**Information in (and not in) the term structure**

**Appendix. Additional results**

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## Properties of the estimated five-factor model

No stationary term structure model is credible unless it implies unconditional properties of yields and returns that are in the ballpark of observed sample properties. Similarly, model-implied principal components of yields must look like principal components in the data. This appendix discusses these basic term structure properties for the estimated five-factor model. The main conclusion is that the model does a good job reproducing the relevant properties of the yields used in estimating the model. A ten-year bond yield, which is not included in estimation, reveals evidence of model misspecification that is not easily addressed. This yield is produced by the Federal Reserve Board

The solid lines in Panels A and B of Figure A1 are model-implied unconditional means and standard deviations of bond yields. The diamonds are sample values. The two sets of means line up closely for those maturities used in estimation. For example, the largest difference between model-implied and sample means is 26 basis points (the three-month bond). The model does less well in fitting the mean ten-year yield. Its sample mean of 7.2 percent is considerably higher than the model-implied mean of 6.5 percent, although well within the 95 percent confidence bounds. These bounds, displayed as dashed lines, are wide owing to the high persistence of yields. The model is not quite as successful at fitting unconditional standard deviations. The inverse relation between volatility and maturity is stronger in the model than in the data, which is most noticeable at the ten-year maturity.

Panel C of the figure reports model-implied unconditional Sharpe ratios for annual log returns. The ratio is

$$\text{Sharpe}_m = \frac{E\left(xr^{(m)}_{t,t+12}\right) + \frac{1}{2}\left(\text{Var}_t(xr^{(m)}_{t,t+12}\right)}{\text{Var}_t\left(xr^{(m)}_{t,t+12}\right)}.$$

The subscript on the variance denotes a conditional variance. Since the model is Gaussian, this does not vary across $t$. The diamonds are corresponding sample values, although they are computed by replacing the conditional variances in the above equation with unconditional variances. The two sets of values roughly coincide. However, the model-implied Sharpe ratios decline more sharply with maturity than do sample Sharpe ratios. (There is no sample Sharpe ratio at ten years because a nine-year yield is not used.) Panel D of Figure A1 is the serial correlation function of the risk premium factor.

Figure A2 is the paper's Figure 1, modified to include the ten-year yield. It displays the loadings of observed bond yields on the factors, and its construction is discussed in the text of the paper. For the purposes of this appendix, the main feature of this figure is the divergence between the fitted and actual loadings for the ten-year bond yield. Panels C and D show clear differences between sample and analytic loadings of the ten-year yield on the fourth and fifth factors. In economic terms, the differences are small; around five basis points of annualized yields for a one-standard-deviation shock to a factor. But the differences also point to a limitation of this affine class of models.

Certain types of shocks to the term structure are ruled out owing to the assumed VAR dynamics of the factors. Consider, for example, the loadings in Panel D for the fifth factor. A reasonable description of the sample values (the diamonds) is that they are approximately zero for maturities below five years, and between three and five basis points for five-year and ten-year yields. That description cannot be reproduced by the analytic loadings. In this class of models that assume VAR dynamics, a shock that affects yields at long maturities must also affect yields at short maturities. The only difference in loadings across maturities is the exponent on $K^q$. Thus to fit the sample loadings for the fourth and fifth factors, the estimated $K^q$ must generate cycles as the exponent increases. At the maximum likelihood estimates, the cycles that fit maturities through five years do not fit the ten-year loading.

What happens if the ten-year yield is included in estimation? The results are summarized in Alternative Figure A1 and Alternative Figure A2. The analytic loadings closely reproduce all of the sample loadings (AF A1). But this modification comes at a substantial cost in fitting the physical dynamics of the term structure. Recall that with the parsimonious risk specification used here, the feedback matrices in the physical and equivalent-martingale measures share 20 parameters. The values needed to fit the sample loadings produce wildly unrealistic behavior of the short rate. In AF A1, the unconditional mean is close to minus one percent and the unconditional standard deviation is about 6.5 percent (both expressed in annual terms). Annual unconditional Sharpe ratios at the short end of the term structure are about one. These results account for my choice to exclude the ten-year bond yield when estimating the model.

## Parameter estimates for three-factor and four-factor models

Point estimates of the three-factor and four-factor models are contained in Tables A1 and A2, respectively. No standard errors are reported because I did not perform Monte Carlo simulations with these models.

## The macroeconomic link to the risk premium factor

Table 6 in the paper explains the hidden component of the risk premium factor with macroeconomic variables. Table A3 repeats the regressions, replacing the hidden component with the entire risk premium factor.

Table A1. Three-factor model

See the notes to Table 1 in the text.

| | Factor 1 | Factor 2 | Factor 3 |
|---|---|---|---|
| Loading of short rate on factors | 0.125 | 0.409 | 0.411 |
| $K^q$ | 0.981 | 0.117 | 0.198 |
| | 0.021 | 0.845 | −0.406 |
| | 0.007 | −0.021 | 0.885 |
| $\text{diag}(\Omega^{1/2}) \times 10^4$ | 27.842 | 7.497 | 2.568 |
| $\lambda_0 \times 10^4$ | −3.427 | −1.637 | 0.0 |
| $\lambda_{1(L)}$ | −0.024 | 0.346 | 0.460 |
| Constant term in short rate ($\times 10^2$) | | 1.016 | |
| Std dev of measurement error ($\times 10^4$) | | 0.638 | |

Table A2. Four-factor model

See the notes to Table 1 in the text.

| | Factor 1 | Factor 2 | Factor 3 | Factor 4 |
|---|---|---|---|---|
| Loading of short rate on factors | 0.124 | 0.407 | 0.506 | 0.262 |
| $K^q$ | 0.975 | 0.115 | 0.269 | 0.007 |
| | 0.022 | 0.849 | −0.528 | −0.341 |
| | 0.007 | −0.016 | 0.781 | −0.344 |
| | −0.002 | 0.005 | 0.020 | 0.991 |
| $\text{diag}(\Omega^{1/2}) \times 10^4$ | 27.816 | 7.538 | 3.092 | 0.783 |
| $\lambda_0 \times 10^4$ | −3.910 | −2.153 | 0.0 | 0.0 |
| $\lambda_{1(L)}$ | −0.030 | 0.343 | 0.610 | 0.239 |
| Constant term in short rate ($\times 10^2$) | | 1.069 | | |
| Std dev of measurement error ($\times 10^4$) | | 0.547 | | |

Table A3. Projections of the risk premium factor on macroeconomic variables, 1964 through 2007

A five-factor term structure model is estimated with the Kalman filter. Bond risk premia are constrained to vary with a single risk premium factor. Monthly smoothed estimates of the risk premium factor are regressed on contemporaneous realizations of other variables. Industrial production growth and CPI inflation are both month-$t$ predictions of month-$(t+1)$ values, from individual ARMA(1,1) models. Ludvigson-Ng construct eight principal components of many macro and financial time series. The first is a "real activity factor," which here is normalized to positively covary with industrial production growth. Each variable used in the table is normalized to have a unit standard deviation. The table reports point estimates and $t$-statistics. The latter are adjusted for 15 lags of moving average residuals. $P$-values of joint tests that coefficients on Ludvigson-Ng factors two through eight equal zero are in square brackets. The column labeled $\rho_{15}$ is the serial correlation of residuals at the 15th lag. The sample is January 1964 through December 2007.

| Include LN factors 2-8? | Ind. prod. growth | Inflation | LN real activity | LN (real activity)$^3$ | $\rho_{15}$ | $R^2$ |
|---|---|---|---|---|---|---|
| No | −0.14 | −0.22 | - | - | −0.03 | 0.05 |
| | (−1.23) | (−1.40) | | | | |
| No | - | - | −0.14 | - | 0.00 | 0.02 |
| | | | (−0.86) | | | |
| Yes | - | - | −0.14 | - | 0.02 | 0.11 |
| [0.156] | | | (−0.91) | | | |
| No | - | - | −0.29 | 0.24 | 0.01 | 0.05 |
| | | | (−1.83) | (4.06) | | |
| Yes | - | - | −0.30 | 0.24 | 0.05 | 0.14 |
| [0.147] | | | (−1.94) | (3.51) | | |

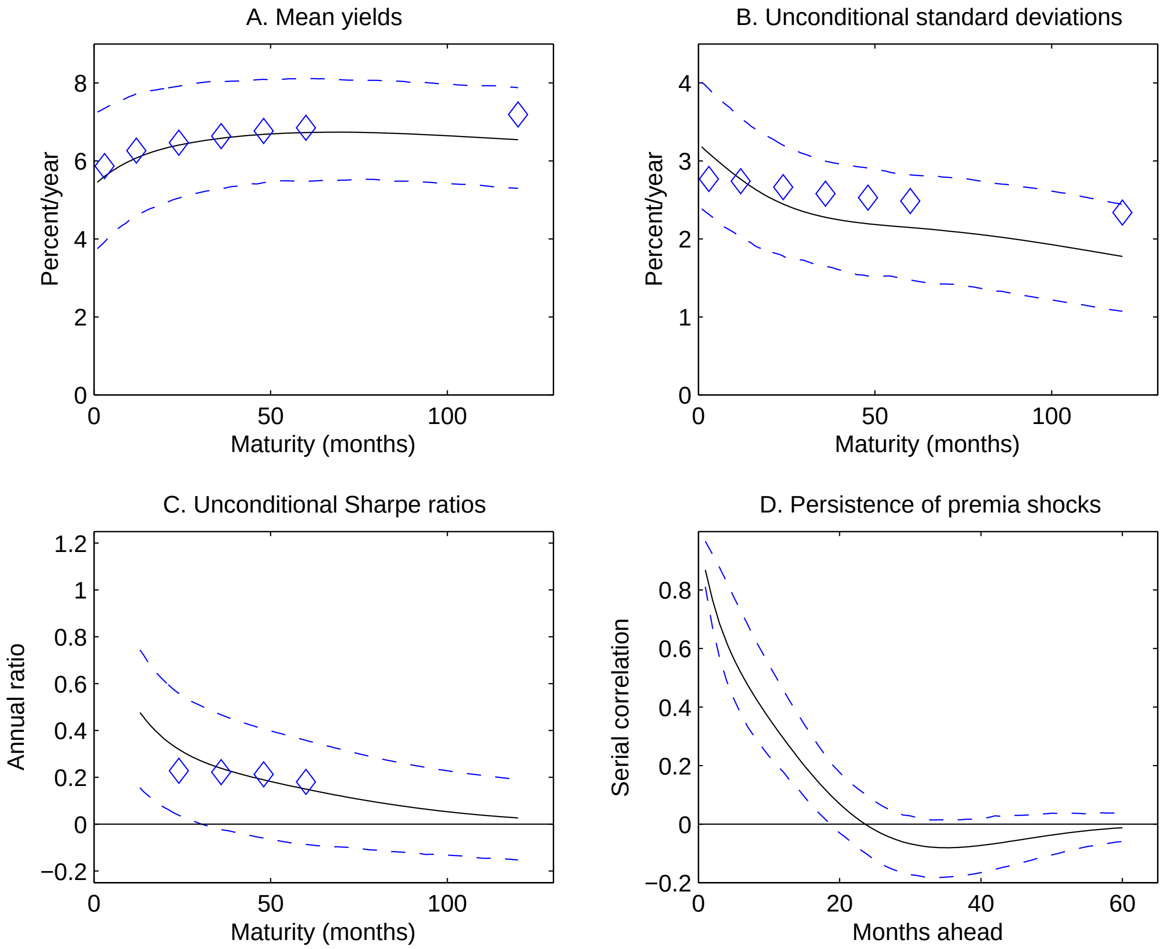



Fig. A1. Properties of an estimated five-factor Gaussian term structure model estimated with monthly data from 1964 to 2007. Sample values calculated using the same data are displayed with diamonds. The dashed lines are two-sided 95 percent confidence intervals calculated from Monte Carlo simulations. The Sharpe ratios in Panel C are for annual log returns in excess of the one-year bond yield. In the model, a single factor drives variation over time in bond risk premia. Panel D reports the model-implied serial correlation of the factor.

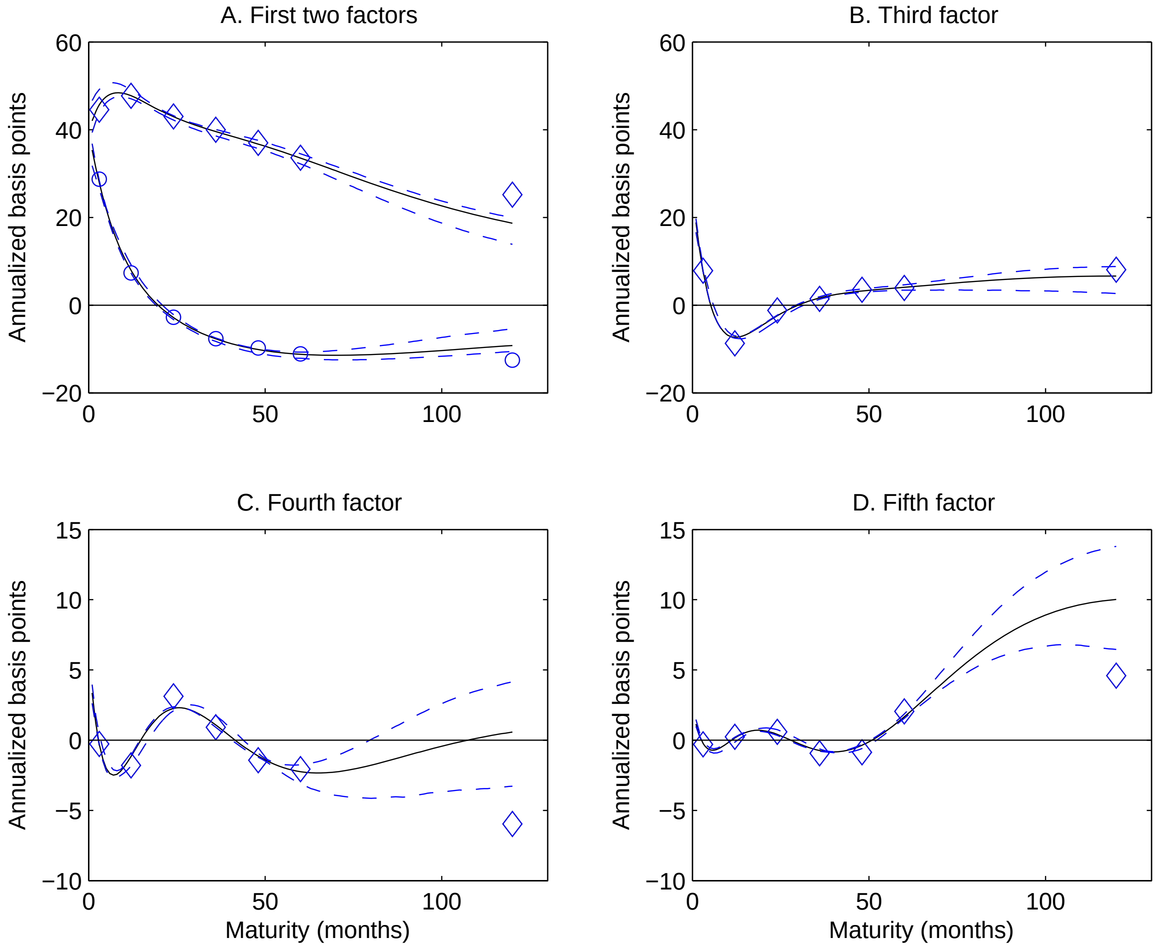
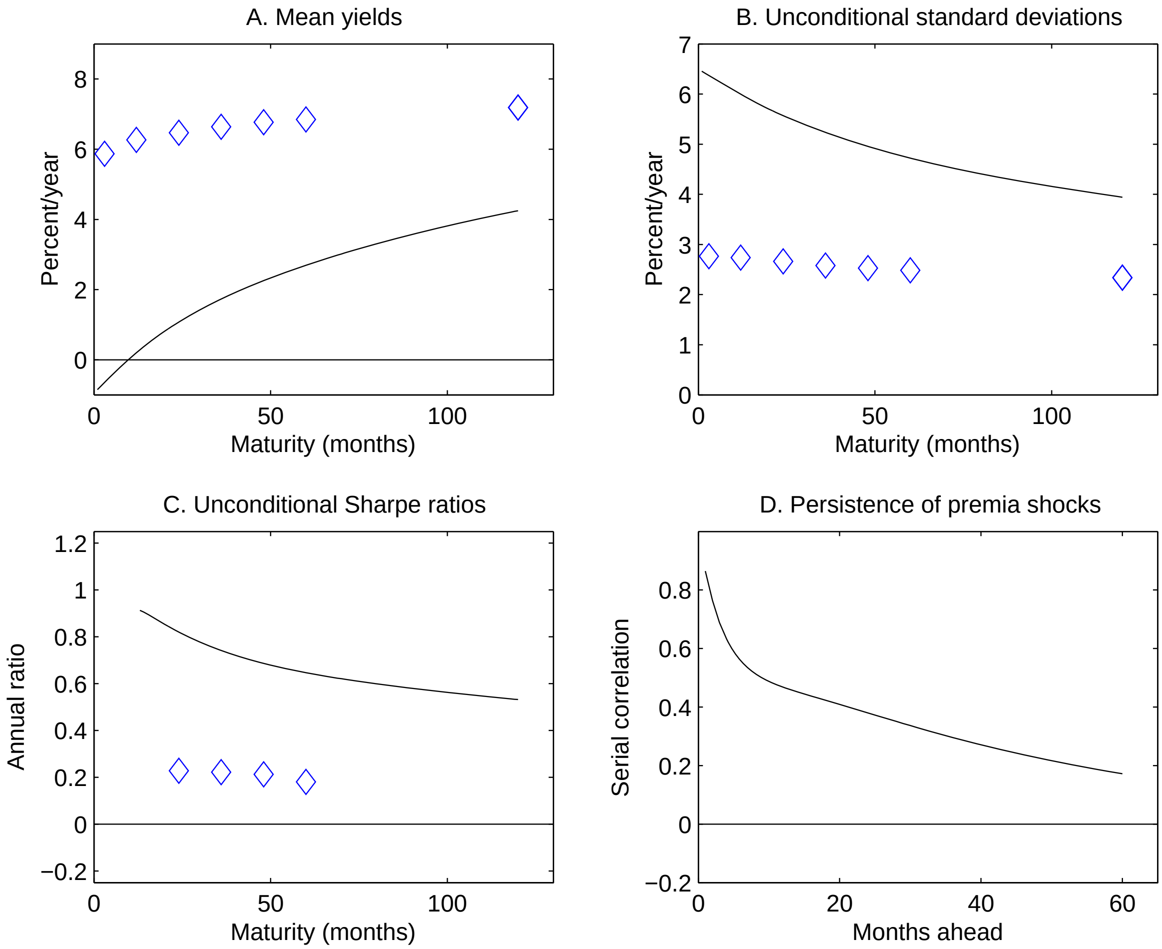
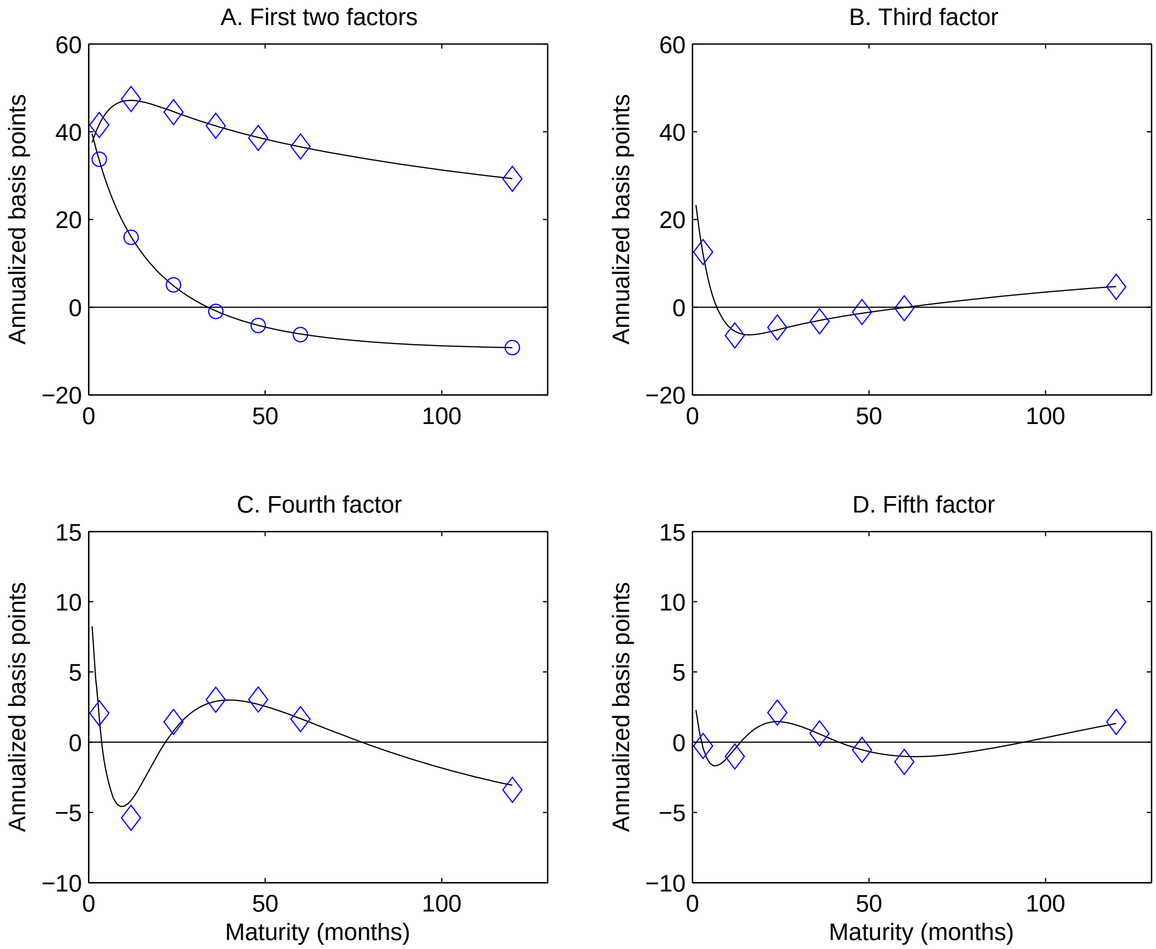



Fig. A2. Estimated yield loadings for a five-factor Gaussian term structure model estimated with monthly data from 1964 to 2007. The factors are principal components of shocks to the term structure. They are scaled by estimated standard deviations of the shocks. The diamonds are coefficients from regressions of observed yields on smoothed estimates of the factors. The dashed lines are two-sided 95 percent confidence intervals calculated from Monte Carlo simulations. Note the vertical scales of Panels A and B differ from those of Panels C and D.




Alternative Fig. A1. Properties of an estimated five-factor Gaussian term structure model estimated with monthly data from 1964 to 2007. The model summarized here is estimated using a ten-year bond yield in addition to the bonds used in estimating the original model. Sample values calculated using the same data are displayed with "o." The dashed lines are two-sided 95 percent confidence intervals calculated from Monte Carlo simulations. The Sharpe ratios in Panel C are for annual log returns in excess of the one-year bond yield. In the model, a single factor drives variation over time in bond risk premia. Panel D reports the model-implied serial correlation of the factor.




Alternative Fig. A2. Estimated yield loadings for a five-factor Gaussian term structure model estimated with monthly data from 1964 to 2007. The model summarized here is estimated using a ten-year bond yield in addition to the bonds used in estimating the original model. Sample The factors are principal components of shocks to the term structure. They are scaled by estimated standard deviations of the shocks. The diamonds are coefficients from regressions of observed yields on smoothed estimates of the factors. The dashed lines are two-sided 95 percent confidence intervals calculated from Monte Carlo simulations. Note the vertical scales of Panels A and B differ from those of Panels C and D.