# Awareness of Unawareness Begets Indecisiveness

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**Abstract**

This paper models indecisiveness displayed by choice behavior under uncertainty attributed to decision makers' awareness that their actions may result in unpredictable consequences. It presents axiomatic characterizations of the representations of incomplete preference relations that capture decision makers' inability to compare state-contingent monetary payoffs because the states may include unknown consequences, whose impact on their welfare they are unable to assess.

**Keywords**: Awareness of unawareness, incomplete preferences, indecisiveness, unawareness.
JEL classification: D8; D81; D83

## 1 Introduction

The inability of decision makers to compare alternatives courses of actions is manifested by indecisive choice behavior. The modeling paradigm that interprets individual choice behavior as an expression of underlying preference relations attributes indecisiveness to the incompleteness of the preference relations.[1]

Sources of incompleteness of preference relations depend on the context. In the case of choice under certainty and under risk, the source of incompleteness is decision makers' ambiguous tastes.[2] In the case of decision making under uncertainty, it stems from ambiguous beliefs, ambiguous tastes, or both.[3] In the case of choice under certainty or under risk, the lack of completeness of preference relations is represented by sets of utility functions. In the case of choice under uncertainty, it is represented by a set of probability distributions or a set of utility-probability pairs. In every case, ranking alternative courses of action requires that the utilities and probabilities that figure in the representations of the preference relations agree on the ranking of the alternatives under consideration. Lack of agreement is the source of indecisiveness.

The literature provides various reasons for the difficulty of decision makers' to compare alternatives, including complexity, multi-dimensionality, and lack of familiarity with the alternatives being considered. In this paper I explore a different reason for decision makers' inability to compare

[1] For a recent review of the literature dealing with the modeling, representations, and implications of incomplete preferences, see Karni (2026).

[2] See Ok (2002), Dubra, Maccheroni, and Ok (2004); Shapley and Baucells (2008); Ozgure and Ok (2011); and Ozgur (2014).

[3] See Bewley (1986); Seidenfeld, Schervish, and Kadane (1995); Nau (2006); Ok, Ortoleva and Riella (2012); Galaabaatar and Karni (2013); Riella (2015); Hara and Riella (2023), and Karni (2020).

alternatives: awareness that their choices may result in unpredictable outcomes.[4] The intuition underlying this explanation is that decision makers are better able to evaluate and form beliefs about the likelihoods of outcomes they encountered and experienced before than they are able to evaluate, or form beliefs about, the likelihood of outcomes the existence and nature of which are unkown. The ambiguity surrounding the nature of unanticipated outcomes is the novel source of indecisiveness examined in the paper.

The following example illustrates the idea being explored. During the COVID pandemic, several vaccines, were approved for use following successful clinical trials. Those trials revealed some side effects. However, given the short term during which the participants in the clinical trials were followed, it is impossible to rule out additional, unpredictable effects that may show up in the longer run.

The same can be said about the COVID virus. The immediate effects, and to some degree their likelihoods, were observed. But possible long-term effects, whose nature is unpredictable, may yet to be discovered.

Individuals facing the decision between vaccinating and not vaccinating and, if they decide to vaccinate, which vaccine to choose, may be aware of their unawareness of possible unpredictable side-effects of the vaccine or the long-term effects of COVID. Concerns about the unpredictable nature of these effects affect their decisions. It is conceivable that, in view of the ambiguity surrounding consequences whose nature and prevalence are unknown, decision makers exhibit indecisiveness when choosing whether to vaccinate and which vaccine to choose.

The main objective of this paper is to develop axiomatic representations of preferences relations exhibiting the indecisiveness depicted in the example above. The paper combines concepts and results developed in Karni and Vierø' s (2017) model of decision makers' choice behavior when they are aware of their unawareness; in Karni and Schmeidler' (2016) subjective expected utility model with state-dependent preferences; and in Karni's (2014) model of choice under uncertainty with event-dependent incomplete preferences.

The next section presents the model and provides an axiomatic characterization of incomplete, state-dependent, preference relations under uncertainty represented by subjective expect multi utility-prior pairs. Section 3 provides an axiomatic characterization of incomplete, state-dependent, preference relations represented by subjective expect utility on the event that consists of fully specified states and expected multi utility-prior pairs representation on the event that consists of states in which unanticipated consequences may obtain. Section 4 includes some concluding remarks and discusses of the meaning of the state-space employed here and the expected multi utility-prior pairs representation. Section 5 presents the proofs.

# 2 The Model

## 2.1 The State Space and Analytical Framework

**The state space.** The primitives of the model consist of a finite, nonempty, set $F$ of *feasible acts* and a finite, nonempty, set $X$ of *feasible consequences*. Depending on the context, the acts represent courses of action available to the decision makers (e.g., vaccinating, choosing a surgeon) or actions taken by other agents (e.g., running horse races, organizing presidential elections).

[4]Decision making under unawareness is modeled in Karni and Vierø(2012, 2015, 2017); Dominiak and Tserenjigmid (2018); Karni, Vierø, and Valenzuela-Stooky (2021).

Decision makers are supposed to be aware of the feasible acts and the nature of the feasible consequences. At any given time, the set of feasible consequences consists of outcomes discovered when they obtained for the first time in the wake of implementation of feasible acts. The sequential discovery of new, unanticipated, consequences the nature of which was unknown before they were observed for the first time is presumed to alert decision makers to the potential discovery of additional consequences the nature of which they are unaware. Following Karni and Vierø(2014), I model this "awareness of unawareness" by introducing abstract consequences defined negatively as "none of the feasible consequences." These abstract consequences may or may not exist, if they exist, their nature is unpredictable. Formally, I denote by $\hat{x} := \neg X$ the notion of consequences not in $X$ and let $\hat{X} = X \cup \{\hat{x}\}$.

A *state* is a resolution of uncertainty in the sense of specifying the consequences of every feasible act. Formally, a state is a mapping of from the set $F$ of feasible acts onto the set $\hat{X}$ of extended consequences.[5] The state space $S = \hat{X}^F$ consists of all such mappings. Events are subsets of the state space. The event $E := \{s \in S \mid s(f) \in X, \forall f \in F\}$ is referred to as the *set of fully describable states*. The complementary event $E^c$ in $S$ consists of states that include $\hat{x}$ as conceivable consequence of some acts. Elements $E^c$ are referred to as *partially describable states*.

The following example illustrates this idea. Let $F = \{f_1, f_2\}$ and $X = \{x_1, x_2\}$. The corresponding state space consists of nine states:

| $S$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ | $s_5$ | $s_6$ | $s_7$ | $s_8$ | $s_9$ |
|---|---|---|---|---|---|---|---|---|---|
| $f_1$ | $x_1$ | $x_2$ | $x_1$ | $x_2$ | $\hat{x}$ | $\hat{x}$ | $x_1$ | $x_2$ | $\hat{x}$ |
| $f_2$ | $x_1$ | $x_1$ | $x_2$ | $x_2$ | $x_1$ | $x_2$ | $\hat{x}$ | $\hat{x}$ | $\hat{x}$ |

The subset of fully describable states in this example is $E = \{s_1, ..., s_4\}$, and the set of partially describable state is $E^c = \{s_5, ..., s_9\}$.

A state is an abstract theoretical construct. It takes on concrete meaning when the acts and corresponding realized consequences are observable and objectively verifiable. Put differently, to be meaningful, scientific concepts states or events must be verifiable by independent observers who must agree on the feasible sets of acts and consequences.[6]

**Prize-state lotteries.** Let $Y$ be a finite set of real numbers representing monetary prizes. Define $\underline{y} = minY, \bar{y} = maxY$. Denote by $\Delta(Y \times S)$ the set of joint distributions on the product set $Y \times S$. Following Karni and Schmeidler (2016), I refer to elements of $\Delta(Y \times S)$ as *prize-state lotteries*.

Because the state probabilities (i.e., the marginal distribution on the state space) cannot be chosen or manipulated, the set of -prize-state lotteries is a hypothetical construct. Consequently, preference relations on this set are hypothetical and are supposed to be expressed verbally. For example, a decision maker may be asked how he would choose between two prize-state probabilities that have different marginal probabilities on the set of states. Verbal responses to such hypothetical questions would be meaningful and informative if they stimulate the same mental process that governs the choice among feasible state-contingent monetary payoffs.

Expected utility theory is a model of choice among probability distributions over arbitrary set of conceivable consequences. To evaluate its merits and develop intuitive understanding to what it

[5] This formulation of the state space appears in Schmeidler and Wakker (1987) and Karni and Schmeidler (1991).

[6] Machina (2003) provides an insightful discussion of the meaning of the state space in decision theory. Karni (2017) gives different take on this subject which is more in line of the approach taken here. Further discussion of the construction of the state space is provided in the concluding section.

entails, the literature resorts to its application to hypothetical situations. Perhaps the most famous example is the lottery depicted by the St.-Petersburg paradox. By itself, the fact that this lottery is hypothetical, does not render meaningless the response to the question "how much would you pay to participate that lottery?" Invoking verbal responses regarding the choice among state-prize lotteries in the present framework should be understood in this spirit.

**Bets.** Bets are mappings from the state space to the set of lotteries with monetary outcomes. Formally, the set of bets is $B := \{b \in [0,1]^{Y\times S} \mid \sum_{y\in Y} b(y,s) = 1, \forall s \in S\}$.[7] Unlike the feasible acts that define the state space, bets are derived actions that do not exist, or may even be conceived of, without the existence of the underlying state space. Depending on the context, some bets are infeasible. If the feasible acts are mutually exclusive, (e.g., choosing not to vaccinate precludes being vaccinated; having an operation performed by one surgeon excludes it being performed by another), then bets can be placed only on events defined by the states the images of which under the chosen act are the outcomes. Formally, for every $(x, f) \in \hat{X} \times F$, let $E(x; f) := \{s \in S \mid s(f) = x\}$. Then, given $f \in F$, bets may be placed only on the events $\{E(x; f) \mid x \in \hat{X}\}$. For example, suppose that the acts that figure in the depiction of the state space above are mutually exclusive. If the decision maker chooses the act $f_1$, there are three distinguishable verifiable events: $E(x_1; f_1) = \{s_1, s_3, s_7\}$, $E(x_2; f_1) = \{s_2, s_4, s_8\}$, and $E(x_3; f_1) = \{s_5, s_6, s_9\}$. The decision maker may place bets only on these events. Put differently, if the implementation of $f_1$ precludes that of $f_2$, then, given $f_1$, the decision maker set of feasible bets is restricted to those bets which payoffs are the same in all the states belonging to the same event.

Define $b(y, E(x; f)) = \sum_{s\in E(x;f)} b(y, s)$ and let $b_f \in [0,1]^{Y\times\hat{X}}$ be such that $\sum_{y\in Y} b(y, E(x; f)) = 1, \forall x \in \hat{X}$. Denote by $B_f$ the subset of bets that are feasible given $f$ (i.e., $B_f := \{b_f \mid b \in B\}$).

**The analytical framework** consists of two *choice sets*, $\Delta(Y \times S)$ and $\mathbb{C} := \{(f, b) \in F \times B \mid b \in B_f\}$. If the feasible acts are not a subject of choice of the decision maker, then the second choice set is $B$.

Corresponding to these choice sets are transitive and reflexive binary relations $\succcurlyeq^*$ on $\Delta(Y \times S)$ and, depending on the context, $\succcurlyeq$ on $\mathbb{C}$ or on $B$. These binary relations are referred to as *preference relations*.

For $\ell, \ell' \in \Delta(Y \times S)$ and $\alpha \in [0,1]$, define $\alpha\ell + (1-\alpha)\ell' \in \Delta(Y \times S)$ by $(\alpha\ell + (1-\alpha)\ell')(y,s) = \alpha\ell(y,s) + (1-\alpha)\ell')(y,s)$, for all $(y,s) \in Y \times S$. Similarly, for all $b, b' \in B$, define $\alpha b + (1-\alpha)b' \in B$ by $(\alpha b + (1-\alpha)b')(y,s) = \alpha b(y,s) + (1-\alpha)b'(y,s))$, for all $(y,s) \in Y \times S$ and, for every $f \in F$ and all $b, b' \in B$, define $\alpha b_f + (1-\alpha)b'_f \in B_f$ by $(\alpha b_f + (1-\alpha)b'_f)(y,s) = \alpha b_f(y,s) + (1-\alpha)b'_f(y,s))$, for all $(y,s) \in Y \times S$. Then the sets $\Delta(Y \times S)$, $B$, and $B_f$, $f \in F$, are convex subsets of linear space $\mathbb{R}^{|Y|\times|S|}$.

## 2.2 The structure of the preference relations and the basic representation theorem

Let $C$ be a convex subset of a linear space, and denote by $\trianglerighteq$ a generic preference relation on $C$. Denote by $\triangleright$ the asymmetric part of $\trianglerighteq$. The preference relation is said to be *nondegenerate* if there are $c, c' \in C$ such that $c \triangleright c'$. The structure of $\trianglerighteq$ is depicted by the following well-known axioms.

(A.1) **Archimedean** For all $c, c', c'' \in C$ such that $c \triangleright c' \triangleright c''$, there are $\alpha, \beta \in (0,1)$ such that $\alpha c + (1-\alpha)c'' \triangleright c'$ and $c' \triangleright \beta c + (1-\beta)c''$.

[7]Bets is analogous to the set of Anscombe-Aumann (1963) acts, which they dubbed horse lotteries.

(A.2) **Independence** For all $c, c', c'' \in C$ and $\alpha \in (0,1]$, $c \trianglerighteq c'$ if and only if $\alpha c + (1-\alpha)c'' \trianglerighteq \alpha c' + (1-\alpha)c''$.

Assume that the set $C$ is $\trianglerighteq$- bounded (i.e., there exist $\underline{c}, \bar{c} \in C$ such that $\bar{c} \vartriangleright c \vartriangleright \underline{c}$ for all $c \in C \diagdown \{\bar{c}, \underline{c}\}$).

The fundamental representation theorem is the following.

**BASIC REPRESENTATION THEOREM (BRT)** *Let $\trianglerighteq$ be a binary relation on a convex, nonempty subset $C$, of a linear space. Then the following conditions are equivalent:*
*(i) $C$ is $\trianglerighteq$- bounded, and $\trianglerighteq$ is a reflexive and transitive satisfying the Archimedean and Independence axioms.*
*(ii) There exists a nonempty closed set $\mathcal{V}$ of affine real-valued functions on $C$ such that for all $V \in \mathcal{V}$,*

$$V(\bar{c})) \vartriangleright V(c) \vartriangleright V(\underline{c})$$

*for all $c \in C \diagdown \{\bar{c}, \underline{c}\}$ and, for all $c, c' \in C$,*

$$c \trianglerighteq c' \Leftrightarrow V(c) \geq V(c'),$$

*Moreover, if $\bar{\mathcal{V}}$ represents $\trianglerighteq$ in the sense above, then $\langle \bar{\mathcal{V}} \rangle = \langle \mathcal{V} \rangle$, where the operator $\langle \cdot \rangle : 2^{\mathbb{R}^n} \to 2^{\mathbb{R}^n}$ is defined as*

$$\langle \mathcal{V} \rangle := cl(cone(\mathcal{V}) + \theta\{1_{Y \times S}\}_{\theta \in \mathbb{R}}),$$

*where the closure operator is with respect to the product topology on $\mathbb{R}^n$ and $cone(\mathcal{V})$ denotes the convex cone generated by $\mathcal{V}$.*[8]

Applied to the present analytical framework, the affinity of the functions $\mathcal{V}$ in the BRT implies that:

(a) There exists a nonempty closed set $\mathcal{U}$ of real-valued functions $u$ on $Y \times S$ such that for all $\ell, \ell' \in \Delta(S \times Y)$,

$$\ell \succcurlyeq^* \ell' \Leftrightarrow \sum_{s \in S} \sum_{y \in Y} u(y,s)[\ell(y,s) - \ell'(y,s)] \geq 0, \forall u \in \mathcal{U}. \tag{1}$$

This result is implied by the Expected Multi-Utility Theorem of Baucells and Shapley (2008).[9]

(b) There exists a nonempty closed set $\mathcal{W}$ of real-valued functions $w$ on $Y \times S$ such that for all $b, b' \in B$,

$$b \succcurlyeq b' \Leftrightarrow \sum_{s \in S} \sum_{y \in Y} w(y,s)[b(y,s) - b'(y,s)] \geq 0, \forall w \in \mathcal{W}. \tag{2}$$

This result is implied by the proof of Lemma 1 in Galaabaatar and Karni (2013).

Representation (2) pertains to situations in which all the states are observable. However, in the case of mutually exclusive acts, the choice of act restricts the set of feasible bets. In this case, the BRT implies that there exists a set $\hat{\mathcal{W}}$ of real-valued functions on $Y \times \{E(x;f) \mid x \in \hat{X}\}$ such that for every $f \in F$,

$$b_f \mapsto \sum_{x \in \hat{X}} \sum_{y \in Y} \hat{w}(y, E(x;f)) b_f(y, E(x;f)), \forall \hat{w} \in \hat{\mathcal{W}}. \tag{3}$$

[8]These notations and definition are from Dubra et al. (2004).

[9]Baucells and Shapley invoke a different continuity axiom which is implied by Archimedean and Independence.

## 2.3 Consistency

The following definitions and axiom are taken from Karni and Schmeidler (2016). A prize-state lottery, $\ell \in \Delta(Y \times S)$, is said to be *semipositive* if $\sum_{y\in Y} \ell(y,s) > 0$ for every $s \in S$. Denote by $\Delta_{sp}(Y \times S)$ the subset of semipositive state-prize lotteries. Define a function $H : \Delta_{sp}(Y \times S) \to B$ by

$$H(\ell(y,s)) = \frac{\ell(y,s)}{\sum_{y\in Y} \ell(y,s)}, \forall (y,s) \in Y \times S.$$

Given $b, b' \in B$ and $s \in S$, $b$ is said to be *equals* $b'$ *outside* $s$ if, for all $s' \neq s$ and $y \in Y$, $b(y,s') = b'(y,s')$. Similarly, given $\ell, \ell' \in \Delta(Y \times S)$ and $s \in S$, $\ell$ is said to be *equals* $\ell'$ *outside* $s$ if, for all $s' \neq s$ and $y \in Y$, $\ell(y,s') = \ell'(y,s')$.

It is common in decision theory to refer to a state or event as being null to convey the idea that the decision maker consider the said state or event impossible to obtain. In the revealed preference methodology, it is standard practice to formalize this idea requiring that, conditional on the state or event under consideration, the decision maker displays indifference among all the alternatives. This practice presumes that the preference relation exhibits state-independence. If instead the preference relation is state-dependent, as is the case here, it is possible the decision maker is indifferent to the alternatives conditional on a state, not because he believes that the state is impossible to obtain but because, given the nature of the state, he is indifferent among the feasible alternatives. For example, conditional on the event of a plane crush, a passenger who has no bequest motive may be indifferent among all flight insurance policies, no matter what their terms.

In view of this observation, and the fact that state-dependence is an essential ingredient of the approach pursued here, a state $s \in S$ is said to be $\succcurlyeq$ - *nonnull* if there are $b, b' \in B$ such that $b$ equals $b'$ outside $s$ and $b \succ b'$. It is said to be $\succcurlyeq$- *null* if $b \sim b'$ for all $b, b' \in B$ that are equal outside $s$. Similarly, a state $s \in S$ is said to be $\succcurlyeq^*$ - *nonnull* if there are $\ell, \ell' \in \Delta(Y \times S)$ such that $\ell$ equals $\ell'$ outside $s$ and $\ell \succ^* \ell'$. It is is said to be $\succcurlyeq^*$- *null* if $\ell \sim^* \ell'$ for all $\ell, \ell' \in \Delta(Y, S)$ that are equal outside $s$. Only if a state is $\succcurlyeq$- *null* and $\succcurlyeq^*$ - *nonnull* we can conclude that the decision maker believes that the state is impossible to obtain.[10]

Let $\Delta(Y)$ be the simplex in $\mathbb{R}^{|Y|}$. The next axiom requires that, for any $\succcurlyeq$ - *nonnull* $s \in S$, the preference relations $\succcurlyeq$ and $\succcurlyeq^*$ conditional on $s$ rank lotteries in $\Delta(Y)$ identically. Formally,

(A.3) **Consistency** For all $s \in S$ and all semipositive $\ell, \ell' \in \Delta(Y \times S)$ such that $\ell$ equals $\ell'$ outside $s$, $H(\ell) \succ H(\ell')$ implies that $\ell \succ^* \ell'$.

## 2.4 First representation theorem

The following theorem generalizes Theorem 2 of Karni and Schmeidler (2016) to include incomplete preferences.

**THEOREM 1** *Let $\succcurlyeq$ on $B$ and $\succcurlyeq^*$ on $\Delta(Y \times S)$ be binary relations, then conditions (i), and (ii) below are equivalent and condition (ii) implies condition (iii).*

[10]Evidently, $\succcurlyeq^*$ - *nonnull* implies that the decision maker is not indifferent among all the alternatives conditional on the given state. But $\succcurlyeq$- *null* implies that he is. This indifference must be attributed to the decision maker's belief that the given state cannot obtain.

*(i) The binary relations $\succcurlyeq$ on $B$ and $\succcurlyeq^*$ on $\Delta(Y \times S)$ are reflexive and transitive satisfying the Archimedean and Independence axioms, $\succcurlyeq$ is nondegenerate, and jointly they satisfy the Consistency axiom.*

*(ii) There exist a set $\mathcal{U}$ of real-valued functions on $Y \times S$ and for each $u \in \mathcal{U}$ there is a corresponding unique probability distribution, $\pi^u$ on $S$, such that, for all $b, b' \in B$,*

$$b \succcurlyeq b' \Leftrightarrow \sum_{s \in S} \sum_{y \in Y} \pi^u(s) u(y,s) b(y,s) \geq \sum_{s \in S} \sum_{y \in Y} \pi^u(s) u(y,s) b'(y,s), \forall u \in \mathcal{U},, \tag{4}$$

*and, for all $\ell, \ell' \in \Delta(Y \times S)$,*

$$\ell \succcurlyeq^* \ell' \Leftrightarrow \sum_{s \in S} \sum_{y \in Y} u(y,s) \ell(y,s) \geq \sum_{s \in S} \sum_{y \in Y} u(y,s) \ell(y,s), \forall u \in \mathcal{U}. \tag{5}$$

*(iii) (a) If a state $s$ is $\succcurlyeq$ - null and $\succcurlyeq^*$- nonnull, then $\pi^u(s) = 0, \forall u \in \mathcal{U}$ (b) If a state $s$ is $\succcurlyeq$ - nonnull then $\pi^u(s) > 0, \forall u \in \mathcal{U}$.*

*Moreover, $\mathcal{V}$ represents the binary relations $\succcurlyeq$ on $B$ and $\succcurlyeq^*$ on $\Delta(Y \times S)$ in the sense of above if and only if $\langle \mathcal{U} \rangle = \langle \mathcal{V} \rangle$.*[11]

Define $\succ_s^*$ on $\Delta(Y \times S)$ and $\succ_s$ on $B$ to be the asymmetric parts of $\succcurlyeq^*$ and $\succcurlyeq$ restricted to prize-state lotteries and bets that are equal outside $s$. If a state $s$ is $\succcurlyeq$- *null* and $\succcurlyeq^*$- *nonnull*, the decision maker is not indifferent among all the prize-state lotteries that are equal outside $s$ but indifferent among bets that are equal outside $s$. This conjunction of preferences implies that the decision maker regards the state $s$ as impossible to obtain and assigns it zero probability. If $\succ_s$ and $\succ_s^*$ are both nonempty (i.e., $s$ is nonnull according to both $\succcurlyeq$ and $\succcurlyeq^*$), then the subjective probability of $s$ is strictly positive. If they are both null, since the set of lotteries is semipositive, the decision maker is indifferent among all the lotteries conditional on the state. However, he may also believe that the state is impossible to obtain. As there is no way of knowing which is the cause, the subjective probability can take any value in $[0, 1)$.

The representation in Theorem 1 pertains to situations in which the state-space is defined by acts taken by independent agents over which the decision maker has no control. Consequently, it applies to betting on extraneous events. Examples of this kind of betting include betting on sporting events, election results, the temperature measured by the weather service, and the performance of the New York Exchange as measured by Dow Jones Industrial index, among others.

Situations in which the decision maker exercises control over the feasible events (i.e., events that are verifiable) by his choice from a set of mutually exclusive acts are prevalent and significant. They include the decision to vaccinate, to install a lightning rod, and to take a flight rather than drive. In this kind of situations –choosing an act and thereby excluding alternative acts – render certain states impossible to observe. Consequently, bets whose payoffs depend on these states are not feasible. In particular, the choice of an act that excludes all others restricts the feasible bets to events defined by the outcomes of the chosen act. In such a case, if an act $f \in F$ is chosen, then the set of feasible bets is $B_f$.

The choice of an act may not be free. Consider a situation in which the choice of alternative acts entails different monetary cost $c(f), f \in F$. For example, surgeons and hospitals may charge different prices for the same procedure, airlines may charge different prices for flights on the same route. In such cases, it is natural to incorporate the cost into the payoff of the bets. For example,

[11] Note that if $\mathcal{U}$ represents $\succcurlyeq^*$ on $\Delta(X \times S)$ so does $\langle \mathcal{U} \rangle$.

a bet $b_f$ is modified to include the cost so that it assigns to the payoff $y - c(f)$ the probability $b_f(y, E(x; f))$.

To state the next result I introduce the following notations and definitions. For every $f \in F$ and $x \in \hat{X}$ let $\pi^u(E(x; f)) := \sum_{s \in E(x;F)} \pi^u(s)$ and $\pi^u(s \mid E(x, f)) = \frac{\pi^u(s)}{\pi^u(E(x;f))}$. Define a function $\bar{u}(\cdot \mid E(x; f)) : Y \to \mathbb{R}$ by $\bar{u}(y \mid E(x; f)) = \sum_{s \in E(x,f)} \pi^u(s \mid E(x, f))u(y - c(f), s)$ for all $y \in Y$. Let $\bar{\mathcal{U}} := \{\bar{u}(\cdot \mid E(x; f)) \mid u \in \mathcal{U}\}$.

**COROLLARY 1** *Let $\succcurlyeq$ on $\mathbb{C}$ and $\succcurlyeq^*$ on $\Delta(X \times S)$ be binary relations, then conditions (i) and (ii) below are equivalent.*

*(i) The binary relations $\succcurlyeq$ on $\mathbb{C}$ and $\succcurlyeq^*$ on $\Delta(Y \times S)$ are reflexive and transitive satisfying the Archimedean and Independence axioms, $\succcurlyeq$ is nondegenerate, and jointly they satisfy the Consistency axiom.*

*(ii) There exists a set, $\mathcal{U}$, of real-valued functions on $Y \times S$ and a corresponding set, $\bar{\mathcal{U}}$, of real-valued functions on $Y \times \{E(x; f) \mid x \in \hat{X}, f \in F\}$. For each $\bar{u} \in \bar{\mathcal{U}}$, there is a corresponding unique probability distribution, $\pi^u$, on $S$ such that for all $(f, b_f), (f', b_{f'}) \in \mathbb{C}$,*

$$(f, b_f) \succcurlyeq (f', b_{f'})$$

*if and only if*

$$\sum_{x \in \hat{X}} \pi^u(E(x, f)) \sum_{y \in Y} [\bar{u}(y \mid E(x; f))b_f(y, E(x; f)) - \bar{u}(y \mid E(x; f'))b_{f'}(y, E(x; f')] \geq 0, \forall \bar{u} \in \bar{\mathcal{U}} \quad (6)$$

*and for all $\ell, \ell' \in \Delta(Y \times S)$,*

$$\ell \succcurlyeq^* \ell' \Leftrightarrow \sum_{s \in S} \sum_{y \in Y} u(y, s)\ell(y, s) \geq \sum_{s \in S} \sum_{y \in Y} u(y, s)\ell(y, s), \forall u \in \mathcal{U}. \quad (7)$$

*Moreover, $\mathcal{V}$ represents the binary relations $\succcurlyeq$ on $\mathbb{C}$ and $\succcurlyeq^*$ on $\Delta(Y \times S)$ in the sense of above if and only if $\langle \mathcal{U} \rangle = \langle \mathcal{V} \rangle$.*

The source of indecisiveness in this model is the decision maker's inability to compare some state-prize lotteries. This general observation notwithstanding, given that the prizes are monetary, it seems natural to suppose that when a state is fully specified (i.e., when the consequences that define the state are known), the decision maker is in better position to evaluate the prize-state pairs than when a state involves unknown consequences. In practice, decision makers may exhibit indecisiveness when facing choice between lotteries with monetary payoffs, because, for instance, their complexity. To highlight the focal issue of this study, I assume that decision makers have no difficulty comparing prize-state lotteries that are equal outside fully specified states. In the next section, I examine the implications of this possibility for the representations of the preference relations.

# 3 Indecisiveness and Unawareness

## 3.1 Partial completeness

For every event, $E \in 2^S$ and $b, \hat{b} \in B$, define $b_E\hat{b} \in B$ by $(b_E\hat{b})(y, s) = b(y, s)$ for all $s \in E$ and $(b_E\hat{b})(y, s) = \hat{b}(y, s)$ for all $s \in E^c$, where $E^c$ denotes the complement of $E$ in $S$. Denote by $B_{\hat{b}}(E)$ the subset of bets that are equal to $\hat{b}$ outside the event $E$. Formally, $B_{\hat{b}}(E) = \{b_E\hat{b} \in B \mid b \in B\}$.

The next axiom requires that if $E$ is the set of fully describable states, the restriction of the preference relation $\succcurlyeq$ to $B_{\hat{b}}(E)$ be complete.

(A.5) **Partial Completeness** Let $E$ be the event that consists of all fully describable states. Then for every given $\hat{b} \in B$ and $b_E\hat{b}, b'_E\hat{b} \in B_{\hat{b}}(E)$, either $b_E\hat{b} \succcurlyeq b'_E\hat{b}$ or $b'_E\hat{b} \succcurlyeq b_E\hat{b}$.

## 3.2 Second representation theorem

The map $H$ from the set of semipositive state-prize lotteries to the set of bets is one-to-one and onto. Denote by $H^{-1}$ the inverse function of $H$. Then for each $b_E\hat{b} \in B$ there is $\ell \in \Delta_{sp}(Y \times S)$ such that $\ell = H^{-1}(b_E\hat{b})$. The Consistency axiom asserts that for all $b_E\hat{b}$ and $b'_E\hat{b}$ in $B$, $b_E\hat{b} \succ b'_E\hat{b}$ implies that $H^{-1}(b_E\hat{b}) \succ^* H^{-1}(b_E\hat{b})$.

The following theorem captures the idea that indecisiveness about choices between bets is attributed to decision makers' inability to evaluate the monetary payoffs when the nature of the underlying states is not fully specified.

**THEOREM 2** *Let $\succcurlyeq$ on $B$ and $\succcurlyeq^*$ on $\Delta(Y \times S)$ be binary relations. Then conditions (i) and (ii) below are equivalent.*

*(i) The binary relations $\succcurlyeq$ on $B$ and $\succcurlyeq^*$ on $\Delta(Y \times S)$ are reflexive and transitive, satisfying the Archimedean and Independence axioms; $\succcurlyeq$ is nondegenerate and satisfies Partial Completeness, and jointly $\succcurlyeq$ and $\succcurlyeq^*$ satisfy the Consistency axiom.*

*(ii) There exist a real-valued function $\hat{u}$ on $Y \times E$ and a unique probability distribution $\pi$ on $E$, a set $\mathcal{U}$ of real-valued functions on $Y \times E^c$ and, for each $u \in \mathcal{U}$ there is a corresponding probability distribution, $\pi^u$, on $E^c$ such that for all $b, b' \in B$, $b \succcurlyeq b'$ if and only if*

$$\sum_{s \in E} \sum_{y \in Y} \pi(s)\hat{u}(y,s)[b(y,s) - b'(y,s)] + \sum_{s \in E^c} \sum_{y \in Y} \pi^u(s)u(y,s)[b(y,s) - b'(y,s)] \geq 0, \forall u \in \mathcal{U}, \tag{8}$$

*and for all $\ell, \ell' \in \Delta(Y, S)$*

$$\ell \succcurlyeq^* \ell' \Leftrightarrow \sum_{s \in E} \sum_{y \in Y} \hat{u}(y,s)[\ell(y,s) - \ell'(y,s) + \sum_{s \in E^c} \sum_{y \in Y} u(y,s)[\ell(y,s) - \ell(y,s)] \geq 0, \forall u \in \mathcal{U}. \tag{9}$$

*Moreover, $\hat{v}$ and $\mathcal{V}$ represent the binary relations $\succcurlyeq$ on $B$ and $\succcurlyeq^*$ on $\Delta(Y \times S)$ in the above sense if and only if $\hat{v}(y,s) = \beta\hat{u}(y,s) + \alpha(s), \beta > 0$ and $\langle\mathcal{U}\rangle$=$\langle\mathcal{V}\rangle$.*

If the acts are mutually exclusive, then for every given $f \in F$, the sets of verifiable fully specified states are $E \cap E(x;f), x \in \hat{X}$ and the sets of verifiable partial specifiable states are $E^c \cap E(x;f), x \in \hat{X}$. Define real-valued functions $\hat{U}(\cdot; E \cap \hat{E}(x;f))$ and $\hat{U}(\cdot; E^c \cap E(x;f))$ on $Y$ as follows:

$$\hat{U}(y; E \cap E(x;f)) = \sum_{s \in E \cap E(x;f)} \hat{u}(y - c(f), s)\pi^{\hat{u}}(s),$$

and for every $u \in \mathcal{U}$,

$$\bar{u}(y; E^c \cap E(x;f)) = \sum_{s \in E^c \cap E(x;f)} u(y - c(f), s)\pi^u(s).$$

Define $\bar{\mathcal{U}} = \{\bar{u}(y; E^c \cap E(x;f)) \mid u \in \mathcal{U}\}$. Theorem 2 implies the following:

**Corollary 2** *Let* $\succcurlyeq$ *on* $\mathbb{C}$ *and* $\succcurlyeq^*$ *on* $\Delta(X \times S)$ *be binary relations. Then conditions (i) and (ii) below are equivalent.*

*(i) The binary relations* $\succcurlyeq$ *on* $\mathbb{C}$ *and* $\succcurlyeq^*$ *on* $\Delta(Y \times S)$ *are reflexive and transitive, satisfying the Archimedean and Independence axioms;* $\succcurlyeq$ *is nondegenerate and satisfies Partial Completeness; and jointly they satisfy the Consistency axiom.*

*(ii) There exist a real-valued function* $\hat{u}$ *on* $Y \times E$ *and unique probability distribution* $\pi^{\hat{u}}$ *on* $E$*, a set* $\mathcal{U}$ *of real-valued functions on* $Y \times E^c$*; and, for each* $u \in \mathcal{U}$ *a corresponding probability distribution,* $\pi^u$*, on* $E^c$ *such that for all* $(f, b_f), (f', b_{f'}) \in \mathbb{C}$*,*

$$(f, b_f) \succcurlyeq (f', b_{f'})$$

*if and only if*

$$\sum_{x \in \hat{X}} \sum_{y \in Y} \hat{U}(y; E \cap E(x; f))[b_f(y, E \cap E(x; f)) - b_{f'}(y, E \cap E(x; f))] +$$

$$\textstyle\sum_{x \in \hat{X}} \sum_{y \in Y} \bar{u}(y, E^c \cap E(x; f))[b_f(y, E^c \cap E(x; f)) - b_{f'}(y, E^c \cap E(x; f)) \geq 0, \forall \bar{u} \in \bar{\mathcal{U}}, \quad (10)$$

*and, for all* $\ell, \ell' \in \Delta(Y, S)$*,*

$$\ell \succcurlyeq^* \ell' \Leftrightarrow \sum_{s \in E} \sum_{y \in Y} \hat{u}(y, s)[\ell(y, s) - \ell'(y, s)] + \sum_{s \in E^c} \sum_{y \in Y} u(y, s)[\ell(y, s) - \ell'(y, s)] \geq 0, \forall u \in \mathcal{U}. \quad (11)$$

*Moreover,* $\hat{v}$ *and* $\mathcal{V}$ *represent the binary relations* $\succcurlyeq$ *on* $\mathbb{C}$ *and* $\succcurlyeq^*$ *on* $\Delta(Y \times S)$ *in the above sense if and only if* $\hat{v}(y, s) = \beta \hat{u}(y, s) + \alpha, \beta > 0$ *and* $\langle \mathcal{U} \rangle = \langle \mathcal{V} \rangle$*.*

# 4 Concluding Remarks

## 4.1 The state-space revisited

A critical aspect of the approach taken here is the modeling of the state-space. It captures the concept of states as complete resolution of uncertainty. Such resolution requires that the state that obtains be verifiable ex post. In our formulation, this requirement amounts to being able to observe the outcomes of each and every feasible act. In some situations, such as betting on the conjunction of the outcomes of three roles of a die, a flip of a coin, or the outcome of running a horse race twice, this condition is easy to satisfy. If, however, the acts defining the states are mutually exclusive, only partial resolution is feasible. For instance, if the feasible acts are mutually exclusive medical procedures, then choosing one procedure excludes the possibility of observing the outcome that would have obtained if an alternative procedures. In this case, it is possible to bet on the outcomes of the procedure chosen, which defines events in the conceivable state-space.

Much of the literature in economics and decision theory dispenses with the specification of states as outcomes of feasible acts, opting instead to depict the states as background causes of the observed outcomes of the acts. Machina (2003) provides a typical example by considering the decision to install a lightning rod. The consequences are "house burns down" and "house does not burn down." He then describes three scenarios: (a) House burns down regardless of whether the lighting rod is installed or not. (b) House does not burn down if the lightning rod is installed and does burn down if it is not installed. (c) House does not burn down regardless of whether or not the lightning rod is installed. Machina attributes these outcomes to three possible natural phenomena: "big lighting strike," "small lightning strike," and "no lighting strike," which he defines as states. In our formulation there are four states, the three Machina identifies and (d) "House burns down if a lighting rod is installed and does not burn down if a lighting rod is not installed." Since installing and not installing a lighting rod are mutually exclusive acts, one can bet only on the events "House burns down" and "House does not burn down" conditional on whether the lighting rod is installed or not.

If there is no way of objectively measuring (i.e., using agreed upon instruments and method) the power of the lighting strike and verifying whether or not it occurred, there is no way of placing a bet on the states as defined by Machina. Thus, the way he defines the states have no scientific meaning. If, however, there exists an objective way of measuring the power of the lighting, then the act of measurement is added to the list of feasible acts and the state space must be redefined to include $2^3$ states (i.e., the number of mappings from the three feasible acts to the two possible consequences). If the instruments of measurement improve (i.e., additional feasible acts becomes available), additional states will be generated by the measurement acts even if the set of consequences remains unchanged (i.e., bets may be placed on the results of the measurements).

## 4.2 The role of prize-state lotteries

In the model of decision making under uncertainty of Karni and Schmeidler (2016), prize-state lotteries determine a unique state-dependent utility function and, consequently, a unique subjective probability distribution on the state-space. Karni and Mongin (2000) showed that the subjective probabilities that figure in the representation of this model capture the decision maker's beliefs about the likely realization of the states. In other words, if the preference relation exhibits state-independence and the probability measure that figures in the model of Karni and Schmeidler differs from the one that figures in the model of Anscombe and Aumann (1963), it is the former that represents the decision maker's beliefs.[12]

Invocation of the prize-state lotteries in this paper in conjunction with the consistency axiom imply that, unlike the standard model of decision making under uncertainty with incomplete preferences, the model of this paper does not admit Knightian uncertainty. The standard model admits the possibility that the source of indecisiveness is the decision maker's incomplete beliefs. Correspondingly, the representation includes a single utility function and a set of priors inducing expected utility functionals that must all agree if an act is to be ranked above another.[13]

In the model of this paper, each prior is associated with a single utility function; the model does not admit the possibility that decision makers are characterized by complete tastes and incomplete beliefs (i.e., entertain incomplete beliefs in conjunction with complete tastes). To grasp this assertion, suppose that the preference relation $\succcurlyeq^*$ is complete. Because the function $H$ is a bijection,

[12] By choosing a state-independent utility function –a choice that is not implied by the state-independent preferences – the Anscombe and Aumann model conceals the fact that utility function may be state-dependent while the decision maker's risk attitudes and, consequently, ranking of lotteries, may be state-independent. This fact and the definition of subjective probabilities in their model may result in a definition of subjective probability measure that does not represent the decision maker's beliefs. For further discussion of this point see Karni (2011).

[13] See Bewley (1986) and Karni and Galaabaatar (2013).

by the Consistency axiom, the preference relation $\succcurlyeq$ must be complete, contradicting the existence of multiple priors.

### 4.3 A comment on the motivation

To the sources of decision making indecisiveness that have been explored in the literature, this paper adds an additional source: decision makers' awareness of possibly of encountering consequences the nature of which is unknown. It seems reasonable to suppose that, being confronted with the need to choose among monetary payoffs contingent on states that affect their well-being in a way that is unpredictable, decision makers find it difficult, if not impossible, to express clear preferences.

To grasp this point, consider again the decision to vaccinate against COVID with a newly developed vaccine. If he suspect that COVID, or the vaccine or both may have unanticipated health consequences, the decision maker will be unsure how to evaluate the impact of the acts on his state of health. In addition, if his risk attitudes depend on his state of health, the decision maker will be unsure what his risk attitudes are, or ought to be. As a result, he may display decisiveness when choosing among bets conditional on the fully specified states (i.e., bets in a given $B_{\hat{b}}(E)$) and indecisiveness when choosing among unconditional bets.

## 5 Proofs

### 5.1 Proof of Theorem 1

$(i) \Rightarrow (ii)$. Let $\ell, \ell' \in \Delta(Y \times S)$ be equal outside $s$, $b = H(\ell)$ and $b' = H(\ell')$. Then $b$ and $b'$ are equal outside $s$. By definition, for every $s \in S$, $b(y, s) = H(\ell(y, s)) = \frac{\ell(y,s)}{\sum_{y' \in Y} \ell(y',s)}$. But $\ell$ equals $\ell'$ outside $s$ implies that $\sum_{y' \in Y} \ell(y', s) = \sum_{y' \in Y} \ell'(y', s) := k(s) > 0$, where the inequality is implied by semipositivity of $\ell$ and $\ell'$. For each $\ell \in \Delta(Y \times S)$ and $s \in S$ , define $\hat{\ell}(\cdot, s) \in \Delta(Y)$ as follows: $\hat{\ell}(y, s) = \ell(y, s)/k(s)$.

The preference relation $\succcurlyeq$ is reflexive and transitive, satisfying the Archimedean and Independence axioms. Hence, by the BRT, it satisfies (2). Substituting the definition of $H(\ell(y, s))$ in (2) we get that, for all $\ell$ and $\ell'$ that are equal outside $s$ and $b = H(\ell), b' = H(\ell')$ ,

$$b \succcurlyeq b' \Leftrightarrow \sum_{y \in Y} w(y, s)[\hat{\ell}(y, s) - \hat{\ell}'(y, s)] \geq 0, \forall w \in \mathcal{W}. \tag{12}$$

By Consistency, $b \succcurlyeq b'$ implies that $\ell \succcurlyeq^* \ell'$. Hence, by (1) and the definition of $\hat{\ell}(\cdot, s)$, it holds that

$$\ell \succcurlyeq^* \ell' \Leftrightarrow \sum_{y \in Y} u(y, s)[\ell(y, s) - \ell'(y, s)] \geq 0, \forall u \in \mathcal{U}. \tag{13}$$

By the uniqueness of the representations, (1) and (2) hold with $\langle \mathcal{U} \rangle$ and $\langle \mathcal{W} \rangle$ instead of $\mathcal{U}$ and $\mathcal{W}$, respectively, and $\langle \mathcal{W} \rangle = \langle \mathcal{U} \rangle$. With some abuse of notation, let $\mathcal{W}$ and $\mathcal{U}$ be the sets of extreme rays that generate cone(W) and cone(U). Hence, $\mathcal{W}$ and $\mathcal{U}$ consist of the same functions. Thus, there is a map $\varphi : \mathcal{W} \to \mathcal{U}$ such that $w$ and $u = \varphi(w)$ agree on the ranking of the elements of $\Delta(Y)$ (i.e., they are the same utility representations of the preferences restricted to $\Delta(Y)$).

Fix $w \in \mathcal{W}$ and let $u = \varphi(w)$. We need to show that there are $\alpha > 0$, $\beta : S \to \mathbb{R}$ and $\pi^u : S \to \mathbb{R}_+$ such that

$$\pi^u(s)u(y,s) = \alpha w(y,s) + \beta(s).^{14}$$

Nondegeneracy of $\succcurlyeq$ implies that there exists a state $t$ such that, $\max_{y \in Y} w(y,t) > \min_{y \in Y} w(y,t)$. Let $\underline{y}$ and $\overline{y}$ be such that $w(\underline{y},t) = \min_{y \in Y} w(y,t)$, and $w(\overline{y},t) = \max_{y \in Y} w(y,t)$. Let $b$ and $b'$ be equal outside $t$, and $b(\overline{y},t) = 1 = b'(\underline{y},t)$. Then

$$\Sigma_{s \in S} \Sigma_{y \in Y} w(y,s)[b(y,s) - b'(y,s)] = w(\bar{y},t) - w(\underline{y},t) > 0. \tag{14}$$

Because $w(\cdot,\cdot)$ represents $\succcurlyeq$, $b \succ b'$. Since the function $H$ maps $\Delta(Y \times S)$ onto $B$, any $\hat{b} \in B$ is an image of an $\hat{\ell} \in \Delta(Y \times S)$ defined by dividing all values of $\hat{b}$ by $\mid S \mid$ (i.e., for all $(y,s) \in (Y \times S)$, $\hat{\ell}(y,s) = \hat{b}(y,s)/ \mid S \mid$.) Deriving in this way $\ell^b, \ell^{b'} \in \Delta(Y \times S)$ from $b$ and $b'$, we get that $\ell^b$ and $\ell^{b'}$ are equal outside $t$. By Consistency $\ell^b \succ \ell^{b'}$ and representation (13), $u(\underline{x},t) < u(\overline{x},t)$. Hence there are $\theta, \tau \in \mathbb{R}$, $\theta > 0$ such that, $\theta w(\bar{y},t) + \tau = u(\bar{y},t)$ and $\theta w(\underline{x},t) + \tau = u(\underline{y},t)$.

Define $w^1$ on $Y$ by

$$w^1(y,s) = \begin{cases} \theta w(y,s) + \tau & s = t \\ \theta w(y,s) & s \neq t \end{cases}$$

By the uniqueness of the representation in the BRT, (2), $w^1$ and $w$ are two representations of $\succcurlyeq$.

**Claim:** For all $y \in Y$, $w^1(y,t) = u(y,t)$.

Proof of Claim: By construction, $w^1(y,t) = u(y,t)$ holds for $y = \underline{y}$ and $y = \overline{y}$. Let $y \in Y \diagdown \{\underline{y}, \overline{y}\}$. By way of negation, suppose that $w^1(y,t) < u(y,t)$. (The opposite inequality is treated analogically.) Since $w^1(\underline{y},t) \leq w^1(y,t) \leq w^1(\overline{y},t)$, at least one of these inequalities is strict.

Assume that $w^1(y,t) < w^1(\overline{y},t)$, and choose a $\mu \in (w^1(y,t), u(y,t))$. Define $b$ and $b'$ in $B$ to be equal outside $t$, and for $y \in Y$

$$b(x,t) = \begin{cases} \mu & y = \overline{y} \\ 1 - \mu & y = \underline{y} \\ 0 & y \neq \underline{y}, \overline{y} \end{cases}$$

$$b'(y',t) = \begin{cases} 1 & y' = y \\ 0 & y' \neq y \end{cases}$$

Evaluating $b$ and $b'$ with $w^1$ we get that $\mu > w^1(y,t)$. Thus $b \succ b'$. Defining $\ell^b, \ell^{b'} \in \Delta(Y \times S)$ by $\ell^b(x,s) = b(x,s)/ \mid S \mid$, and $\ell^{b'}(x,s) = b'(x,s)/ \mid S \mid$; and evaluating $\ell^b$ and $\ell^{b'}$ with $u$ we get the opposite inequality (i.e., $\mu < u(y,t)$). Thus, $\ell^{b'} \succ^* \ell^b$. Since $H(\ell^b) = b$ and $H(\ell^{b'}) = b'$, this contradicts Consistency.

If $w^1(y,t) > w^1(\overline{x},t)$ a similar construction leads to a contradiction. This concludes the proof of the claim. $\square$

At this stage, we have $w^1$ representing $\succcurlyeq$ . The valuating function, $w^1$ differs from the one derived by the BRT from the axioms on $\succcurlyeq$ in two respects. First, $w^1(\cdot,s) = \theta w(\cdot,s)$, for all $s \neq t$. Second, $w^1(\cdot,t) = \theta w(\cdot,t) + \tau = u(\cdot,t) = q(t)u(\cdot,t)$, where $q(t) = 1$.

[14] The argument that follows reproduces the proof of Theorem 2 in Karni and Schmeidler (2016). It is brought up here to make the exposition self-contained.

Next assume that there is another $\succcurlyeq$-nonnull state, $r$ in $S$. Let $\underline{y}(r)$ and $\overline{y}(r)$ be a minimizer and a maximizer of the function $w^1(\cdot, r)$ on $Y$.

Constructing the appropriate acts and the corresponding lotteries, we conclude (via Consistency) that $u(\underline{x}(r), r) < u(\overline{x}(r), r)$. Hence there are $\theta(r), \tau(r) \in \mathbb{R}$, $\theta(r) > 0$ such that, $\theta(r)w^1(\bar{x}, t) + \tau(r) = u(\bar{x}, t)$ and $\theta(r)w^1(\underline{x}, t) + \tau(r) = u(\underline{x}, t)$. Denote $q(r) = \theta(r)$ and define $w^2$ on $Y$ by

$$w^2(y, s) = \begin{cases} q(r)w^1(y, s) + \tau(r) & s = r \\ q(r)w^1(y, s) & s \neq r \end{cases}$$

It is easy to see that the Claim holds when $r$ replaces $t$. Thus, $w^2$ represents $\succcurlyeq$, and for $s \in \{t, r\}$, $w^2(\cdot, s) = q(s)u(\cdot, s)$. This procedure can be applied to all $\succcurlyeq$-nonnull states. For any $\succcurlyeq$-null state, $\hat{s} \in S$, define $q(\hat{s}) = 0$ and let $w(\cdot, \hat{s}) = u(\cdot, \hat{s})$. Normalizing the vector $q$ yields representation (4) in *(ii)*, where $\pi^u(s) = q(s)/\sum_{s' \in S} q(s')$ for all $s \in S$.

Repeating the same argument for each $w \in \mathcal{W}$ and $u = \varphi(w)$, we get that each $w \in \mathcal{W}$ corresponds to a function $u = \varphi(w)$ and a probability distribution $\pi^u$ on $S$. Let $\mathcal{U} := \{u = \varphi(w) \mid w \in \mathcal{W}\}$.

This complete the proof that (i) implies (ii).

The proof that (ii) implies (i) is straightforward. The uniqueness of the representation is implied by the uniqueness theorem of Dubra et al. (2004).

To show that *(ii)* implies *(iii)* it suffices to observe that a state $s$ is $\succcurlyeq$- null if and only if at least one of the following conditions holds: (A) for all $y, y' \in Y$, $u(y, s) = u(y's), \forall u \in \mathcal{U}$ or (B) $\pi^u(s) = 0, \forall u \in \mathcal{U}$. The representation of $\succcurlyeq^*$ implies that a state $s$ is $\succcurlyeq^*$-null if and only if $u(y, s) = u(y's), \forall u \in \mathcal{U}$ (i.e., if and only if (A) holds). Hence, if $s$ is $\succcurlyeq$-null and $\succcurlyeq *$-nonnull it must hold that $\pi^u(s) = 0, \forall u \in \mathcal{U}$. Hence, *(iii)(a)* holds. If $s$ is $\succcurlyeq$- nonnull, then (B) does not hold. Hence, $\pi^u(s) > 0, \forall u \in \mathcal{U}$. $\square$

## 5.2 Proof of Theorem 2

*(i)* $\Rightarrow$ *(ii)*. Let $\succcurlyeq$ on $B$ and $\succcurlyeq^*$ on $\Delta(Y \times S)$ be binary relations. Denote by $E$ the set of fully describable states.

Define $L_{\hat{\ell}}(E) = \{\ell \in \Delta(Y \times S) \mid \ell(y, s) = \hat{\ell}(y, s), \forall (y, s) \in Y \times E^c\}$ (i.e., $L_{\hat{\ell}}(E)$ is the set of prize-state lotteries that are equal to $\hat{\ell}$ on $E^c$. Then $L_{\hat{\ell}}(E)$ and $B_{\hat{b}(E)}$ are convex subsets of linear spaces.

Fix $\hat{b} \in B$, and $\hat{\ell} \in \Delta(Y \times S)$ and denote by $\succcurlyeq_E$ and $\succcurlyeq^*_E$ the restrictions of $\succcurlyeq$ to $B_{\hat{b}}(E)$ and $\succcurlyeq^*$ to $L_{\hat{\ell}}(E)$. By the separability imposed by the Independence axiom, the restricted preference relations are independent of the choice of $\hat{b}$.

By Partial Completeness and Consistency, $\succcurlyeq_E$ and $\succcurlyeq^*_E$ are weak-orders (i.e., complete and transitive) preference relations satisfying Archimedean and Independence. By the BRT, there exist real-valued functions $w$ and $\hat{u}$ on $Y \times S$ such that, for every $\hat{b}$ and all $b, \hat{b} \in B_{\hat{b}}(E)$

$$b \succcurlyeq_E b' \Leftrightarrow \sum_{s \in E} \sum_{y \in Y} w(y, s)b(y, s) \geq \sum_{s \in E} \sum_{y \in Y} w(y, s)b'(y, s). \tag{15}$$

and for all $\ell . \ell' \in L_{\hat{\ell}}(E)$

$$\ell \succcurlyeq^*_E \ell' \Leftrightarrow \sum_{s \in E} \sum_{y \in Y} \hat{u}(y, s)\ell(y, s) \geq \sum_{s \in E} \sum_{y \in Y} \hat{u}(y, s)\ell'(y, s). \tag{16}$$

Moreover, both $w$ and $\hat{u}$ are unique up to multiplication by a positive constant and addition of state-dependent numbers.

By the proof of Theorem 2 in Karni and Schmeidler (2016), there exists a unique set of nonnegative numbers $\{\hat{q}(s) \mid s \in E\}$ such that

$$\sum_{s\in E}\sum_{y\in Y} w(y,s)b(y,s) = \sum_{s\in E}\sum_{y\in Y} q(s)\hat{u}(y,s)b(y,s).$$

Normalizing we get that

$$\sum_{s\in E}\sum_{y\in Y} w(y,s)b(y,s) = \sum_{s\in E}\sum_{y\in Y} \pi_E(s)\hat{u}(y,s)b(y,s),$$

where $\pi_E$ is unique conditional probability distribution on $E$. Consequently, for all $b, b' \in B_{\hat{b}}(E)$,

$$b \succcurlyeq_E b' \Leftrightarrow \sum_{s\in E}\pi_E(s)\sum_{y\in Y}\hat{u}(y,s)[b(y,s) - b'(y,s)] \geq 0. \tag{17}$$

Define $L_{\hat{\ell}}(E^c) = \{\ell \in \Delta(Y \times S) \mid \ell(y,s) = \hat{\ell}(y,s), \forall (y,s) \in Y \times E\}$ (i.e., $L_{\hat{\ell}}(E^c)$ as the set of prize-state lotteries that are equal to $\hat{\ell}$ on $E$). Similarly, define $B_{\hat{b}(E^c)} = \{b \in B \mid b(y,s) = \hat{b}(y,s), \forall (y,s) \in Y \times E\}$. Then $L_{\hat{\ell}}(E^c)$ and $B_{\hat{b}}(E^c)$ are convex subsets of linear spaces.

Fix $\hat{b} \in B$ and $\hat{\ell} \in \Delta(Y \times S)$ and denote by $\succcurlyeq_{E^c}$ and $\succcurlyeq^*_{E^c}$ the restrictions of $\succcurlyeq$ to $B_{\hat{b}}(E^c)$ and $\succcurlyeq^*$ to $L_{\hat{\ell}}(E^c)$.

By Theorem 1 there is a set, $\mathcal{U}$, of real-valued functions on $Y \times E^c$ such that it holds that for all $b, b' \in B_{\hat{b}}(E^c)$,

$$b \succcurlyeq_{E^c} b' \Leftrightarrow \sum_{s\in E^c}\sum_{y\in Y}\pi^u_{E^c}(s)u(y,s)[b(y,s) - b'(y,s)] \geq 0, \forall u \in \mathcal{U}, \tag{18}$$

and for all $\ell, \ell' \in L_{\hat{\ell}}(E^c)$,

$$\ell \succcurlyeq^*_{E^c} \ell' \Leftrightarrow \sum_{s\in E^c}\sum_{y\in Y} u(y,s)[\ell(y,s) - \ell'(y,s)] \geq 0, \forall u \in \mathcal{U}. \tag{19}$$

Let $\pi(E) := \sum_{s\in E}\pi_E(s)$, and for every $u \in \mathcal{U}$, let $\pi^u(E^c) := \sum_{s\in E^c}\pi^u_{E^c}(s)$. Define $\pi(s) = \pi(E)\pi_E(s)$ for all $s \in E$ and $\pi^u(s) = \pi(E^c)\pi^u_E(s)$ for all $s \in E^c$ and $u \in \mathcal{U}$. Then for all $b, b \in B$

$$b \succcurlyeq b' \Leftrightarrow \sum_{s\in E}\pi(s)\sum_{y\in Y}\hat{u}(y,s)[b(y,s) - b'(y,s] + \sum_{s\in E^c}\pi^u(s)\sum_{y\in Y}u(y,s)[b(y,s) - b'(y,s] \geq 0, \forall u \in \mathcal{U}. \tag{20}$$

Combining (16) and (19) we get

$$\ell \succcurlyeq^*_E \ell' \Leftrightarrow \sum_{s\in E}\sum_{y\in Y}\hat{u}(y,s)[\ell(y,s) - \ell'(y,s)] + \sum_{s\in E^c}\sum_{y\in Y}u(y,s)[\ell(y,s) - \ell'(y,s)] \geq 0, \forall u \in \mathcal{U}. \tag{21}$$

This completes the proof that $(i)$ implies $(ii)$.

Another pair $(\hat{v}, \mathcal{V})$ of a real-valued function of $Y \times E$ and a set of real-valued functions on $Y \times E^c$ represents the preferences relation $\succcurlyeq$ and $\succcurlyeq^*$ in the sense above, if and only if, by the uniqueness of the expected utility representation $\hat{v} = \beta\hat{u} + \alpha, \beta > 0$ and, by Theorem 1, $\langle\mathcal{V}\rangle = \langle\mathcal{U}\rangle$.
□

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