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On a diversity of perspectives and world views: Learning under Bayesian vis-á-vis DeGroot updating☆

Aniruddha Ghosh *, M. Ali Khan

Department of Economics, Johns Hopkins University, Baltimore, MD 21218, United States of America

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ABSTRACT

In an influential recent paper, Mailath–Samuelson formalize learning and reasoning through "model-based inference" and Bayesian updating. In this announcement, we substitute DeGroot's heuristic for Bayesian updating by (i) furnishing a plausible interaction matrix that agents use to weigh each other's beliefs, and by (ii) using this matrix to derive properties of the process for the DeGroot updating of beliefs by agents and oracles. The alternative argumentation that we provide facilitates bridging the literature on networks and that on model-based *learning* and *inference*; and it identifies productive and ongoing directions for further investigation.

> A crowd, Le Bon argued, was more than just the sum of its members ... a kind of independent organism [with] an identity and a will of its own, [acting] in ways that no one within the crowd intended. [It] can never accomplish acts demanding a high degree of intelligence, [and is] always intellectually inferior to the isolated individual. Gustave Le Bon had things exactly backward.
>
> [Surowiecki (2005)][1]

1. Introduction

How does a crowd become a mob and a protest become a riot? How does a rumor get translated into purposeful mass action? How does a crowd form in the first place? Is it solely a matter of information aggregation? These questions have been addressed in both sociology and anthropology in the classic and time-honored works of Mackay and Le Bon, and more recently by literary criticism pertaining to the English literature of the Romantic and Victorian periods in the 19th century.[2] However, one can argue that they are also fundamental to cooperative game theory and to the economics of information: they lie at the heart of the Nash program of providing non-cooperative foundations to cooperative solutions. However, as vernaculars go, the word *crowd* has been used in economics more as a verb than as a noun,[3] and an exception to this is only in the recent work of Mailath and Samuelson (2020) (hereafter MS) that can be read as taking the aphorism "two is a company and three is a crowd" as its point of departure.

In their stimulating contribution, MS write:

> [P]eople work with models that are deliberately incomplete, including the most salient variables and excluding others Different people work with different models. Civil engineers building bridges and electrical engineers designing quantum computers persist with models that are incomplete.

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* Corresponding author.
E-mail addresses: aghosh23@jh.edu (A. Ghosh), akhan@jh.edu (M.A. Khan).

1 See his Introduction and also the Afterword. In the latter, he writes, "Even after I'd written an entire book about collective wisdom, there was still a part of me that instinctively questioned whether it would really work. This story is, I think, a testament to how uncanny-and counterintuitive-the wisdom of crowds really is." The phrase "wisdom of crowds" is used as a section heading in Mailath and Samuelson (2020), and in the title of Golub and Jackson (2010).

2 See Plotz (2000) and Menschel (2002) and their references to Mackay, Le Bon and others.

3 A quick search on Google Ngram shows the peak usage of the term 'crowding-out' to be in 1984. We also invite the reader to search for 'crowding-out' on JSTOR and browse the papers also with 'crowding-in', both being qualified by adjectives such as, 'dysfunctional' and 'financial'

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With this difference in individual perspectives as their underlying subtext, the authors ask how this difference is mediated by an exchange of information and beliefs when the agents are,"hampered by different incomplete views of the world"?

> People routinely interact, exchanging information and beliefs. These exchanges seldom lead to complete agreement, but people do learn from each other. How do they do this when hampered by different incomplete views of the world? We address this question by developing and analyzing a model of "model-based inference".

In this exploratory announcement, we complement the MS conception by asking how it would be transformed by a non-Bayesian perspective, by the substitution of DeGroot updating for the Bayesian one.[4] The results are of substantive interest: not only are the MS results overturned but intuitions such as of "oracular beliefs" take alternative formulations with theorems in one becoming trivialities in the other.

Our argumentation revolves around the construction of a plausible weighting matrix that captures *model-based reasoning* and interaction. We are also motivated by some recent empirical evidence in Giacomini et al. (2020) suggesting non-Bayesian updating by *inattentive* agents in crisis years.[5] For specificity, note that Berger (1981) presents a necessary and sufficient condition under which a consensus on a common subjective probability distribution for an unknown parameter by a group of k individuals will be reached by using DeGroot's method.[6] In more recent work, (Romeijn and Roy, 2018), study DeGroot updating as "iterated linear pooling: upon learning what others believe each agent forms her new belief by taking a linear combination of the opinions of herself and of others, weighted by how much she trusts or respects them. By iterating this process sufficiently often the agents will converge to a fixed point in the space of opinions".[7]

The paper is structured as follows. Section 2 provides the conceptual preliminaries. Section 3 constructs the weighting matrix, and Section 4 presents the results: after cataloging the basic properties of the matrix in with Propositions 1 and 2 addresses model-based beliefs, updated under DeGroot's heuristic, and Proposition 3 provides a condition for the "wisdom of the crowd" to hold. Section 5 replicates an MS-example adapted to our framework. Section 6 concludes by indicating directions for further and ongoing research.

2. Conceptual preliminaries

In keeping with the MS-notation, let $X \subset \mathbb{R}$ denote the finite set of consequences, $N \subset \mathbb{N}$ to be finite and $\Omega = X^N$, the states of the world. Nature draws a state ω from Ω according to a prior ρ on Ω. An event $F \subset \Omega$ is represented by χ_F,

$$\chi_F(\omega) = \begin{cases} 1 & \omega \in F \\ 0 & \omega \notin F \end{cases}. \tag{1}$$

Next, consider a group of $K = \{1, 2, \dots, k\}$ agents (finite) where each agent $i \in K$ is characterized by a model $M_i \subseteq N$, information set $I_i \subseteq M_i$ and a corresponding theory f^i which are used to form beliefs about the occurrence of event F. That is, each agent i is characterized by a triplet of objects, (M_i, I_i, f^i) where

$$f^i : X^{M_i} \times X^{N-M_i} \to [0, 1]. \tag{2}$$

For agent i, his model M_i partitions the full state space X^N into equivalence classes, of the form $\{\omega_{M_i}\}\times X^{N-M_i}$. Therefore, for notational convenience, we shall only mention the model(information) space and its realization in the argument of agent's theory. Next, we state a key assumption in MS that establishes the consistency between agent i's theory, f^i and the event F indicator function, χ_F.

Assumption (*MS*)**.** The probability agent $i's$ theory forms of the event F conditional[8] on the variables in M_i is given by,

$$f^i(\omega_{M_i}) = \sum_{\omega\in\Omega} \chi_F(\omega)\rho(\omega|\{\omega_{M_i}\} \times X^{N-M_i}). \tag{3}$$

Each agent i then generates their interim belief of an event F by updating their theory f^i with respect to their information set I^i, i.e. the interim belief is a mapping from $X^{I_i} \times X^{N-I_i}$ to $[0, 1]$ and is calculated as,

$$\beta_i(\omega_{I_i}) = \sum_{\omega\in\Omega} f^i(\omega_{M_i})\rho(\omega|\{\omega_{I_i}\} \times X^{N-I_i}), \tag{4}$$

where $\rho(\omega|\{\omega_{I_i}\} \times X^{N-I_i})$ is the prior ρ on Ω but now conditional on ω_{I_i} (for which $\rho(\{\omega_{I_i}\} \times X^{N-I_i}) > 0$).[9] The interim beliefs for k agents is represented by a $k \times 1$ vector, $\beta(\omega)$ where $\beta(\omega) = (\beta_1(\omega_{I_1}), \dots, \beta_k(\omega_{I_k}))^T$. That is, for each $\omega \in \Omega$, we have a *subjective* probability distribution $\beta_i(\omega_{I_i})$ of the event F for each agent $i \in K$ based on their theory, model and information available.

Once interim beliefs about the event F are formed, agents simultaneously and truthfully announce them.[10] However, instead of updating their interim beliefs in a *Bayesian* manner as in MS, we make our agents update them by *suitably* weighting all the announcements àla *DeGroot's* heuristic. More formally, for $i, j \in K$ let t_{ij} denote the weight that i assigns to the announcement of j. Further, $t_{ij} \geqslant 0$ for every value of i and j, and $\sum_{j=1}^k t_{ij} = 1$ for each $i \in K$. Upon hearing the announcements of interim beliefs of all the other agents, agent i revises its own distribution in the first round from $\beta_i(\omega_{I_i})$ to

$$\beta_{i1}(\omega) = \sum_{j=1}^{k} t_{ij}\beta_j(\omega_{I_j}), \tag{5}$$

where $\beta_{i1}(\cdot)$ denotes the belief of the ith agent at round 1. Therefore, using (6), for each $\omega \in \Omega$,

$$\beta^1(\omega) = \mathbf{T}\beta(\omega), \tag{6}$$

where $\beta^1(\omega) = (\beta_{11}(\omega), \dots, \beta_{k1}(\omega))^T$ is the first-round update of the K agents, $\beta(\omega) = (\beta_1(\omega_{I_1}), \dots, \beta_k(\omega_{I_k}))^T$ is the k-dimensional vector of interim beliefs and $\mathbf{T} = (t_{ij})_{k\times k}$ is a matrix of weights. Further, iterating on (6), the nth round of updating is,

$$\beta^n(\omega) = \mathbf{T}\beta^{n-1}(\omega) = \mathbf{T}^n\beta(\omega). \tag{7}$$

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4 In addition to DeGroot (1974), Berger (1981) and Molavi et al. (2018).

5 Already also noted by Mailath and Samuelson (2020, p. 1772).

6 For the record, he also corrects the original paper, and emphasizes it easy verifiability.

7 Romeijn and Roy (2018) compare the two updating methods, and write "In Bayesian models of agreement, upon learning what the others believe, [and] under the assumption of a common starting point and under the further assumption that agents know the type of information, but not the information content, iteratively announcing the posteriors will lead the agents to agree;" in addition to Aumann, see their reference to Geanakoplos and Polemarchakis (1982).

8 This is defined for every ω_{M_i} for which $\rho(\{\omega_{M_i}\} \times X^{N-M_i})$ is positive where $\rho(\omega|\{\omega_{M_i}\} \times X^{N-M_i})$ is the prior ρ on Ω conditional on ω_{M_i}.

9 The process to form interim beliefs is Bayesian. We interpret it this way. While the agent is sophisticated in deciphering his information and model, he is not sophisticated enough (naive) to process the announcements of the other agents in a Bayesian way.

10 This popular information exchange protocol is due to Geanakoplos and Polemarchakis (1982).

Next, we adapt the definition of the different types of oracles as introduced in MS.

Definition (*Oracles*)**.** An agent with a state space Ω with a prior distribution ρ, is said to be an oracle if the number of variables in his model equals N, that is, an agent's theory for an event F is the indicator function, χ_F. Further, an oracle is said to be a/an

(i) omniscient oracle if he knows the realization of the state;
(ii) universal oracle if he has access to the information sets of all the k-agents, i.e. the information set for the universal oracle, I_u, equals $\bigcup_{i=1}^k I_i$;
(iii) agent-i oracle if he has access to the information set of the ith agent and the entire sequence of announcements, where his oracular belief at the nth round is given by, $\beta_a^n(\omega) = \sum_{j=1}^k t_{ij}^a \beta_{jn}(\omega)$, $t_{ij}^a = (t_{i1}^a, \ldots, t_{ik}^a)$ is the vector of weights attached to the announcements of the k-agents;
(iv) public oracle if he has access to the entire sequence of announcements but no access to any information set, where his belief at the nth round is given by $\beta_p^n(\omega) = \sum_{j=1}^k t_j^p \beta_{jn}$ (ω), $t_j^p = (t_1^p, \ldots, t_k^p)$ is the vector of weights attached to the announcements of the k-agents.[11]

In (ii), we denote the universal oracular's belief by $\beta_u(\omega_{I_u}) \equiv \sum_{\omega\in\Omega} \chi_F \rho(\omega|\{\omega_{I_u}\} \times X^{N-I_u})$.

3. A plausible weighting matrix

In this section, we construct a plausible weighting matrix $\mathbf{T_A}$ needed for DeGroot's averaging method, based on the interaction environment of MS. $\mathbf{T_A}$ is reasonable in the sense that instead of placing weights arbitrarily on each others' announcements, agents derive them from each others' models and their respective information structures.

A group of k agents represented by a symmetric k-by-k matrix $\mathbf{A}$ with entries in $\{0, 1\}$. The entries in matrix $\mathbf{A}$ are defined as,

$$\mathbf{A} = (a_{ij})_{k\times k} = \begin{cases} a_{ij} = 1, & M_i \cap M_j \neq \emptyset \\ a_{ij} = 0, & M_i \cap M_j = \emptyset. \end{cases} \tag{8}$$

An entry a_{ij} in the matrix $\mathbf{A}$ takes value 0 or 1 depending on whether agents i and j share any component of their model in common. If $a_{ij} = 0$, then agent i and j have nothing to learn from each other's announcements and hence weights allotted to each other's announcements will be zero. If $a_{ij} = 1$, then agent i and j *may* have variables to learn from each other's announcements and weights accorded to each other's announcements *may* be positive.[12] But we first furnish an assumption that is crucial for this construction.

Assumption 1. MS offer two interpretations for its modeling framework. Their first interpretation is that of *"unknown sense"*, that is, agent i need know nothing about j's model. The other interpretation is of *"known nonsense"* in which each agent knows the models of other agents. We rely on the second interpretation here. Further, we assume that the agents know the variables contained in the information sets of the other agents but certainly they do not know the realization of those variables.

We construct $\mathbf{T_A}$ using the following three steps.[13]

(i) **Step 1:** Individual i generates a feasible model space, M_i^f where $M_i^f = M_i \cap (\bigcup_{i=1}^k I_i)$. M_i^f is the set containing those variables in i's model about which i can learn from the k information sets available. If $M_i = N$, then $N^f = N \cap (\bigcup_{i=1}^k I_i) = (\bigcup_{i=1}^k I_i)$ since $\bigcup_{i=1}^k I_i \subseteq N$.[14]
(ii) **Step 2:** Agent i places a weight $t_{ii} = \dfrac{|I_i|}{|M_i^f|}$ on his own announcement where $|\cdot|$ is the cardinality of the set. That is, agent i checks how many variables it has in its information set and normalizes this by the cardinality of the feasible model space.[15]
(iii) **Step 3:** Next, agent i allots the remaining weight $\left(1 - \dfrac{|I_i|}{|M_i^f|}\right)$ to the rest of the $k-1$ agents using the following method. For each agent $j \in K \setminus \{i\}$, i calculates the number of variables common between his feasible model space adjusted for his information set and j's information set. This is normalized by summing the cardinality of such common variables across all j's. We repeat this process for all k agents.

We formalize the above construction in Definition 1 with the formalization of belief convergence in Definition 2 using the preliminaries presented in Section 2.

Definition 1. Given a matrix $\mathbf{A}$, the entries of the weighting matrix $\mathbf{T_A}$ depend on (I, M) and they are calculated as

$$\mathbf{T_A}(I, M) = (t_{ij})_{k\times k} = \begin{cases} \dfrac{|I_i|}{|M_i^f|}, & j = i \\ \left(1 - \dfrac{|I_i|}{|M_i^f|}\right) \dfrac{|(M_i^f - I_i) \cap I_j|}{\sum_{j\in K} |(M_i^f - I_i) \cap I_j|} & j \neq i, \end{cases} \tag{9}$$

where I and M denote the collection of k information sets and k models, respectively. For the rest of our analysis, to keep the notation consistent with Section 2 we refer to $\mathbf{T_A}(I, M)$ as $\mathbf{T}$, unless specified.[16]

Definition 2. For each $\omega \in \Omega$, the beliefs of the k agents is said to converge to each other if the weighting matrix $\mathbf{T}$ converges, i.e., $\lim_{n\to\infty} \beta^n(\omega) = \lim_{n\to\infty} \mathbf{T}^n \beta(\omega) = \mathbf{T}^* \beta(\omega)$, where $\lim_{t\to\infty} \mathbf{T}^n = \mathbf{T}^*$ and $\mathbf{T}^*$ is the matrix of limiting weights with each of its rows given by $(t_1^*, t_2^* \ldots t_k^*)$.

In keeping with this definition, limit beliefs are said to converge (agree) if for each ω, the common subjective distribution of each of the k members of the group, $\beta^*(\omega)$ equals the product of the limiting weights and the interim beliefs, i.e., $\beta^*(\omega) = \sum_{i=1}^k t_i^* \beta_i(\omega_{I_i})$.

11 In MS, the corresponding beliefs of a universal oracle, an agent-i oracle and public oracle are given by $\mathbb{E}[\chi_F|\mathcal{I}_1, \ldots, \mathcal{I}_k], \mathbb{E}[\chi_F|\mathcal{I}_i, \mathcal{B}_n]$ and $\mathbb{E}[\chi_F|\mathcal{B}_n]$, respectively where $\mathcal{B}_n$ is the sigma algebra induced by the nth round of announcements. The limiting beliefs, therefore, for agent-i oracle and public oracle are given by $\mathbb{E}[\chi_F|\mathcal{I}_i, \mathcal{B}_\infty]$ and $\mathbb{E}[\chi_F|\mathcal{B}_\infty]$, respectively. We will shortly define the limit beliefs in this context. The oracular beliefs in parts (iii) and (iv) are defined for $n > 1$. For $n = 1$, it is simply the interim beliefs weighted by the vectors defined.

12 If agent i is fully informed about his model, then according to our construction even if agents i and j have a common model, i assigns zero weight to j's announcements since he is fully informed.

13 It is assumed that $M_i^f \neq \emptyset$ since the setup demands that there is some learning (information needed for models) through interaction between the agents.

14 Therefore, we can conveniently derive the weighting vectors for different oracles.

15 This setup can be easily extended to the case where variables in the models are weighted unequally, say due to payoff considerations.

16 Later in the analysis, we evaluate $\mathbf{T}$ matrix for specific instances of I and activate dependence on it accordingly. However we drop $\mathbf{A}$ throughout.

We end this section by two remarks. Remark 1 specifies a weighting matrix that we would need for one of our main results. Remark 2 describes the weighting vectors for agent-i oracle and the public oracle.

Remark 1. If $I_i = \emptyset$, then by (9) weight put on i's belief by all the agents is zero. Therefore, the ith column of such a matrix is a column of zeros. We denote such an instance of weighting matrix with $\mathbf{T}(I_i = \emptyset, I_{-i}, M)$ by $\mathbf{T}'$.

Remark 2. From (9), the weighting vectors, t^a_{ij} and t^p_j, of the agent oracles and the public oracles respectively are as follows:

$$t^a_{ii} = \frac{|I_i|}{|N^f|},$$
$$t^a_{ij} = \left(1 - \frac{|I_i|}{|N^f|}\right) \frac{|(N^f - I_i) \cap I_j|}{\sum_{j \in K} |(N^f - I_i) \cap I_j|} \text{ for } j \in K \setminus i, \text{ and}$$
$$t^p_j = \frac{|N^f \cap I_j|}{\sum_{j \in K} |N^f \cap I_j|} \text{ for } j \in K.$$

4. Main results

In this section, we present three propositions, using the conceptual setup and notation developed above, that sharply contrast the MS-results by the substitution DeGroot's heuristic for of Bayesian updating. Proposition 1 presents the properties of our weighting matrix, Proposition 2 provides properties of model-based beliefs, and Proposition 3 provides a condition for the "wisdom of the crowd" result in MS to hold in our alternative setup.

Proposition 1. *The matrix* $\mathbf{T}$ *has the following properties.*

(a) If agents i and j have disjoint models, then the weights assigned to each other's announcements is zero i.e. if $a_{ij} = 0$*, then* $t_{ij} = t_{ji} = 0$ *for* $i, j \in K$.

(b) The elements of $\mathbf{T}$ *have values between* 0 *and* 1*, i.e.,* $0 \leqslant t_{ij} \leqslant 1$ *for* $i, j \in K$.

(c) $\mathbf{T}$ *is a row-stochastic matrix, i.e.,* $\sum_{j=1}^{k} t_{ij} = 1$ *for all* $i \in K$.

(d) If the information sets of all the agents contain the information needed for their feasible model space, i.e., $|I_i| = |M_i^f|$ *for all i then* $\mathbf{T}$ *is an identity matrix.*

Proof of Proposition 1. (a). If $a_{ij} = 0$ then $(M_i^f - I_i) \cap I_j = \emptyset$ since $I_j \subseteq M_j$ and $M_i \cap M_j = \emptyset$, (b), (c) and (d) follow from the construction in (9). ■

With the properties of $\mathbf{T}$ established, we turn to the properties of the beliefs under DeGroot's heuristic.

Proposition 2. *Model-based beliefs under DeGroot's heuristic have the following properties.*

(a) If $\mathbf{T}$ *is an irreducible matrix, then* $\mathbf{T}$ *converges to* $\mathbf{T}^*$*(updating process terminates) for all* $\omega \in \Omega$ *if and only if* $\mathbf{T}$ *is aperiodic.*[17]

(b) If $\mathbf{T}$ *converges to* $\mathbf{T}^*$ *for all* $\omega \in \Omega$*, limiting beliefs of all the agents converge and equal* $\beta^*(\omega)$*. Moreover, agent's limit belief equals public oracular belief and agent-i oracular belief, that is,* $\beta^*(\omega) = \sum_{j=1}^{k} t^p_j \beta^*(\omega) = \sum_{j=1}^{k} t^a_{ij} \beta^*(\omega)$*. Furthermore, if* $\mathbf{T}'$ *converges to* $\mathbf{T}'^*$ *and* $t^*_i \beta_i(\omega_{I_i}) = \sum_{j \in K/\{i\}} (t^*_j - t'^*_j) \beta_j(\omega_{I_j})$*, then agent i's private information is pooled in the limit.*

[17] Irreducibility and aperiodicity are standard terms in Markov chain theory; See Chapters 4 and 5 in Meyn and Tweedie (2012).

Proof of Proposition 2. (a) See Kemeny and Snell (1976), Golub and Jackson (2010). (b) The convergence and therefore, agreement/equalization of the limiting beliefs of the agents follows from the definitions under the convergence of $\mathbf{T}$. Moreover, because of the convergence of the limiting beliefs of all the agents, the limit beliefs of the agent-i oracular and public oracular are equal. The last part in (b) simply follows from the rearrangement of the terms. ■

Remark 3. Part (a) of Proposition 2 presents conditions under which limit beliefs converge. In MS, under Bayesian updating, finite models always result in the termination of the updating process, even though the limiting beliefs may not converge. However, this is not the case with DeGroot's method. For example, for a two agent setup with the weighting matrix $\mathbf{T} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, the updating process neither terminates nor do the limiting beliefs converge.

Remark 4. Part (b) presents three properties that further reveal the chasm between the two methods; under DeGroot's heuristic, (i) the limiting beliefs of the agents are always equal, (ii) an agent's limiting belief trivially equals agent-oracular and public-oracular belief,[18] and, (iii) the condition needed for the pooling of information in the limit is altered. The interested reader may want to compare these properties with Proposition 1 in MS.

Finally, we provide a version of the wisdom of crowd result as in MS, modified for the DeGroot's heuristic.

Proposition 3. *Suppose* $\beta^*(\omega)$ *equals universal oracle's belief, i.e., for all* $\omega \in \Omega$, $\beta^*(\omega) = \beta_u(\omega_{I_u})$ *then, for all* $\omega \in \Omega$*, the public and universal oracular beliefs coincide.*[19]

Remark 5. Under DeGroot's method, we require equality between the limiting belief and the universal oracular belief for the wisdom of the crowd to hold. In keeping with the thrust in Remark 4, we invite the reader to compare Equation (11) in MS with our simple condition established in Proposition 3.

5. An illustration

In this section, we demonstrate our construction using Example 2 of MS in Table 1 given below. For each $\omega \in \Omega$, we replicate Mailath–Samuelson's Figure 2 with DeGroot's heuristic for weights given by $\mathbf{T}$. Columns (7) and (8) show round-one beliefs for the agents after averaging the interim beliefs according to weights in $\mathbf{T}$. Column (9) calculates the limit belief using $\mathbf{T}^*$.

Example 1. Given $N = M_1 = \{1, 2, 3, 4\}, I_1 = \{2, 3\}, M_2 = \{3, 4\}, I_2 = \{4\}$, we can calculate the weights of $\mathbf{T}$ using the construction outlined in Section 3. Hence $M_1^f = M_1 \cap (\bigcup_{i=1}^{2} I_i) = \{2, 3, 4\}, |M_1^f| = 3, |I_1| = 2$ and therefore $t_{11} = \frac{2}{3}$ and $t_{12} = \frac{1}{3}$. Similarly, we have $t_{21} = t_{22} = \frac{1}{2}$. The weighting matrix $\mathbf{T} = \begin{pmatrix} 2/3 & 1/3 \\ 1/2 & 1/2 \end{pmatrix}$ has all positive entries and by Kemeny and Snell (1976, Theorem 4.1.4, p.7), is convergent to $\mathbf{T}^* = \begin{pmatrix} 3/5 & 2/5 \\ 3/5 & 2/5 \end{pmatrix}$.

[18] It is also trivially true that under DeGroot's method, even if an agent's belief equals 0 or 1 under any round of updating, i.e. they agree with the omniscient oracle for some state ω, they may be subsequently revised because of weighting with other non-degenerate beliefs. See (iii) in Proposition 1 of MS.

[19] In fact, under the convergence of $\mathbf{T}$, all the belief types, agent or oracular are equal trivially if the hypothesis holds in Proposition 3.

Table 1
Beliefs with DeGroot's heuristic for weights $t_{11} = 2/3, t_{12} = 1/3, t_{21} = 0.5, t_{22} = 0.5$.

State	Prior		2's theory	Interim		Round one		Limit beliefs
$(\omega_1, \omega_2, \omega_3, \omega_4)$	ρ	$f(\omega)$	$f^2(\omega_{M_2})$	$\beta^1(\omega_{I_1})$	$\beta^2(\omega_{I_2})$	$\beta^1(\omega_{I_1}, b_0^2)$	$\beta^2(\omega_{I_2}, b_0^1)$	$\beta^*(\omega)$
(0,0,0,0)	9/64	0	3/8	1/16	14/32	3/16	1/4	17/80
(0,0,0,1)	3/64	0	5/8	1/16	29/32	11/32	31/64	2/5
(0,0,1,0)	3/64	0	5/8	13/16	14/32	11/16	5/8	53/80
(0,1,0,0)	3/64	0	3/8	13/16	14/32	11/16	5/8	53/80
(1,0,0,0)	3/64	0	3/8	1/16	14/32	3/16	1/4	17/80
(0,0,1,1)	9/64	1	1	13/16	29/32	27/32	55/64	17/20
(0,1,0,1)	1/64	1	5/8	13/16	29/32	27/32	55/64	17/20
(1,0,0,1)	1/64	1	5/8	1/16	29/32	11/32	31/64	2/5
(0,1,1,0)	1/64	1	5/8	1	14/32	13/16	23/32	31/40
(1,0,1,0)	1/64	1	5/8	13/16	14/32	11/16	5/8	53/80
(1,1,0,0)	9/64	1	3/8	13/16	14/32	11/16	5/8	53/80
(1,1,1,0)	3/64	1	5/8	1	14/32	13/16	23/32	31/40
(1,1,0,1)	3/64	1	5/8	13/16	29/32	27/32	55/64	17/20
(1,0,1,1)	3/64	1	1	13/16	29/32	27/32	55/64	17/20
(0,1,1,1)	3/64	1	1	1	29/32	31/32	61/64	77/80
(1,1,1,1)	9/64	1	1	1	29/32	31/32	61/64	77/80

6. Open directions for further work

The MS-conceptualization of "model-based inference" under Bayesian updating relies heavily on the measure-theoretic register, and by switching away from Bayesian updating, we give up powerful tools of measure theory. If one has to pick one fundamental theorem that MS's results crucially hinge upon, it would be Egorov's theorem on the uniform convergence of a pointwise convergent sequence of measurable functions. With DeGroot's method, however, a plausible conceptualization of the MS environment in the form of weights allows the importation of the technical apparatus of Markov chain theory for the iterations of beliefs and inferences.[20]

With this observation in hand, we can list at least three ongoing research directions to complement the results reported in this exploratory announcement: (i) consensus times, (ii) heterogeneous updaters, (iii) chaotic dynamics. With regard to (i), Golub and Jackson (2012) have pioneered important results on *speeds of convergence* in networks with DeGroot's updating, and we would take their analyses to place a bound on the number of rounds (iterations) needed for the beliefs to reach within a specified *distance* of their limiting beliefs. As to (ii), it is surely of interest how learning takes place between agents with different updating methods, an environment with *model based reasoners*, populated by both Bayesian and DeGroot updaters.[21] Finally, with respect to (iii), it is also of interest to explore the properties of the interaction matrix and the requirements that it imposes, for example, on instances where limit beliefs agree at even iterations and disagree at odd iterations: all this is a segway for us to apply the rich literature on dynamical systems to the setup introduced here.

[20] This application is already available in network economics in the work of Matthew Jackson and his followers; see Jackson (2010), Golub and Jackson (2010) and the references therein.

[21] See Bradley (2018) and Romeijn and Roy (2018) for conditions under which DeGroot's heuristic can be rationalized from a Bayesian perspective. Also, see Chandrasekhar et al. (2020) for an empirical application where a society is composed of a mixture of naive (DeGroot) and sophisticated (Bayesian) agents.

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