversion. One agent is said to display greater ambiguity aversion than another if, for any given action, the set of priors of the former contains that of the latter. Inspection of equations (10) reveals that, ceteris paribus, the effect of greater ambiguity aversion depends on the coefficient \( \alpha_1 / (\alpha_1 - \alpha_0) \) and \( (1 - \alpha_1) / (\alpha_1 - \alpha_0) \). Specifically, increasing ambiguity aversion corresponds to a decline in the values of \( \alpha_i, i = 0, 1 \). Hence if the shift, \( \alpha_1 - \alpha_0 \), due to the change of action, is independent of the agent’s ambiguity aversion, then greater ambiguity aversion implies a larger spread between the payoffs associated with success and failure. Be-
cause the agent is risk averse, the mean payoff required to induce the agent to implement \( a_1 \) must increase, and the principal is made worse off. Since the utility of the outside option of the agent remains the same, the overall result is Pareto inferior.

### 5.3 Ambiguity-averse principal and agent

Suppose that the principal and the agent are both ambiguity averse and that \( C^P(a_i) = C^A(a_i) := C(a_i), i = 0, 1 \). As in the preceding case, denote by \( \hat{w} = (\hat{w}_s, \hat{w}_f) \) the contract required to implement \( a_1 \) (the solution to equation (10)) and let \( \bar{w} = (\bar{w}, \bar{w}) \) be the contract required to implement \( a_0 \). What are the consequences of the ambiguity aversion of the principal? In particular, is it possible that, ceteris paribus, an ambiguity-neutral principal would choose to implement an action different from that of an ambiguity-averse principal?

Recall that \( \hat{w}_s > \hat{w}_f \) and suppose that \( (x_s - \hat{w}_s) > (x_f - \hat{w}_f) \). If, in addition, \( p_s(a_1) - p_s(a_0) > \alpha_1 - \alpha_0 \), it is possible that

\[
p_s(a_1) (x_s - \hat{w}_s) + (1 - p_s(a_1)) (x_f - \hat{w}_f) > p_s(a_0) x_s + (1 - p_s(a_0)) x_f - \bar{w},
\]

(13)

and

\[
\alpha_0 x_s + (1 - \alpha_0) x_f - \bar{w} > \alpha_1 (x_s - \hat{w}_s) + (1 - \alpha_1) (x_f - \hat{w}_f).
\]

(14)

In other words, it is possible that an ambiguity-neutral principal would induce the agent to implement \( a_1 \) while an ambiguity-averse principal, in the same situation, would prefer that the agent implement \( a_0 \).
6 Summary

This paper axiomatizes the parametrized distributions formulation of agency model, in which the agent are representable by the maxmin expected utility maximizers, displaying ambiguity aversion. The two features that distinguish maxmin expected utility representations in this paper from the original maxmin expected utility representation of Gilboa and Schmeidler (1989) are: the action-dependent sets of priors and effect-dependent utility functions of money. These features reflect the distinct analytical frameworks more than the characteristics of the individual preferences.

Using a simple set-up, this paper also discusses some considerations that need to be addressed when modeling principal-agent relations in which either party or both are maxmin expected utility players. Within this framework the paper examines some implications of ambiguity aversion for the design of incentive contracts, illustrating complications and welfare implications that arise due to ambiguity aversion.

7 Proofs

7.1 Proof of Theorem 1

That \((i) \Rightarrow (ii)\) is implied by the following lemmata.
Lemma 4 There exist an affine function \( \bar{U} : B^{cv} \to \mathbb{R} \) such that, for all \( \bar{b}, \bar{b}' \in B^{cv} \), \( \bar{b} \succ \bar{b}' \) if and only if \( \bar{U}(\bar{b}) \geq \bar{U}(\bar{b}') \). Moreover, \( \bar{U} \) is unique up to positive affine transformation.

Proof of Lemma 4. By Axiom (A.6) \( B^{cv} \) is a convex set. Axiom (A.5) implies that the independence axiom of expected utility theory holds on \( B^{cv} \). Hence, by the von Neumann-Morgenstern theorem, axioms (A.1), (A.2) and (A.5) imply that, for every \( a \in A \), there exists an affine function, \( U(\cdot, a) : B^{cv} \to \mathbb{R} \), unique up to positive affine transformation, such that, for all \( \bar{b}, \bar{b}' \in B^{cv} \),

\[
(a, \bar{b}) \succ (a, \bar{b}') \iff U(\bar{b}, a) \geq U(\bar{b}', a) .
\]  

(15)

Invoking the uniqueness of \( U(\cdot, a) \), for every \( a \in A \), let \( U(\bar{b}, a) = 0 \) and \( U(\bar{b}^{**}, a) = 1 \). By unique solvability, for every \( \bar{b} \in B^{cv} \) and \( a \in A \), there is a unique \( \alpha(\bar{b}, a) \in [0, 1] \) satisfying \( \bar{b} \sim \alpha(\bar{b}, a) \bar{b}^{**} + (1 - \alpha(\bar{b}, a)) \bar{b}^{*} \).\(^{10}\) Hence, by equation (15) and the normalization, \( U(\bar{b}, a) = \alpha(\bar{b}, a) = \alpha(\bar{b}) \) for all \( a \in A \). The affinity of \( U(\cdot, a) \) implies that \( \alpha(\cdot) \) is affine and, by definition, \( \alpha(\bar{b}^{*}) = 0 \) and \( \alpha(\bar{b}^{**}) = 1 \).

Define \( \bar{U} : B^{cv} \to \mathbb{R} \) by \( \bar{U}(\bar{b}) = \alpha(\bar{b}) \) for all \( \bar{b} \in B^{cv} \). Then, by equation (15), \( \bar{b} \succ \bar{b}' \) if and only if \( \bar{U}(\bar{b}) \geq \bar{U}(\bar{b}') \), and \( \bar{U} \) is affine. The uniqueness of \( \bar{U} \) is immediate. \( \square \)

Lemma 5 Given a function \( \bar{U} \) from Lemma 4, there exists a unique \( J : C \to \mathbb{R} \) such that:

\(^{10}\)Unique solvability asserts that, if \( (a, \bar{b}) \succ (a, \bar{b}') \succ (a, \bar{b}'') \) and \( (a, \bar{b}) \succ (a, \bar{b}'') \) then there exists a unique \( \alpha \in [0, 1] \) such that \( (a, \bar{b}') \sim (a, \alpha \bar{b} + (1 - \alpha) \bar{b}'') \). Unique solvability is implied by Axioms (A.1), (A.2) and (A.5).
(i) \( (a, b) \succ (a', b') \) if and only if \( J(a, b) \geq J(a', b') \) for all \((a, b), (a', b') \in \mathbb{C} \).

(ii) For every \( \bar{b} \in B^{cv} \), \( J(a, \bar{b}) = \bar{U}(\bar{b}) \) for all \( a \in A \).

**Proof of Lemma 5.** On \( A \times B^{cv} \) \( J \) is uniquely determined by (ii) and is independent of \( a \).

To extend \( J \) to \( \mathbb{C} \), take \( (a, b) \in \mathbb{C} \) then \( \bar{b}^{**} \succ (a, b) \succ \bar{b}^{*} \). By unique solvability there exists a unique \( \alpha(a, b) \in [0, 1] \) such that \( (a, b) \sim (a, \alpha(a, b) \bar{b}^{**} + (1 - \alpha(a, b)) \bar{b}^{*}) \). By the convexity of \( B^{cv} \), \( \alpha(a, b) \bar{b}^{**} + (1 - \alpha(a, b)) \bar{b}^{*} \in B^{cv} \). Define

\[
J(a, b) = J(a, \alpha(a, b) \bar{b}^{**} + (1 - \alpha(a, b)) \bar{b}^{*}) = \bar{U}(\alpha(a, b) \bar{b}^{**} + (1 - \alpha(a, b)) \bar{b}^{*} \).
\]

But, by transitivity, \( (a, b) \succ (a', b') \) if and only if \( \alpha(a, b) \bar{b}^{**} + (1 - \alpha(a, b)) \bar{b}^{*} \succ \alpha(a', b') \bar{b}^{**} + (1 - \alpha(a', b')) \bar{b}^{*} \). By Lemma 4 and equation (16),

\[
\bar{U}(\alpha(a, b) \bar{b}^{**} + (1 - \alpha(a, b)) \bar{b}^{*}) \geq \bar{U}(\alpha(a', b') \bar{b}^{**} + (1 - \alpha(a', b')) \bar{b}^{*}) \iff J(a, b) \geq J(a', b').
\]

Thus \( (a, b) \succ (a', b') \) if and only if \( J(a, b) \geq J(a', b') \). Hence \( J \) satisfies (i) and is also unique.

\( \square \)

Choose a utility function such that the utility of \( \bar{b}^{*} \) smaller than \(-1\) and that of \( \bar{b}^{**} \) is greater than \(1\). (This choice is possible due to (A.0), and the uniqueness of \( \bar{U} \) in Lemma 5).

For expository convenience, with slight abuse of notation, I denote this function by \( \bar{U} \).

Let \( F \) be the set of all real-valued functions on \( \Theta \). Let \( K = \bar{U}(B^{cv}) \) and denote by \( F(K) \) the set of real-valued functions on \( \Theta \) with values in \( K \). Given \( b \in B \), let \( \bar{U} \otimes b \in F(K)^{\Theta} \) be defined by: \( \bar{U} \otimes b := \left( \bar{U}(b_{\theta \cdot \phi}(\theta)) \right)_{\theta \in \Theta} \). Note that, because \( \bar{U} \) is affine, for all \( b, b' \in B \) and
\[\alpha \in [0, 1], \bar{U} \otimes (\alpha b + (1 - \alpha) b') = \alpha \bar{U} \otimes b + (1 - \alpha) \bar{U} \otimes b'.\] (To see this, observe that
\[
\bar{U} \otimes (\alpha b + (1 - \alpha) b') = \left( \bar{U} \left( b_{-\theta} (\alpha b + (1 - \alpha) b') (\theta) \right) \right)_{\theta \in \Theta} = (\bar{U} \alpha b_{-\theta} (\theta) + (1 - \alpha) b_{-\theta} b' (\theta))_{\theta \in \Theta}
\]
\[= (\alpha \bar{U} \left( b_{-\theta} b (\theta) \right) + (1 - \alpha) \bar{U} \left( b_{-\theta} b' (\theta) \right))_{\theta \in \Theta} = \alpha \bar{U} \otimes b + (1 - \alpha) \bar{U} \otimes b',\]
where use has been made of the convexity of \(B_{cv}\) and Definition 1).

For every \(r \in \mathbb{R}\) denote by \(r_c\) the constant function in \(F\) whose value is \(r\). Let \(\bar{b}^1 \in B_{cv}\) be such that \(\bar{U} \otimes \bar{b}^1 = (\bar{U} (\bar{b}^1), \ldots, \bar{U} (\bar{b}^1)) = 1_c\).

**Lemma 6** There exists a functional, \(I : A \times F \to \mathbb{R}\), such that:

(i) For all \((a, b) \in C\), \(I (a, \bar{U} \otimes b) = J (a, b)\) (hence \(I (a, \bar{U} \otimes \bar{b}^1) = 1\) for all \(a \in A\)).

For all \(a \in A\),

(ii) \(I (a, \cdot)\) is monotonic (that is, for all \(f, g \in F\), \(f \geq g \iff I (a, f) \geq I (a, g)\)).

(iii) \(I (a, \cdot)\) is superadditive and homogeneous of degree 1.

(iv) \(I (a, \cdot)\) is constant independent (that is, \(I (a, f + r_c) = I (a, f) + I (a, r_c)\) for all \(f \in F\) and \(r \in \mathbb{R}\)).

**Proof of Lemma 6.** Let \(I\) be defined by condition (i) then, by Lemma 5 and (A.4), \(I\) is well defined.\(^\text{11}\) Let \(f, g \in F (K)\) satisfy \(f = \alpha g\), for some \(\alpha \in (0, 1]\). The proof that \(I (a, \cdot)\) is constant independent for \(\alpha \in (0, 1]\) and \(f \geq g\) is similar to the proof of Lemma 5 and (A.4).

\(^{11}\)To see this, note that, by Lemma 4, \(\bar{U} \otimes b = \bar{U} \otimes b'\) implies that \(b_{-\theta} b (\theta) \sim b_{-\theta} b' (\theta)\) for all \(\theta \in \Theta\). Thus, by definition of \(\succ \theta\), \(b (\theta) \sim b' (\theta)\), for all \(\theta \in \Theta\). By (A.4), this implies \((a, b) \sim (a, b')\). Whence (i) implies \(I (a, \bar{U} \otimes b) = J (a, b) = J (a, b') = I (a, \bar{U} \otimes b')\), and \(I\) is well-defined.
linear homogeneous requires showing that $I (a, f) = \alpha I (a, g)$. (This will imply the equality for $\alpha > 1$). Let $b \in B$ be such that $\bar{U} \otimes b = g$ and let $\bar{b}^0 \in B^{cv}$ satisfy $J (a, \bar{b}^0) = \bar{U} (\bar{b}^0) = 0$ (the first equality is implied by Lemma 5). Define $b' = ab + (1 - \alpha) \bar{b}^0$. By Lemma 4, $\bar{U}$ is affine. Hence $\bar{U} \otimes b' = \alpha \bar{U} \otimes b + (1 - \alpha) \bar{U} \otimes \bar{b}^0 = \alpha g = f$, where use has been made of the fact that $\bar{U} \otimes \bar{b}^0$ is a constant function whose value is $\bar{U} (\bar{b}^0) = 0$. By (i) $I (a, f) = J (a, b')$.

Let $\bar{b} \in B^{cv}$ satisfy $\bar{b} \sim (a, b)$ (hence $J (a, \bar{b}) = J (a, b) = I (a, g)$). Then, by axiom (A.5), $(a, \alpha \bar{b} + (1 - \alpha) \bar{b}^0) \sim (a', ab + (1 - \alpha) \bar{b}^0)$. Invoking the convexity of $B^{cv}$ and the affinity of $\bar{U}$, by Lemma 5, (ii),

$$J (a, b') = J (a, \alpha \bar{b} + (1 - \alpha) \bar{b}^0) = \alpha J (a, \bar{b}) + (1 - \alpha) J (a, \bar{b}^0) = \alpha J (a, \bar{b}). \quad (17)$$

Thus

$$I (a, f) = J (a, b') = \alpha J (a, \bar{b}) = \alpha J (a, b) = \alpha I (a, g) \quad (18)$$

and $I (a, \cdot)$ is linear homogeneous.

Using homogeneity, extend $I (a, \cdot)$ to $F$. To prove (ii), take $f, g \in F (K)$ such that $f \geq g$. Take $b, b' \in B$ such that $t \bar{U} \otimes b = f$ and $t \bar{U} \otimes b' = g$. Then $\bar{U} \otimes \bar{b} (\theta) \bar{b} (\theta) \geq \bar{U} \otimes \bar{b} (\theta) b' (\theta) \bar{b} (\theta)$ for all $\theta \in \Theta$. By Lemma 4 and the definition of $\geq \theta$, $b (\theta) \geq b' (\theta)$ for all $\theta \in \Theta$. Axiom (A.4) implies that $(a, b) \geq (a, b')$ for all $a \in A$. By Lemma 5, the definition of $I (a, \cdot)$, and its homogeneity this implies $I (a, f) / t = J (a, b) \geq J (a, b') = I (a, g) / t$, and $I (a, \cdot)$ satisfies (ii).

Next consider condition (iv). Let there be given $f \in F$ and $r \in \mathbb{R}$. Invoking homogeneity, without loss of generality, assume that $2f, 2r_c \in F (K)$. Define $\gamma = I (a, 2f) = 2I (a, f)$. Let $b \in B$ satisfy $\bar{U} \otimes b = 2f$ and let $p, q \in \Delta (I (\theta))$ satisfy $\bar{U} (\bar{b} \theta p) = \gamma$ and $\bar{U} (\bar{b} \theta q) = 2r$. 31
Since, by Lemma 5 and (i), \((a, b) \sim (\overline{a}, \overline{b}q)\), axiom (A.5) implies that
\[
\left(\frac{1}{2}a + \frac{1}{2}bq\right) \sim \left(\frac{1}{2}a + \frac{1}{2}bq\right).
\] (19)

Hence, by Lemma 5, \(J\left(\frac{1}{2}a + \frac{1}{2}bq\right) = J\left(\frac{1}{2}a + \frac{1}{2}bq\right)\). By (i) this implies,
\[
I(a, f + r_c) = I\left(\frac{1}{2}a + \frac{1}{2}bq\right) = I\left(\frac{1}{2}a + \frac{1}{2}bq\right) = I(a, f + r_c) = \frac{1}{2}\gamma + r = I(a, f) + r
\] (20)

where used has been made of
\[
I\left(\frac{1}{2}a + \frac{1}{2}bq\right) = \frac{1}{2}I(a, f) + \frac{1}{2}I(a, g)
\] (21)

Thus \(I(a, c)\) satisfies (iv).

To show that \(I(a, c)\) is superadditive let \(f, g \in F\) be given. By homogeneity it suffices to prove that \(I(a, \frac{1}{2}f + \frac{1}{2}g) \geq \frac{1}{2}I(a, f) + \frac{1}{2}I(a, g)\). Let \(b, b' \in B\) satisfy \(\overline{b} \otimes b = f\) and \(\overline{b} \otimes b' = g\). If \(I(a, f) = I(a, g)\) then \((a, b) \sim (a, b')\) and, by axiom (A.3), \((a, \frac{1}{2}b + \frac{1}{2}b') \succ (a, b)\) which, in turn, implies \(I(a, \frac{1}{2}f + \frac{1}{2}g) \geq I(a, f) = \frac{1}{2}I(a, f) + \frac{1}{2}I(a, g)\).

Assume that \(I(a, f) > I(a, g)\) and let \(\beta = I(a, f) - I(a, g)\). Set \(h = g + \beta e\) then, by (iv), \(I(a, h) = I(a, g) + \beta = I(a, f)\). Hence, by (iv) and the superadditivity for the case above above,
\[
I\left(a, \frac{1}{2}f + \frac{1}{2}g\right) + \frac{1}{2}\beta = I\left(a, \frac{1}{2}f + \frac{1}{2}h\right) \geq \frac{1}{2}I(a, f) + \frac{1}{2}I(a, h) = \frac{1}{2}I(a, f) + \frac{1}{2}I(a, g) + \frac{1}{2}\beta.
\] (22)

Thus \(I(a, c)\) is superadditive. \(\square\)
Lemma 7 Let $I$ be the functional in Lemma 6. Then, there exist a family $\{C(a)\}_{a \in A}$ of closed and convex sets of probability measures on $\Theta$ such that, for all $(a, f) \in A \times F$,

$$I(a, f) = \min \{ \sum_{\theta \in \Theta} f(\theta) \pi(\theta) \mid \pi \in C(a) \}.$$  

Proof of Lemma 7. Let $(a, g) \in A \times F$ satisfy $I(a, g) > 0$. The next step is to construct a probability measure, $h_{(a, g)}$ on $\Theta$, such that $I(a, f) \leq \sum_{\theta \in \Theta} f(\theta) h_{(a, g)}(\theta)$, for all $f \in F$, and $I(a, g) = \sum_{\theta \in \Theta} g(\theta) h_{(a, g)}(\theta)$. Define

$$D_1(a) = \{ f \in F \mid I(a, f) > 1 \}$$

and

$$D_2(a) = \text{conv} \left( \{ f \in F \mid f \leq 1_c \} \cup \{ f \in F \mid f \leq g/I(a, g) \} \right).$$

The sets $D_1(a)$ and $D_2(a)$ are disjoint. To see that, let $d_2 \in D_2(a)$ satisfy $d_2 = \alpha f_1 + (1 - \alpha) f_2$, where $f_1 \leq 1_c$, $f_2 \leq g/I(a, g)$ and $\alpha \in [0, 1]$. Then, by monotonicity, homogeneity and constant-valuation independence,

$$I(a, d_2) \leq I(a, \alpha 1_c + (1 - \alpha) f_2) = \alpha + (1 - \alpha) I(a, f_2) \leq 1. \tag{23}$$

Thus $d_2 \notin D_1(a)$ whence $D_1(a) \cap D_2(a) = \emptyset$.

Since both $D_1(a)$ and $D_2(a)$ are convex sets with nonempty interiors, there exist a hyperplane $(h_{(a, g)}, \beta)$ separating $D_1(a)$ and $D_2(a)$. That is, for all $d_1 \in D_1(a)$ and $d_2 \in D_2(a)$,

$$h_{(a, g)} \cdot d_1 \geq \beta \geq h_{(a, g)} \cdot d_2. \tag{24}$$
Since the unit ball of \( F \) is contained in \( D_2(a) \), \( \beta > 0 \). (Otherwise \( h_{(a,g)} = 0 \).) Henceforth assume, without loss of generality, that \( \beta = 1 \).

Since \( 1_c \in D_2(a) \) inequalities (24) imply that \( h_{(a,g)} \cdot 1_c \leq 1 \). Because \( I(a,1_c) = 1 \), \( 1_c \) is a limit point of \( D_1(a) \), thus \( h_{(a,g)} \cdot 1_c \geq 1 \). Hence \( h_{(a,g)} \cdot 1_c = 1 \). Let \( E \subset \Theta \) and \( 1_E \) be the indicator function, then \( h_{(a,g)} \cdot 1_E + h_{(a,g)} \cdot (1_c - 1_E) = h_{(a,g)} \cdot 1_c = 1 \). But \( (1_c - 1_E) \in D_2(a) \) hence \( h_{(a,g)} \cdot 1_E \geq 0 \) for all \( E \subset \Theta \). Thus \( h_{(a,g)} \) is a probability measure on \( \Theta \).

I show next that \( h_{(a,g)} \cdot f \geq I(a,f) \) for all \( f \in F \) and \( h_{(a,g)} \cdot g = I(a,g) \). If \( I(a,f) > 0 \) then \( f/I(a,f) + (1/n)c \in D_1(a) \) for all \( n = 1,2,\ldots \). Hence inequalities (24) imply that \( h_{(a,g)} \cdot f \geq I(a,f) \). If \( I(a,f) \leq 0 \), there is \( r \in \mathbb{R} \) such that \( I(a,f + rc) = I(a,f) + r > 0 \), since \( I \) satisfies constant independence. Apply the argument above to \( I(a,f + rc) \) to obtain:

\[
h_{(a,g)} \cdot (f + rc) = h_{(a,g)} \cdot f + r \geq I(a,f + rc) = I(a,f) + r.
\]

(25)

Hence \( h_{(a,g)} \cdot f \geq I(a,f) \).

Since \( g/I(a,g) \in D_2(a) \) inequalities (24) imply the converse inequality for \( g \). Hence \( h_{(a,g)} \cdot g = I(a,g) \).

Let \( C(a) \) be the closure of the convex hull of \( \{h_{(a,g)} \mid I(a,g) > 0, g \in F\} \). Then

\[
I(a,f) \leq \min\{\sum_{\theta \in \Theta} f(\theta) \pi(\theta) \mid \pi \in C(a)\}.
\]

For \( f \) such that \( I(a,f) > 0 \), it was shown that the converse inequality holds as well. For \( f \) such that \( I(a,f) \leq 0 \) the converse inequality follows from constant independence. \( \Box \)
For every $\theta \in \Theta$ define a real-valued function, $u(\cdot, \theta)$ on $\Delta(I(\theta))$, by $u(b(\theta), \theta) = \bar{U}(b_{-\theta}b(\theta))$. Then Lemmata 4–7 imply that, for all $(a, b), (a', b') \in C$,

$$(a, b) \succeq (a', b') \iff \min_{\pi \in C(a)} \sum_{\theta \in \Theta} u(b(\theta), \theta) \pi(\theta) \geq \min_{\pi \in C(a')} \sum_{\theta \in \Theta} u(b'(\theta), \theta) \pi(\theta) \tag{26}$$

Furthermore,

$$u(\alpha p + (1 - \alpha) q, \theta) = \bar{U}(b_{-\theta}(\alpha p + (1 - \alpha) q)) = \alpha \bar{U}(b_{-\theta}p) + (1 - \alpha) \bar{U}(b_{-\theta}q) \tag{27}$$

where use is made of the fact that $B^{cv}$ is convex, $\bar{U}$ is affine, and

$$\bar{U}(b_{-\theta}(\alpha p + (1 - \alpha) q)) = \bar{U}(\alpha(b_{-\theta}p) + (1 - \alpha)(b_{-\theta}q)).$$

Hence

$$u(\alpha p + (1 - \alpha) q, \theta) = \alpha u(p, \theta) + (1 - \alpha) u(q, \theta) \tag{28}$$

Thus $u(\cdot, \theta)$ is affine. This completes the proof that $(i) \Rightarrow (ii)$.

$(ii) \Rightarrow (i)$. Let $I$ be a real-valued function on $A \times F$ that, for every given $a \in A$, is defined by $I(a, f) = \min_{\pi \in C(a)} \sum_{\theta \in \Theta} f(\theta) \pi(\theta)$, where $C(a)$ is compact and convex. Then $I(a, \cdot)$ is continuous and satisfy $(i) - (iv)$ of Lemma 6. Let $\{u(\cdot; \theta) : \Delta(I(\theta)) \to \mathbb{R}\}_{\theta \in \Theta}$ be a family of affine functions. Define a function $\bar{U} : B^{cv} \to \mathbb{R}$ by $\bar{U} \otimes b_{-\theta}p := u(p, \theta), \theta \in \Theta$.

Define a preference relation $\preceq$ on $C$ by $(a, b) \preceq (a', b')$ if and only if $I\left(a, \left(\bar{U} \otimes b_{-\theta}b(\theta)\right)_{\theta \in \Theta}\right) \geq I\left(a', \left(\bar{U} \otimes b_{-\theta}b'(\theta)\right)_{\theta \in \Theta}\right)$. Then $\preceq$ satisfies axioms (A.1), (A.2), (A.3), (A.4) and (A.5). To show that (A.6) holds, let $\bar{b}, \bar{b}' \in B^{cv}$ we need to show that $\alpha \bar{b} + (1 - \alpha) \bar{b}' \in B^{cv}$. Let $\bar{U} \otimes \bar{b} = r_c$ and $\bar{U} \otimes \bar{b}' = r'_c$ then $I(a, ar_c + (1 - \alpha) r'_c) = ar + (1 - \alpha) r' = \alpha \bar{U} \otimes \bar{b} + (1 - \alpha) \bar{U} \otimes \bar{b}'$ for all
\( a \in A \). Since \( u ( \cdot , \theta ) , \theta \in \Theta \), are affine functions, \( \bar{U} \) is affine. Hence \( \alpha \bar{U} \otimes \bar{b} + (1 - \alpha) \bar{U} \otimes \bar{b}' = \bar{U} \otimes (\alpha \bar{b} + (1 - \alpha) \bar{b}') \). Thus

\[
I (a, \bar{U} \otimes (\alpha \bar{b} + (1 - \alpha) \bar{b}')) = \alpha r + (1 - \alpha) r' = I (a', \bar{U} \otimes (\alpha \bar{b} + (1 - \alpha) \bar{b}')) ,
\]

for all \( a, a' \in A \). Thus, by definition, \((a, \alpha \bar{b} + (1 - \alpha) \bar{b}') \sim (a', \alpha \bar{b} + (1 - \alpha) \bar{b}')\) for all \( a, a' \in A \). Hence \( \alpha \bar{b} + (1 - \alpha) \bar{b}' \in B^{cv} \). This completes the proof of the existence of the representation.

The uniqueness of \( \{ u ( \cdot , \theta ) \}_{\theta \in \Theta} \) follows from the uniqueness of \( \bar{U} \) in Lemma 4.

By (A.0), \( \succeq \) is nondegenerate. Assume that, for some \( a \in A \), there are \( C_1 (a) \neq C_2 (a) \) both non-empty, closed, convex sets, such that the two functions on \( B \):

\[
H_1 (b; a) = \min_{\pi \in C_1 (a)} \sum_{\theta \in \Theta} u (b (\theta) , \theta ) \pi (\theta) \quad \text{and} \quad H_2 (b; a) = \min_{\pi \in C_2 (a)} \sum_{\theta \in \Theta} u (b (\theta) , \theta ) \pi (\theta)
\]

both represent \( \succeq \). Without loss of generality assume that there exist \( \hat{\pi} \in C_1 (a) \setminus C_2 (a) \). Then, by a separating hyperplane theorem, there exist \( h \in \mathbb{R}^{|\Theta|} \) such that

\[
\sum_{\theta \in \Theta} h (\theta) \hat{\pi} (\theta) < \min_{\pi \in C_2 (a)} \sum_{\theta \in \Theta} h (\theta) \pi (\theta) .
\]

Without loss of generality, assume that \( h \in F (K) \). Then there exist \( b \in B \) such that \( H_1 (b; a) < H_2 (b; a) \). Let \( \bar{b} \in B^{cv} \) satisfy \( \bar{b} \sim (a, b) \) then \( H_1 (\bar{b}; a) = H_1 (b; a) < H_2 (b; a) = H_2 (\bar{b}; a) \), a contradiction.

\[ \square \]

7.2 Proof of Theorem 2

The proof of Theorem 2 is similar to that of Theorem 1 and is only outlined here.
Lemma 8 There exists jointly cardinal functions $V: \tilde{B} \to \mathbb{R}$ and $\varsigma: A \to \mathbb{R}$ such that, for all $\left(a, \tilde{b}\right), \left(a', \tilde{b}'\right) \in A \times \tilde{B}$, $\left(a, \tilde{b}\right) \succeq \left(a', \tilde{b}'\right)$ if and only if $V\left(\tilde{b}\right) + \varsigma\left(a\right) \geq V\left(\tilde{b}'\right) + \varsigma\left(a'\right)$. Moreover, the function $V$ is affine.

Proof of Lemma 8: The restriction of $\succeq$ to $A \times \tilde{B}$ is a continuous weak order satisfying axioms (A.7) – (A.9) if and only if there exist jointly cardinal, continuous, real-valued functions $V$ on $\tilde{B}$ and $\varsigma$ on $A$ such that $\succeq$ on $A \times \tilde{B}$ is represented by $\left(a, \tilde{b}\right) \mapsto V\left(\tilde{b}\right) + \varsigma\left(a\right)$ (Wakker (1989) Theorem III.4.1).

Because $\tilde{B}$ is a convex set and $\succeq$ is a continuous weak order, axiom (A.5') implies that, on $\tilde{B}$, the von Neumann-Morgenstern expected utility theorem applies. Hence $V$ is affine. \(\square\)

Lemma 9 Given the functions $(V, \varsigma)$ from Lemma 8, there exist a unique function $J: A \times B \to \mathbb{R}$ such that:

(i) $(a, b) \succeq (a', b')$ if and only if $J\left(a, b\right) \geq J\left(a', b'\right)$ for all $(a, b), (a', b') \in C$.

(ii) For every $(a, \tilde{b}) \in A \times \tilde{B}$, $J\left(a, \tilde{b}\right) = V\left(\tilde{b}\right) + \varsigma\left(a\right)$.

Proof of Lemma 9. On $A \times \tilde{B}$ let $J$ be uniquely determined by $(V, \varsigma)$ according to (ii).

Let $\tilde{b}^{**}, \tilde{b}^* \in \tilde{B}$ satisfy $V\left(\tilde{b}^{**}\right) \geq V\left(\tilde{b}\right) \geq V\left(\tilde{b}^*\right)$ for all $\tilde{b} \in \tilde{B}$. Hence $(a, \tilde{b}^{**}) \succ (a, b) \succ (a, \tilde{b}^*)$, for all $(a, b) \in C$. Invoking the uniqueness of $V$, without loss of generality, normalize
Choose a utility function $V$ such that $V\left(\tilde{b}^{**}\right) < -1$ and $V(\tilde{b}^{**}) > 1$. Let $L = V(\tilde{B})$ and denote by $F(L)$ the set of real-valued functions on $\Theta$ with values in $L$. Given $b \in B$, let $V \otimes b \in F(L)^{\Theta}$ be defined by: $V \otimes b := \left( V\left( b_{-\theta}p\left(\theta\right)\right) \right)_{\theta \in \Theta}$, where $b_{-\theta}p$ is the element of $\tilde{B}$ whose $\theta$–coordinate is $p$. Note that, because $V$ is affine, $\tilde{B}$ is convex, and the uniqueness of the elements of $\tilde{B}$, $V \otimes (\alpha b + (1 - \alpha) b') = \alpha V \otimes b + (1 - \alpha) V \otimes b'$ for all $b, b' \in B$ and $\alpha \in [0,1]$. (To see this, note that, by the convexity and uniqueness of $\tilde{B}$, for each $\theta' \in \Theta$, $b_{-\theta}(\alpha p + (1 - \alpha) q)\left(\theta'\right)$ and $\alpha b_{-\theta}p\left(\theta'\right) + (1 - \alpha) b_{-\theta}q\left(\theta'\right)$ are elements of $\mathcal{I}\left(\left(\alpha p + (1 - \alpha) q\right); \theta, \tilde{b}, a\right)$. Hence, by Lemma 8,

$$V\left( b_{-\theta}(\alpha p + (1 - \alpha) q)\right) = V\left( \alpha b_{-\theta}p\left(\theta'\right) + (1 - \alpha) b_{-\theta}q\left(\theta'\right)\right) = \alpha V\left( b_{-\theta}p\left(\theta'\right)\right) + (1 - \alpha) V\left( b_{-\theta}q\left(\theta'\right)\right).$$

The affinity of $V \otimes$ follows from its definition.)
The remainder of the proof uses the same arguments as in the proof of Theorem 1, with the obvious changes, that is, replace $B^\text{sw}$ with $\tilde{B}$, $\tilde{b}$ with $\tilde{b}$. By the same arguments as in the proof of Lemma 6, there is a functional, $I : A \times F \rightarrow \mathbb{R}$, defined by $I (a, V \otimes b) = J (a, b) - \varsigma (a)$. Then, for all $(a, b) \in C$, $I (a, \cdot)$ is monotonic, super additive, homogeneous of degree 1, and satisfies constant-independence.$^{12}$

Let $I$ be the functional in Lemma 10. Then, by the same argument as in the proof of Lemma 7, there exists a family $\{C (a)\}_{a \in A}$ of closed and convex sets of probability measures on $\Theta$ such that, for all $(a, f) \in A \times F$,

$$I (a, f) = \min \{ \sum_{\theta \in \Theta} f (\theta) \pi (\theta) \mid \pi \in C (a) \}.$$ 

For every $\theta \in \Theta$ define a real-valued function, $v (\cdot, \theta)$ on $\Delta (I (\theta))$, by $v (b (\theta), \theta) = V \left( \tilde{b} (\theta) \right)$. Then, Lemmas 8, 9 and the definition of $I$ imply that, for all $(a, b), (a', b') \in C$,

$$(a, b) \succ (a', b') \iff \min_{\pi \in C (a)} \sum_{\theta \in \Theta} v (b (\theta), \theta) \pi (\theta) + \varsigma (a) \geq \min_{\pi \in C (a')} \sum_{\theta \in \Theta} v (b' (\theta), \theta) \pi (\theta) + \varsigma (a') \quad (31)$$

Furthermore, because $\tilde{B}$ is convex, $V$ is affine and

$$V (b_{-\theta} (\alpha p + (1 - \alpha) q)) = V \left( \alpha (b_{-\theta} p) + (1 - \alpha) (b_{-\theta} q) \right).$$

Because $V$ is affine,

$$v (\alpha p + (1 - \alpha) q, \theta) = \alpha V (b_{-\theta} p) + (1 - \alpha) V (b_{-\theta} q) = \alpha v (p, \theta) + (1 - \alpha) v (q, \theta) \quad (32)$$

Thus $v (\cdot, \theta)$ is affine. This completes the proof that $(i) \Rightarrow (ii)$.

$^{12}$See Lemma 10 in the Appendix for details.
That (ii) implies (A.7) - (A.9) is an implication of Lemma 8. That \( \succeq \) is a continuous weak order satisfying (A.3), (A.4'), (A.5') follows using the same arguments as in the proof of Theorem 1.

That the functions \( V := \sum_{\theta \in \Theta} v \left( \tilde{b}(\theta), \theta \right) \) on \( \tilde{B} \) and \( \varsigma \) are jointly cardinal is an implication of Lemma 8. Moreover, for all \( \theta \in \Theta \), \( v (\cdot, \theta) \) is affine. Hence if \( \{ \hat{v} (\cdot, \theta) \}_{\theta \in \Theta} \) and \( \hat{\varsigma} \) is another set of functions, representing \( \succeq \) in the sense of (ii), then there are \( \beta > 0, \gamma, \) and \( \delta \), such that \( \hat{v} (\cdot, \theta) = \beta v (\cdot, \theta) + \gamma \) and \( \hat{\varsigma} = \beta \varsigma + \delta \).\(^{13}\) Thus \( \{ v (\cdot, \theta) \}_{\theta \in \Theta} \) are cardinally measurable and fully comparable.

The proof that, for each \( a \in A \), the set \( C(a) \) is unique, and each \( \pi \in C(a) \) has full support, is by the same argument as in the proof of Theorem 1. \( \Box \)

7.3 Proof of Theorem 3

(i) \( \iff \) (ii). By Theorem 2, for all \( (a, b), (a', b') \in C \),

\[
(a, b) \succeq (a', b') \iff \min_{\pi \in C(a)} \sum_{\theta \in \Theta} v (b(\theta), \theta) \pi (\theta) + \varsigma (a) \geq \min_{\pi \in C(a')} \sum_{\theta \in \Theta} v (b(\theta), \theta) \pi (\theta) + \varsigma (a').
\]

\(^{13}\)Note that, \( \{ v (\cdot, \theta) \}_{\theta \in \Theta} \) are jointly cardinal. That is, if \( \{ \hat{v} (\cdot, \theta) \}_{\theta \in \Theta} \) is another set of effect-dependent utility function that, jointly with \( \varsigma \) represent the preference relation, then \( \hat{v} (\cdot, \theta) = \beta v (\cdot, \theta) + \gamma (\theta) \). However, for this representation to hold for every \( a \), it is necessary that \( \gamma (\theta) = \gamma \) for all \( \theta \in \Theta \).
For all $a \in \mathcal{A}$, $\theta, \theta' \in \Theta$, and $p, q \in \Delta$, $v(p, \theta) \geq v(q, \theta)$ if and only if,

$$v(p, \theta) \pi(\theta) + \sum_{\tilde{b} \in \Theta - \{\theta\}} v\left(\tilde{b}^{**}(\tilde{\theta}), \tilde{\theta}^{**}\right) \pi(\tilde{\theta}) \geq v(q, \theta) \pi(\theta) + \sum_{\tilde{b} \in \Theta - \{\theta\}} v\left(\tilde{b}^{**}(\tilde{\theta}), \tilde{\theta}^{**}\right) \pi(\tilde{\theta}) \quad (34)$$

where $\pi \in \arg \min_{C(\theta)} \sum_{\theta \in \Theta} v\left(\left(\tilde{b}^{**}_{\theta} q\right)(\theta), \tilde{\theta}(\theta)\right) \pi(\theta)$. (Note that, because $\left(a, \tilde{b}^{**}\right) \succ (a, \tilde{b}^{**}_{\theta} p) \succ (a, \tilde{b}^{**}_{\theta} q)$, arg min$_{C(\theta)} \sum_{\theta \in \Theta} v\left(\left(\tilde{b}^{**}_{\theta} p\right)(\theta), \tilde{\theta}(\theta)\right) \pi(\theta) = \arg \min_{C(\theta)} \sum_{\theta \in \Theta} v\left(\left(\tilde{b}^{**}_{\theta} q\right)(\theta), \tilde{\theta}(\theta)\right) \pi(\theta)$).

Thus, by (33), this is equivalent to $\left(a, \tilde{b}^{**}_{\theta} p\right) \succ (a, \tilde{b}^{**}_{\theta} q)$. By axiom (A.10) this is equivalent to $\left(a, \tilde{b}^{**}_{\theta} p\right) \succ (a, \tilde{b}^{**}_{\theta} q)$. Hence, by (33), this is equivalent to,

$$v(p, \theta') \pi(\theta') + \sum_{\tilde{b} \in \Theta - \{\theta'\}} v\left(\tilde{b}^{**}(\tilde{\theta}), \tilde{\theta}^{**}\right) \pi(\tilde{\theta}) \geq v(q, \theta') \pi(\theta') + \sum_{\tilde{b} \in \Theta - \{\theta'\}} v\left(\tilde{b}^{**}(\tilde{\theta}), \tilde{\theta}^{**}\right) \pi(\tilde{\theta}) \quad (35)$$

But (35) holds if and only if $v(p, \theta') \geq v(q, \theta')$. Thus, by the uniqueness of the von Neumann-Morgenstern utility function, for all $\theta \in \Theta$, $v(\cdot, \theta) = \beta(\theta) v(\cdot) + \gamma(\theta)$, $\beta(\theta) > 0$.

However, for all $\tilde{b} \in \tilde{B}$, min$_{\pi \in C(\theta)} \sum_{\theta \in \Theta} \left[\beta(\theta) v\left(\tilde{b}(\theta)\right) + \gamma(\theta)\right] \pi(\theta)$ is independent of $a$.

By the uniqueness of $\tilde{b}$,

$$\beta(\theta) v\left(\tilde{b}(\theta)\right) + \gamma(\theta) = \beta(\theta') v\left(\tilde{b}(\theta')\right) + \gamma(\theta'), \text{ for all } \theta, \theta' \in \Theta \text{ and } \tilde{b} \in \tilde{B} \quad (36)$$

Hence

$$[\beta(\theta) - \beta(\theta')] \left[v\left(\tilde{b}(\theta)\right) - v\left(\tilde{b}'(\theta')\right)\right] = 0 \text{ for all } \theta, \theta' \in \Theta \text{ and } \tilde{b}, \tilde{b}' \in \tilde{B} \quad (37)$$

Thus $\beta(\theta) = \beta$ for all $\theta \in \Theta$. But $v\left(\tilde{b}(\theta)\right) = v\left(\tilde{b}'(\theta)\right)$, thus equations (36) imply that $\gamma(\theta) = \gamma$ for all $\theta \in \Theta$. By the normalization of $V$, and the definition of $v$, $v\left(\tilde{b}^{0}(\theta)\right) = 0$ and $v\left(\tilde{b}^{1}(\theta)\right) = 1$. Hence $\gamma = 0$ and $\beta = 1$.

The uniqueness follows from Theorem 2. \[\square\]
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APPENDIX

Lemma 10 There exists a functional \( I : \mathcal{A} \times F \to \mathbb{R} \) such that:

(i) For all \((a, b) \in \mathbb{C}\), \( I(a, V \otimes b) = J(a, b) - \varsigma(a) \).

(ii) \( I(a, \cdot) \) is monotonic (that is, for all \( f, g \in F, f \geq g \Leftrightarrow I(a, f) \geq I(a, g) \)).

(iii) \( I(a, \cdot) \) is superadditive and homogeneous of degree 1.

(iv) \( I(a, \cdot) \) is constant independent (that is, \( I(a, f + r_c) = I(a, f) + I(a, r_c) \) for all \( f \in F \) and \( r \in \mathbb{R} \)).

Proof of Lemma 10. Let \( I \) be defined by condition (i) then, by Lemma 9, \( I \) is well defined. Let \( f, g \in F(L) \) satisfy \( f = \alpha g \), for some \( \alpha \in (0, 1] \). The proof that \( I(a, \cdot) \) is linear homogeneous requires showing that \( I(a, f) = \alpha I(a, g) \). (This will imply the equality for \( \alpha > 1 \)). Let \( b \in B \) be such that \( V \otimes b = g \) and let \( \tilde{b}^* \in \tilde{B} \) satisfy \( J\left(a, \tilde{b}^*\right) - \varsigma(a) = V\left(\tilde{b}^*\right) = 0 \) (the first equality is implied by Lemma 9 and the second by the normalization of \( V \)). Define \( b' = \alpha b + (1 - \alpha) \tilde{b}^* \). By Lemma 8 \( V \) is affine, hence \( V \otimes b' = \alpha V \otimes b + (1 - \alpha) V \otimes \tilde{b}^* = \alpha g = f \), where use has been made of \( \left(V \otimes \tilde{b}^* (\theta)\right)_{\theta \in \Theta} = \left(V\left(\tilde{b}^*\right), ..., V\left(\tilde{b}^*\right)\right) = 0_c \). By (i) \( I(a, f) = J(a, b') - \varsigma(a) \). Let \( \tilde{b} \in \tilde{B} \) satisfy \( (a, \tilde{b}) \sim (a, b) \) (hence \( J\left(a, \tilde{b}\right) - \varsigma(a) = J(a, b) - \varsigma(a) = I(a, g) \)). But, by axiom (A.5'), \( \alpha \tilde{b} + (1 - \alpha) \tilde{b}^0 \sim \alpha b + (1 - \alpha) \tilde{b}^* \). Hence, invoking the convexity of \( B^{cv} \) and the affinity of \( V, J\left(a, \alpha \tilde{b} + (1 - \alpha) \tilde{b}^0\right) - \varsigma(a) = \alpha J\left(a, \tilde{b}\right) + \)

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\[(1 - \alpha) J \left( a, \tilde{b} \right) - \varsigma(a) = \alpha \left[ J \left( a, \tilde{b} \right) - \varsigma(a) \right].\] Thus
\[J \left( a, b' \right) - \varsigma(a) = J \left( a, \alpha \tilde{b} + (1 - \alpha) \tilde{b}' \right) - \varsigma(a) = \alpha \left[ J \left( a, \tilde{b} \right) - \varsigma(a) \right]. \quad (38)\]

and
\[I \left( a, f \right) = J \left( a, b' \right) - \varsigma(a) = \alpha \left[ J \left( a, \tilde{b} \right) - \varsigma(a) \right] = \alpha \left[ J \left( a, b \right) - \varsigma(a) \right] = \alpha I \left( a, g \right) \quad (39)\]

Hence \(I \left( a, \cdot \right)\) is linear homogeneous.

Using homogeneity extend \(I \left( a, \cdot \right)\) to \(F\). To prove (ii) let \(f, g \in F \left( L \right)\), \(f \geq g\). Let \(b, b' \in B\) be such that \(tV \otimes b = f\) and \(tV \otimes b' = g\). Then \(V \otimes b_\theta b(\theta) \geq V \otimes b_\theta b'(\theta)\) for all \(\theta \in \Theta\). Thus, by Lemma 8, \(\left( a, b_\theta b(\theta) \right) \succ (a, b)\) for all \(\theta \in \Theta\) and \(a \in A\). Hence Axiom (A.4') implies that \((a, b) \succ (a, b')\) for all \(a \in A\). By Lemma 9, the definition of \(I \left( a, \cdot \right)\), and its homogeneity this implies \(I \left( a, f \right)/t = J \left( a, b \right) \geq J \left( a, b' \right) = I \left( a, g \right)/t\), and \(I \left( a, \cdot \right)\) satisfies (ii).

Next consider condition (iv). Let there be given \(f \in F\) and \(r \in \mathbb{R}\). Invoking homogeneity, without loss of generality, assume that \(2f, 2r_c \in F \left( L \right)\). Define \(\gamma = I \left( a, 2f \right) = 2I \left( a, f \right)\). Let \(b \in B\) satisfy \(V \otimes b = 2f\) and let \(p, q \in \Delta \left( I(\theta) \right)\) satisfy \(V \left( b_\theta p \right) = \gamma\) and \(V \left( b_\theta q \right) = 2r\). Since, by Lemma 9 and (i), \((a, b) \sim (a, b_\theta p)\), axiom (A.5') implies that
\[
\left( a, \frac{1}{2} b + \frac{1}{2} b_\theta q \right) \sim \left( a, \frac{1}{2} b_\theta p + \frac{1}{2} b_\theta q \right). \quad (40)
\]

Hence, by (i),
\[
I \left( a, f + r_c \right) = I \left( a, \frac{1}{2} V \otimes b + \frac{1}{2} V \otimes b_\theta q \right) = \quad (41)
\]
\[ I \left( a, \frac{1}{2} V \otimes b_{-\theta p} + \frac{1}{2} V \otimes b_{-\theta q} \right) = I(a, \frac{1}{2} \gamma_c + r_c) = \frac{1}{2} \gamma + r = I(a, f) + r, \]

where used has been made of

\[ I \left( a, \frac{1}{2} V \otimes g_{b_{-\theta p}} - \theta p + 1 \right) \]
\[ = J \left( a, \frac{1}{2} V \otimes g_{b_{-\theta p}} - \theta q \right) = 1 \]
\[ = V \left( \frac{1}{2} b_{-\theta p} + \frac{1}{2} b_{-\theta q} \right) = \frac{1}{2} V \left( b_{-\theta p} \right) + \frac{1}{2} V \left( b_{-\theta q} \right) = \frac{1}{2} \gamma + r. \]

Thus \( I(a, \cdot) \) satisfies (iv).

To show that \( I(a, \cdot) \) is superadditive let \( f, g \in F \) be given. By homogeneity it suffices to prove that \( I(a, \frac{1}{2} f + \frac{1}{2} g) \geq \frac{1}{2} I(a, f) + \frac{1}{2} I(a, g) \). Let \( b, b' \in B \) satisfy \( V \otimes b = f \) and \( V \otimes b' = g \). If \( I(a, f) = I(a, g) \) then \( (a, b) \sim (a, b') \) and, by uncertainty aversion (axiom (A.3)), \( (a, \frac{1}{2} b + \frac{1}{2} b') \succ (a, b) \) which, by Lemma 9, implies \( I(a, \frac{1}{2} f + \frac{1}{2} g) \geq I(a, f) = \frac{1}{2} I(a, f) + \frac{1}{2} I(a, g) \).

Assume that \( I(a, f) > I(a, g) \) and let \( \beta = I(a, f) - I(a, g) \). Set \( \ell = g + \beta \) then, by (iv), \( I(a, \ell) = I(a, g) + \beta = I(a, f) \). Hence, by (iv) and the superadditivity for the case above,

\[ I \left( a, \frac{1}{2} f + \frac{1}{2} g \right) + \frac{1}{2} \beta = I \left( a, \frac{1}{2} f + \frac{1}{2} h \right) \geq \frac{1}{2} I(a, f) + \frac{1}{2} I(a, h) = \frac{1}{2} I(a, f) + \frac{1}{2} I(a, g) + \frac{1}{2} \beta. \]

Thus \( I(a, \cdot) \) is superadditive.

\( (ii) \Rightarrow (i) \). Let \( I \) be a real-valued function on \( \mathcal{A} \times F \) that, for every given \( a \in \mathcal{A} \), is defined by \( I(a, f) = \min_{\pi \in C(a)} \sum_{\theta \in \Theta} f(\theta) \pi(\theta) \), where \( C(a) \) is compact and convex. Then
I (a, ·) is continuous and satisfy (i) – (iv) of Lemma 6. Let \{u (·; θ) : Δ (I (θ)) → \mathbb{R}\}_{θ ∈ Θ} be
a family of affine functions. Define a function \( \bar{U} : B^{cw} → \mathbb{R} \) by \( \bar{U} ⊗ b_θ p := u (p, θ), \ θ ∈ Θ \).

Define a preference relation \( \succ \) on \( \mathbb{C} \) by \( (a, b) \succ (a', b') \) if and only if
\[ I \left( a, \left( \bar{U} ⊗ b_θ (θ) \right)_{θ ∈ Θ} \right) \succ I \left( a', \left( \bar{U} ⊗ b_θ (θ') \right)_{θ ∈ Θ} \right). \]

Then \( \succ \) satisfies axioms (A.1), (A.2), (A.3), (A.4) and (A.5). To show that (A.6) holds, let \( \bar{b}, \bar{b}' ∈ B^{cw} \) we need to show that \( α\bar{b} + (1 - α) \bar{b}' ∈ B^{cw} \). Let \( \bar{U} ⊗ \bar{b} = r_c \) and \( \bar{U} ⊗ \bar{b}' = r'_c \), then
\[ I \left( a, α r_c + (1 - α) r'_c \right) = αr + (1 - α) r' = α \bar{U} ⊗ \bar{b} + (1 - α) \bar{U} ⊗ \bar{b}' \] for all \( a ∈ A \). Since \( u (·; θ), θ ∈ Θ \), are affine functions, \( \bar{U} \) is affine. Hence
\[ \alpha \bar{U} ⊗ \bar{b} + (1 - α) \bar{U} ⊗ \bar{b}' = \bar{U} ⊗ (α \bar{b} + (1 - α) \bar{b}'). \]
Thus
\[ I \left( a, \bar{U} ⊗ (α \bar{b} + (1 - α) \bar{b}') \right) = αr + (1 - α) r' = I \left( a', \bar{U} ⊗ (α \bar{b} + (1 - α) \bar{b}') \right), \]
for all \( a, a' ∈ A \). Thus, by definition, \( (a, α \bar{b} + (1 - α) \bar{b}') \sim (a', α \bar{b} + (1 - α) \bar{b}') \) for all \( a, a' ∈ A \). Hence \( α \bar{b} + (1 - α) \bar{b}' ∈ B^{cw} \). This completes the proof of the existence of the representation.

The uniqueness of \{u (·; θ)\}_{θ ∈ Θ} follows from the uniqueness of \( \bar{U} \) in Lemma 4.

By (A.0), \( \succ \) is nondegenerate. Assume that, for some \( a ∈ A \), there are \( C_1 (a) \neq C_2 (a) \)
both non-empty, closed, convex sets, such that the two functions on \( B : \)
\[ H_1 (b; a) = \min_{π ∈ C_1 (a)} \sum_{θ ∈ Θ} u (b (θ), θ) π (θ) \text{ and } H_2 (b; a) = \min_{π ∈ C_2 (a)} \sum_{θ ∈ Θ} u (b (θ), θ) π (θ) \]
both represent \( \succ \). Without loss of generality assume that there exist \( \hat{π} ∈ C_1 (a) \setminus C_2 (a) \).
Then, by a separating hyperplane theorem, there exist \( h ∈ \mathbb{R}^{\mid Θ \mid} \) such that
\[ \sum_{θ ∈ Θ} h (θ) \hat{π} (θ) < \min_{π ∈ C_2 (a)} \sum_{θ ∈ Θ} h (θ) π (θ). \]
Without loss of generality, assume that $h \in F(K)$. Then there exist $b \in B$ such that $H_1(b; a) < H_2(b; a)$. Let $\bar{b} \in B^{cv}$ satisfy $\bar{b} \sim (a, b)$ then $H_1(\bar{b}; a) = H_1(b; a) < H_2(b; a) = H_2(\bar{b}; a)$, a contradiction. \qed