Separately Convex and Separately Continuous Preferences: On Results of Schmeidler, Shafer and Bergstrom-Parks-Rader^{*†}

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Abstract: We provide necessary and sufficient conditions for a correspondence taking values in a finite-dimensional Euclidean space to be open so as to revisit the pioneering work of Schmeidler (1969), Shafer (1974), Shafer-Sonnenschein (1975) and Bergstrom-Rader-Parks (1976) to answer several questions they and their followers left open. We introduce the notion of *separate convexity* for a correspondence and use it to relate to classical notions of continuity while giving salience to the notion of *separateness* as in the interplay of *separate continuity* and *separate convexity* of binary relations. As such, we provide a consolidation of the convexity-continuity postulates from a broad inter-disciplinary perspective and comment on how the qualified notions proposed here have implications of substantive interest for choice theory. (119 words)

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^{*}The authors take pleasure in noting, and celebrating, the fact that 2024 represents a landmark year: the $(48-50)^{th}$ anniversary of the publication of the papers of Bergstrom, Parks, Rader and Mas-Colell and Shafer, and the 55^{th} anniversary of that of Schmeidler.

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Modo paulatim modo saltatim.¹

The definition of a "thing" is effected by means of continuity and of correlations which have a certain differential independence of other "things." Bertrand Russell $(1915)^2$

1 Introduction

In a remarkable paper, Gerschenkron (1962) writes, "The mathematician's continuity has the indubitable advantage of being an unambiguous concept, even though it can be defined in a variety of more or less stringent fashions." And after furnishing a formal definition lifted from a then standard text in mathematics, and another from one in mathematical economics,³ he continues as follows:

This definition can be rendered even more simply by describing a continuous function as one that is "dense everywhere" in the sense of not having, strictly speaking, any contiguous points. For between any two points of such a function an infinite number of additional points can be placed. Stated still more simply, a continuous function is one that can be drawn in its entirety without lifting pencil from paper and which accordingly shows neither "gaps" nor "jumps."

Everything is so neat and tidy that in his masterful essays, Koopmans (1957), does not even see the need to define the concept,⁴ and as such, is at one with Gerschenkron. And so armed with some clear validation and authority, the latter writes:

Thus the historian who has gone out to scrutinize the mathematical concept of continuity travels far into strange lands and still is likely to return from his journey empty-handed. Continuity in this sense, or senses, does not appear to be a tool historians can profitably put to work; even when it appears in the empirical guise as gradualness of change it eludes

³See his Footnote reference to Hyslop's text, and also to R. G. D Allen's tome on *Mathematical Economics*.

¹This is the epigraph of Gerschenkron (1962) and is difficult to translate by virtue of *saltatim* being associated with a dance metaphor, and *paulatim* a metaphor associated with gradualness. We are grateful to Richard Bett for his rendering, "little by little, jump by jump."

²Address Delivered to the Philosophical Society of Manchester, in February, 1915 and reprinted as Chapter VII of *Mysticism and Logic and Other Essays* from the *Monist*, July, 1915. Russell explains: "That is to say, given a particular in one perspective, there will usually in a neighbouring perspective be a very similar particular, differing from the given particular, to the first order of small quantities, according to a law involving only the difference of position of the two perspectives in perspective space, and not any of the other "things" in the universe. It is this continuity and differential independence in the law of change as we pass from one perspective to another that defines the class of particulars which is to be called "one thing."

⁴See Footnote 4 in Uyanik and Khan (2022) and the text it footnotes. The authors state "After a footnote reference to a continuous function, an adjective he does not define but freely uses, and then to a *continuously representable* preference ordering that he does define, he masks the continuity postulate under the assumption of *local non-satiation*, referring to the latter as "a rather weak continuity property of preferences".

the crucial problem of mensuration and is so thickly covered with the metaphysical paint of inevitability as to be destitute of all usefulness.

Fortunately, all this did not discourage the historian from launching his classic in which five formulations of continuity are investigated for philosophers and scientists, especially in the humanities.⁵

This paper conducts an inquiry parallel to that of Gerschenkron, and its results take as their point of departure the claim that the concept of continuity has many meanings even when projected to the mathematical register. In a series of recent papers, the authors go back to the foundational fact established by Genocchi and Peano (1884) that "separate continuity does not imply continuity," and rely on it to provide a deconstruction and an integration of the concept as it is used in economic theory.⁶ The essential thrust of this work consisted of two related analytical moves:

- a translation of separate and linear continuity of functions to new continuity concepts for correspondences and binary relations, and obtaining new results on continuity of correspondences and binary relations, and
- a consolidation, through this translation, of a variety of unrelated continuity assumptions on binary relations dispersed in the economic literature.

We give salience to convexity, and investigate in particular how notions of "separate" convexity provide not only a consolidation but also answer several questions left open in the pioneering papers. From an analytical point of view, the marginal contribution of this paper then lies not so much in pushing the internal investigation of the topological (continuity) register still further, but in supplementing it with the sister-registers of linearity and order. Thus it looks towards Hardy, Littelwood, and Pólya (1952(1934)), Young (1910) and their followers for the impact on continuity of the convexity and monotonicity postulates. Again, as in the previous inquiries, we take this classical forking on functions and translate it to correspondence and relations. Once

⁵This is not the place to engage Gerschenkron (1962) but it is worthwhile to note his five concepts of, historical change, they are "(a) constancy of direction; (b) periodicity of events; (c) endogenous change; (d) length of causal regress; (e) stability of the rate of change."

⁶The authors note with some satisfaction the complementarity of Gerschenkron's engagement with Leibniz with the concluding discussion in Uyanik and Khan (2022). To be sure, the fact that joint continuity is stronger than separate continuity was, even then in the time of Cauchy in the early part of the 18th century, a standard material in textbooks on multivariate calculus, and its investigation constituted a rich development to which many mathematicians, including Heine, Baire and Lebesgue. It culminated in the benchmark theorem of Rosenthal (1955); see, for example, Ciesielski and Miller (2016) and the references therein. How this work is then followed up and consolidated, specifically in the context of both mathematical psychology and mathematical economics, see Ghosh, Khan, and Uyanik (2022).

redirected in this way, we are led naturally to *separate* as opposed to *joint* (universal or global) convexity of sets and binary relations.⁷

Moving on to mainstream mathematical economics, we single out Schmeidler (1969), Shafer (1974) and Bergstrom, Parks, and Rader (1976), among papers that include Mas-Colell (1974), Gale and Mas-Colell (1975) and Shafer and Sonnenschein (1975), to consider the relationship between section and graph continuity assumptions for binary relations in mathematical economics. Under completeness and transitivity, the two concepts are equivalent. Schmeidler (1969) drops completeness but keeps transitivity and establishes the equivalence under monotonicity. Shafer (1974) shows that under a strong convexity assumption, the section and graph continuity postulates are equivalent for a complete but non-transitive preference relation. Bergstrom, Parks, and Rader (1976) provide generalizations of these results and a comprehensive treatment on the relationship between section and graph continuity postulates. In a recent paper, Gerasimou (2015) works with a transitive and reflexive preference relation and establishes the equivalence between section and graph continuity under the additivity assumption,⁸ and the authors' own work studies the relationship among different continuity assumptions in mathematical economics and decision theory under monotonicity or convexity of preferences.⁹

This paper consists of three theorems that cover binary relations and correspondences. Under *separate* convexity, Theorems 1 and 2 provide a characterization of the open graph property for correspondences and restore the equivalence between several notions of continuity of a correspondence. Considering a binary relation as a graph of the correspondence, the two theorems, along with their corollaries, substantially generalize the results of Schmeidler (1969, 5.1), Shafer (1974, main Lemma) and Bergstrom, Parks, and Rader (1976, Theorem 3) on the continuity of a binary relation by weakening considerably their continuity and convexity/monotonicity assumptions, and by allowing a more general domain on which the binary relation is defined. Moreover, by weakening the convexity assumption, these results complement the following quotation from Bergstrom, Parks, and Rader (1976):

The assumption $P(x) \equiv \{y \in \mathbb{R}^n : y \succ x\}$ is convex demands more convexity than is needed for many purposes in general equilibrium analysis.

⁷The implications of separate convexity for distance functions in optimization problems is studied in Bauschke and Borwein (2001).

⁸Even though additivity is not a standard assumption in mathematical economics, it and its related forms, have been used in decision theory; see Gerasimou (2010, 2013) for its use in the context of continuity of preferences. In a parallel inquiry, Dubra, Maccheroni, and Ok (2004, Proposition 1) show that under the independence assumption, (graph) mixture continuity is equivalent to graph continuity for an incomplete preference relation, and Gilboa, Maccheroni, Marinacci, and Schmeidler (2010) show that under the independence assumption, reflexive and transitive weak preference relation has closed graph if and only if it is mixture continuous.

⁹See for example, Uyanik and Khan (2022), Ghosh, Khan, and Uyanik (2022), Ghosh, Khan, and Uyanik (2023), and their references. See also Karni (2007), Dubra (2011) and Galaabaatar, Khan, and Uyanik (2019) on the relationship among scalar continuity assumptions in decision theory.

Furthermore, we answer in the negative the author's question "whether [Shafer's result] can be generalized to infinite dimensional spaces or to an arbitrary convex subset of \mathbb{R}^{n} ."¹⁰ In Theorem 3, under separate convexity, we provide a relationship among various continuity assumptions in mathematical economics and mathematical psychology for complete and transitive binary relations that is illustrated in Figures 3 and 4.¹¹ We also offer a host of examples to demonstrate the usefulness of our results. Theorems 1–3 establish useful relationships between graph continuity and other continuity postulates for non-ordered preferences, and by weakening the convexity and monotonicity assumptions in earlier work, also generalize it.

The paper is structured as follows. Section 2 presents the two main theorems on the continuity of correspondences. Section 3 presents a portmanteau theorem on the continuity of binary relations. Section 4 discusses some extensions of these results and their applications to consumer and producer theory. Section 5 provides the required proofs.

2 On Continuity of Correspondences

In this section we present our two main theorems on the characterization of open graph property that consolidate and generalize the existing results presented in Schmeidler (1969), Shafer (1974) and Bergstrom, Parks, and Rader (1976).

A correspondence from a topological space X into a set $Y \subseteq \mathbb{R}^n$ is a mapping $F: X \to Y$ that assigns every $x \in X$ to a subset of Y. Define the graph of F as $\operatorname{gr} F = \{(x, y) \in X \times Y \mid y \in F(x)\}$. For every $x \in X$, F(x) denotes upper section of F at x, and for every $y \in Y$, $F^{-1}(y) = \{x \in X \mid y \in F(x)\}$ denotes the lower section of F at y. F has open sections if it has both open upper and lower sections. A correspondence F has separately open upper sections if for every $x \in X$ and every straight line¹² L in Y that is parallel to a coordinate axis, $L \cap F(x)$ is open in L and linearly open upper sections if for every $x \in X$ and every straight line L in Y, $L \cap F(x)$ is open in L. It follows from their definitions that there is a strict nested relationship between the following continuity postulates on a correspondence F:

open graph \implies open sections \implies linearly open sections \implies separately open sections. (1)

¹⁰We note in Section 4 how our results can be generalized to an infinite dimensional setting. Note that Yamazaki (1983) also partially answers this open problem question by providing a generalization to infinite dimensional settings. He also claims in his Corollary 3 that Shafer's result holds also for arbitrary convex sets in \mathbb{R}^n . Example 6 provides a counter-example to Yamazaki's result.

¹¹A detailed elaboration of this figure revolves around how our results generalize antecedent results by (inessentially) weakening continuity and (essentially) weakening the convexity assumption. For elaboration on the notions of *hiddenness* and *essentiality*, deriving from Kim and Richter (1986), see Uyanik and Khan (2022). In this connection, the reader might also note the question of *inconsistency* of axioms as in Assa and Zimper (2018).

¹²A straight line in $Y \subseteq \mathbb{R}^n$ is the intersection of a one-dimensional affine subspace of \mathbb{R}^n and Y. A subset X of a (real) vector space is called *affine* if for all $x, y \in X$ and $\lambda \in \mathbb{R}$, $\lambda x + (1 - \lambda)y \in X$.

A set $A \subseteq Y$ is separately convex if for every straight line L in Y that is parallel to a coordinate axis $i = 1, ..., n, A \cap L$ is convex.

Here we present our first result on graph continuity of correspondence defined on a set that satisfies the following property.

(A) For a convex set $Y \subseteq \mathbb{R}^n$, either Y is open or $Y = \prod_{i=1}^n Y_i$, $Y_i \subseteq \mathbb{R}$ for all i = 1, ..., n.

The following provides a partial converse relationship among the continuity postulates listed in Equation 1.

Theorem 1. Let X be a topological space, $Y \subseteq \mathbb{R}^n$ satisfy property **A**, and $F : X \to Y$ has separately convex upper sections. Then, F has open graph if and only if it has open lower sections and separately open upper sections.

Notice that the separate convexity assumption in Theorem 1 is weaker than assuming that F has convex values. Note that if we define a correspondence $F : Y \to X$ from Y into X and define separately open lower sections analogous to separately open upper sections, then we obtain the following result that is symmetric to Theorem 1: *if the lower sections of* F *are separately convex, then* F *has open graph if and only if it has open upper sections and separately open lower sections.* These two results imply the following converse relationship among the continuity postulates listed in Equation 1.

Corollary 1. Let $X \subseteq \mathbb{R}^n$ satisfy property **A**, and $F : X \twoheadrightarrow X$ has separately convex upper sections and separately convex lower sections. Then F has an open graph if and only if it has separately open lower sections and separately open upper sections.

Examples 1 and 2 are illustrations of the failure of the hypothesis in Theorem 1. While in Example 1, property A fails, in Example 2, the separate convexity assumption fails.

Example 1. Let $X = \{x \in [0,1]^2 | x_2 \ge x_1\}$, $A = \{x \in [0,1]^2 | x_2 > x_1\} \cup \{(1,1)\} \subset X$ and $F: X \to X$ be a correspondence defined as F(X) = A for all $x \in X$. Note that X is a bounded polytope, but is closed and hence fails property **A**. Further, F has open lower sections and separately open upper sections since for any straight line L in X that is parallel to a coordinate axis, $L \cap A$ is open in L. However, the graph of F is not open since every neighborhood of ((1,1), (1,1)) contains a point outside of the graph of F.

Example 2. Let $X = \mathbb{R}^2$, $A = \mathbb{R}^2 \setminus \{x \in \mathbb{R}^2 | x_1 = x_2, x \neq 0\}$ and $F : X \to X$ be a correspondence defined as F(X) = A for all $x \in X$. Clearly, X satisfies property **A**. Notice that F is not separately convex. The intersection of A and any line parallel to a coordinate axis is either the real line or a union of two open intervals. Therefore, F has separately open upper sections. However, it does not have an open graph since every open ball containing the origin in \mathbb{R}^4 contains a point in the complement of the graph of F.



Figure 1: Utility functions in Equations (2) and (3).

The next example gives two instances of non-convex preferences from Halevy, Persitz, and Zrill (2017) that are separately convex.

Example 3. Consider the following two preference relations that are defined on \mathbb{R}^2_+ and are represented by the following utility functions:

$$u_1(x,y) = \begin{cases} x^3y & \text{if } x \ge y\\ xy^3 & \text{if } x \le y \end{cases}$$
(2)

and

$$u_2(x,y) = \sqrt{\max\{x,y\}} + \frac{1}{4}\sqrt{\min\{x,y\}}.$$
(3)

In Figure 1, we illustrate the indifference curves corresponding to the two utility functions above. Notice that while the preferences are not convex, they are separately convex.

Our next result is on the continuity of correspondences defined on subspaces that satisfy the following property.

(B) For a convex set $Y \subseteq \mathbb{R}^n$, either Y is open or a polyhedron, where a polyhedron is the intersection of a finite number of half-spaces.

Theorem 2. Let X be a topological space, Y a convex subset of \mathbb{R}^n satisfies property **B** and $F: X \to Y$ a correspondence such that F(x) is convex for all $x \in X$. Then, F has an open graph if and only if F has open lower sections and linearly open upper sections.

Analogous to Theorem 1, if we define a correspondence $F : Y \to X$ from Y into X, then we obtain the following result that is symmetric to Theorem 2: if the lower sections of F are convex, then F has open graph if and only if it has open upper sections and linearly open lower sections. These two results imply the following converse relationship among the first three continuity postulates listed in Equation 1.

Corollary 2. Let $X \subseteq \mathbb{R}^n$ satisfy property **B**, and $F : X \twoheadrightarrow X$ has convex upper sections and convex lower sections. Then F has open graph if and only if it has linearly open sections.

Note that unlike Theorem 1, open graph property in Theorem 2 is not equivalent to open lower sections and separately open upper sections. For instance, X is a bounded polytope in Example 1 and hence satisfies property **B** but the correspondence F fails to have linearly open upper sections, hence does not have an open graph. Also notice that property **B** is weaker than property **A**, however the convexity and continuity assumptions in Theorem 2 are stronger than those in Theorem 1. Hence, the assumptions in these two theorems are non-nested.

Example 2 above illustrates a correspondence that is separately convex and has separately open upper sections but fails to have linearly open upper sections as well as an open graph. The following example illustrates that if the separate convexity assumption fails, then a correspondence may have both separately open and linearly open upper sections but fails graph continuity.

Example 4. Let $X = \mathbb{R}^2$, $A = \mathbb{R}^2 \setminus \{x \in \mathbb{R}^2 | x_2 = x_1^2, x \neq 0\}$ and $F : X \to X$ be a correspondence defined as F(X) = A for all $x \in X$. Clearly, X satisfies properties **A** and **B** but F is neither separately convex nor has convex upper sections. For any straight line $L, L \cap A$ excludes at most two points of L. Hence, F has both separately open and linearly open upper sections. However, A is not open since every open ball containing 0 contains a point in the complement of A, hence F fails to have open upper sections and an open graph.

The following example illustrates that the convexity assumption in Theorem 2 is essential even for a correspondence defined on an interval in \mathbb{R} .

Example 5. Let X = [0, 1] and $F : X \to X$ such that F(0) = (0, 1] and $F(x) = \{y \in X \mid y > x, \text{ and } y \neq (1 - x)\}$ for all x > 0. It is easy to see that F has open sections but (0, 1) has no open neighborhood contained in the graph of F, hence F does not have an open graph. It is clear that F(x) and $F^{-1}(y)$ are not convex for all $x \in (0, 0.5)$ and for all $y \in (0.5, 1)$.

The next result provides additional relationships among the continuity postulates.

Proposition 1. Let X be a topological space, Y, Z two convex subsets of \mathbb{R}^n that satisfy property A, B, respectively, and $F: X \rightarrow Y, G: X \rightarrow Z$ be two correspondences.

- (a) if F has separately convex upper sections, then the following are equivalent: F has (i) open upper sections, (ii) linearly open upper sections, (iii) separately open upper sections.
- (b) if G has convex upper sections, then G has open upper sections if and only if it has linearly open upper sections.

Note that the additional relationships among the *lower* continuity postulates are obtained by suitably adjusting the domains and the ranges of the correspondences and replacing "upper" in Proposition 1 by "lower."

As in Theorem 1, considering a binary relation as the graph of a correspondence, Theorem 2 provides necessary and sufficient conditions for a binary relation on a convex subspace of \mathbb{R}^n to be continuous. It provides a characterization of the continuity of a binary relation, or a correspondence, by using a topological property similar to the *linear continuity* postulate. As noted in Equation 1, open upper (lower) sections property is stronger than linearly open upper (lower) sections property that is stronger than separately open upper (lower) sections property. Therefore, Theorems 1 and 2 generalize the following result of Shafer (1974, Lemma, pg. 914) and Bergstrom, Parks, and Rader (1976, Theorem 3) on continuity of a binary relation by considerably weakening their continuity and convexity assumptions, and allowing a more general domain on which the binary relation P on X, there exists a unique correspondence $F: X \to X$ such that P = grF. The upper and lower sections of a binary relation P at $x \in X$ is defined as P(x) = F(x) and $P^{-1}(x) = F^{-1}(x)$, respectively.

Corollary 3. Let $X = \mathbb{R}^n_+$ and $P : X \to X$ be a correspondence such that P(x) is convex for all $x \in X$ (or $P^{-1}(x)$ is convex for all $x \in X$). Then, P has an open graph if and only if P has open sections.

Moreover, along with Theorems 1 and 2, Corollary 1 generalize the following result of Schmeidler (1969) by dropping the transitivity assumption, substantially weakening the continuity and monotonicity assumptions,¹³ and allowing a more general domain on which the binary relation is defined.

Corollary 4. Let $X = \mathbb{R}^n_+$ and $P : X \to X$ be a transitive, irreflexive and strongly monotone binary relation on X. Then P has an open graph if and only if P has open sections.

Note that Gerasimou (2015) provides a result on the relationship between sections and graph continuity of a reflexive and transitive binary relation under the additivity assumption.

 $^{^{13}}$ It is easy to observe that if a strict binary relation is strongly monotone, then it has separately convex upper and lower sections, but the converse relationship does not hold as for example the relation need not be complete. A similar relationship holds for weak monotonicity whose proof is provided in Section 5.

The results in this paper focus on separate convexity and do not impose additivity. It is easy to show that additivity and separate convexity assumptions are non-nested for preferences defined in this paper and in Gerasimou (2015). Therefore, our results and the results presented in Gerasimou (2015) are non-nested.

Bergstrom, Parks, and Rader (1976, Theorem 1) provides a generalization of Schmeidler's result for a transitive and order-dense binary relation on general topological spaces. Their result and our results above are non-nested; while we impose weaker continuity assumptions and do not assume transitivity or order-denseness, they do not impose any convexity assumption and allow for a more general domain. Further, they note that the generalization of Theorem 2 to arbitrary convex sets in \mathbb{R}^n is an open problem. In Example 6, we answer this open problem in the negative and also provide a counterexample to Corollary 3 of Yamazaki (1983) on the open problem.¹⁴ The example further demonstrates that the results in this section cannot be generalized to a setting where Y is an arbitrary convex set in \mathbb{R}^n .



Figure 2: A convex binary relation with open sections whose graph is not open

Example 6. Let $X = \{x \in \mathbb{R}^2_+ | x_1^2 + x_2^2 \leq 1\}$. Clearly, X is convex. Consider a homeomorphism $f : [0,1] \rightarrow \{x \in \mathbb{R}^2_+ | x_1^2 + x_2^2 = 1\}$ as illustrated in Figure 2 where f(0) = (0,1) and f(1) = (1,0). Define a correspondence $F : X \rightarrow X$ as follows: $F(1,x_2) = X$ for all $(1,x_2) \in X$ and

¹⁴Yamazaki (1983, Proposition 2) identifies the following property for a subspace of a topological vector space: a subspace X of a topological vector space is *locally finite* if for each $z \in X$ there exists a finite collection of points $\{x^1, \ldots, x^k\}$ in X such that the convex hull of $\{x^1, \ldots, x^k\}$ is a neighborhood of z. Yamazaki's Proposition 2 provides a generalization of Shafer's result to spaces satisfying this property. (He also works with a weakening of convex domains by restricting the preferences to the restricted domain). The local finiteness property has its roots in the proof of Shafer's (1974) result and plays a crucial role. However, Yamazaki's application of his Proposition 2 to solve the open problem mentioned above has a mistake.

 $F(x_1, x_2) = X \setminus \{f(x_1)\}$ for all $x \in X$ with $x_1 \neq 1$. Note that F is constant in the second variable. Moreover, F(x) is convex for all $x \in X$ since it excludes at most one point that cannot be written as a weighted average of two points in F(x). Also, F(x) is open in X for all $x \in X$ since its complement is closed (either an empty set or a singleton). Furthermore,

$$F^{-1}(y) = \begin{cases} X & \text{if } y \notin f([0,1)) \\ X \setminus (\{a\} \times \{z_2 \in \mathbb{R}_+ \mid (a, z_2) \in X\}) & \text{if } y = f(a) \text{ for some } a \in [0,1). \end{cases}$$

Note that for each $y \in \{x \in \mathbb{R}^2_+ | x_1^2 + x_2^2 = 1\}, y \neq (1,0)$, there exists a unique such $a \in [0,1)$ such that y = f(a). Hence, F has open lower sections. However, F does not have an open graph since every neighborhood of $((1,0), (0,1)) \in \operatorname{gr} F$ contains a point in the complement of $\operatorname{gr} F$. In this example, setting $P = \operatorname{gr} F$ implies that $P \subseteq X \times X$ is a binary relation on the convex set X with open sections and convex values. However, P is not open in $X \times X$.

The results we present here either focus on correspondences with open graph, or on open binary relations.¹⁵ Example 7 illustrates that a binary relation need not be closed even if it has closed sections and convex upper sections. There are results in the literature that show equivalence between having closed sections and being closed for a binary relation; see for example, Ward (1954) and Shafer (1974) under completeness or transitivity assumptions.

Example 7. Let X = [0, 1] and $P \subseteq X \times X$ such that $P(x) = \{x\}$ for all x < 1 and $P(1) = \{0\}$. It is clear that P has closed sections (both upper and lower). Also, P(x) is convex-valued since it's a singleton for all $x \in X$. However, P is not closed in $X \times X$ since $(1, 1) \in P^c$ has no open neighborhood contained in P^c .

3 On Continuity of Ordered Preferences

In the previous section, we provide relationships among different continuity assumptions under separate convexity assumption for a binary relation that need not be complete or transitive. In this section, we bring out the additional relationships when the binary relation is ordered, that is, complete and transitive.

Let $X \subseteq \mathbb{R}^n$ be a convex set. A subset \succeq of $X \times X$ denotes a *binary relation* on X. We denote an element $(x, y) \in \succeq$ as $x \succeq y$. The *asymmetric part* \succ of \succeq is defined as $x \succ y$ if $x \succeq y$ and $y \not\succeq x$, and its *symmetric part* \sim is defined as $x \sim y$ if $x \succeq y$ and $y \succeq x$. The inverse of \succeq

¹⁵In a related setting, Zhou (1995, Proposition 2) replaces open lower sections assumption in Shafer's result with lower semicontinuity of a correspondence whose values are in $Y = \mathbb{R}^n$ to provide a characterization of the open graph property; see also Impicciatore and Ruscitti (2012) for a recent treatment. By using the arguments in this paper, it is possible to show that the open and convex upper sections in Zhou's result can be replaced with separately open and separately continuous upper sections.

is defined as $x \preceq y$ if $y \succeq x$. Its asymmetric part \prec is defined analogously and its symmetric part is \sim . For any $x \in X$, let $A_{\succeq}(x) = \{y \in X | y \succeq x\}$ denote the *upper section* of \succeq at x, and $A_{\preceq}(x) = \{y \in X | y \preceq x\}$ its *lower section* at x. For any $x, y \in X$ and $\lambda \in [0, 1]$, let $x \lambda y$ denote $\lambda x + (1 - \lambda)y$. We provide in Table 1 the descriptive adjectives pertaining to a relation in a tabular form for the reader's convenience.¹⁶

reflexive	$x \succcurlyeq x, \; \forall x \in X$
non-trivial	$\exists x, y \in X \text{ such that } x \succ y$
complete	$x \succcurlyeq y \text{ or } y \succcurlyeq x, \ \forall x, y \in X$
transitive	$x\succcurlyeq y\succcurlyeq z\Rightarrow x\succcurlyeq z, \; \forall x,y,z\in X$
monotone	$x > y \implies x \succsim y \; \forall \; x, y \in X \; \text{ or } \; x > y \implies x \precsim y \; \forall \; x, y \in X$
order dense	$x \succ y \implies \exists z \in X \text{ such that } x \succ z \succ y \ \forall x, y \in X$

Table 1: Properties of binary relations

The continuity assumption on a binary relation is one of the standard assumptions in decision theory and mathematical psychology. The following definition lists the many forms it takes in the literature.

Definition 1. A binary relation \succeq defined on a convex set $X \subseteq \mathbb{R}^n$ is

- (a) graph continuous if \succeq is a closed subset of $X \times X$.
- (b) upper continuous if \succeq has closed upper sections, lower continuous if \succeq has closed lower sections, and continuous if \succeq has closed upper and lower sections.
- (c) *linearly upper (lower) continuous* if the restriction of the upper (lower) sections of \succeq to any straight line L in X is closed in L, and *linearly continuous* if \succeq is linearly upper and linearly lower continuous.
- (d) separately upper (lower) continuous if the restriction of the upper (lower) sections of \succeq to any straight line L in X that is parallel to a coordinate axis is closed in L, and separately continuous if \succeq is separately upper and separately lower continuous.
- (e) upper (lower) mixture continuous if for any $x, y, z \in X$, $\{\lambda \in [0,1] | x \lambda y \succeq z\}$ is closed $(\{\lambda \in [0,1] | x \lambda y \preceq z\}$ is closed) in the unit interval [0,1], and mixture continuous if \succeq is upper mixture and lower mixture continuous.
- (f) upper (lower) Archimedean if for any $x, y, z \in X, x \succ y$ implies that there exists $\lambda \in (0, 1)$ ($\delta \in (0, 1)$) such that $x\lambda z \succ y$ ($x \succ y\delta z$), and Archimedean if it is upper and lower Archimedean.

¹⁶For vectors x and y, " $x \ge y$ " means $x_i \ge y_i$ in every component; "x > y" means $x \ge y$ and $x \ne y$; and " $x \gg y$ " means $x_i > y_i$ in every component.

- (g) Wold-continuous if it is order-dense and $x \succ z \succ y$ implies that any curve¹⁷ joining x to y meets the indifference class of z.
- (h) weakly Wold-continuous if it is order-dense and $x \succ z \succ y$ implies that the straight line joining x to y meets the indifference class of z.
- (i) restricted solvable if for all $i \in \{1, ..., n\}$, all $x, y \in X$ and all $(a_i, y_{-i}), (b_i, y_{-i}) \in X$ with $(a_i, y_{-i}) \succeq x \succeq (b_i, y_{-i})$, there exists $c \in X$ with $(c_i, y_{-i}) \in X$ such that $x \sim (c_i, y_{-i})$.

Moreover, \succeq possesses the

- (j) intermediate value property (IVP) if for all $x, y, z \in X$ with $x \succeq y \succeq z$, there exists $\lambda \in [0, 1]$ such that $x \lambda z \sim y$.
- (k) strong *IVP* if for all $x, y, z \in X$ with $x \succeq y \succeq z$ and all curves C_{xy} connecting x and z, there exists $c \in C_{xy}$ such that $c \sim y$.

In his pioneering work, Wold (1943–44) developed consumer theory and provided a result on numerical representation of binary relations by a continuous function,¹⁸ based on the Wold continuity assumption. In subsequent work, Wold and Jureen (1953) uses a weaker version of Wold continuity, the weak-Wold assumption. This weaker continuity assumption is first used in decision theory independently by Nash (1950) and Marschak (1950) who provide a complete axiomatization of the expected utility theory initiated by von Neumann and Morgenstern (1947).¹⁹ These Wold-continuity postulates are analogous to restricted solvability and the two IVP postulates. The linear and separate continuity postulates are motivated by their classical counterparts for functions in mathematics. Note that linear continuity is equivalent to mixture continuity. The remaining continuity postulates are standard in mathematical economics and decision theory.

Theorem 3. Let \succeq be a complete and transitive binary relation on a non-empty and convex set $X \subseteq \mathbb{R}^n$ with property **A**.

(a) If the upper (lower) sections of ≿ are separately convex, then the following are equivalent for ≿: upper (lower) continuity, upper (lower) mixture continuity, upper (lower) linear continuity, upper (lower) separate continuity, lower (upper) Archimedean.

¹⁷A curve on X is the image of a continuous injective function $m: [0,1] \to X$.

 $^{^{18}}$ Wold's representation theorem and the well-known representation theorem of Eilenberg (1941) are independent discoveries, and the proofs are different; see Banerjee and Mitra (2018), Uyanik and Khan (2022) and Ghosh, Khan, and Uyanik (2023) for details.

¹⁹von Neumann and Morgenstern (1947) use a version of the Archimedean assumption in their theory. There are many versions of the axiomatics of the expected utility theory. Nash and Marschak use weak Wold-continuity while Herstein and Milnor (1953) use mixture continuity; see Bleichrodt, Li, Moscati, and Wakker (2016) for a history of axiomatics of the expected utility theory.

(b) If ≿ is order dense and its sections are separately convex, then the following are equivalent for ≿: graph continuity, continuity, linear continuity, mixture continuity, Archimedean, strong IVP, IVP, Wold-continuity, weak Wold-continuity, restricted solvability, and separate continuity.

The results in the previous section provide equivalence relationship among separate continuity, linear continuity, continuity, and graph continuity of a preference relation that need not be complete or transitive. Theorem 3(a) adds Archimedean and mixture continuity into this relationship when the upper or the lower sections of the relation are separately convex. Part (b) of the theorem shows that if the separate convexity assumption holds for both upper and lower sections, then further equivalence relationships hold among the defined continuity postulates.

Note that the convexity assumption in part (b) of the theorem is equivalent to the following monotonicity property: \succeq is separately monotone if $x_i > y_i$ implies $(x_i, z_{-i}) \succeq (y_i, z_{-i})$ for all i = 1, ..., n and all $(x_i, z_{-i}), (y_i, z_{-i}) \in X$, or $x_i > y_i$ implies $(x_i, z_{-i}) \succeq (y_i, z_{-i})$ for all i = 1, ..., n and all $(x_i, z_{-i}), (y_i, z_{-i}) \in X$. Moreover, it is not difficult to show that a transitive binary relation \succeq is separately monotone if and only if it is monotone. We provide proof of these claims in the next section. Ghosh, Khan, and Uyanik (2022) provide an equivalence relation among different continuity postulates under monotonicity and convexity. Theorem 3(b) provides a generalization by allowing a more general domain and an alternative proof of their result and connects it to separate convexity.

Figures 3 and 4 provide pictorial representations of the results we report in Theorem 3. The reader should note in particular how the notion of *separate convexity* allows the execution and the completion of several equivalences.



Figure 3: Relationship among continuity postulates in Theorem 3(a).

Example 8. This example illustrates that separate convexity is not enough to obtain equivalence between Archimedeanity and mixture continuity and between separate continuity and continuity



Figure 4: Relationship among continuity postulates in Theorem 3(b).

as well as restricted solvability and continuity. It is motivated by the classic counterexample of Genocchi and Peano (1884).²⁰ Let $X = \mathbb{R}^2$ and $f : X \to \mathbb{R}$ defined as follows:

$$f(x_1, x_2) = \frac{2x_1x_2}{x_1^2 + x_2^2} + \min\{x_1, x_2\}$$
 if $x \in \mathbb{R}^2_+ \setminus \{0, 0\}$, and $f(x) = 0$ otherwise.

Let \succeq be the binary relation on X induced by f, that is $x \succeq y$ if and only if $f(x) \ge f(y)$. In this case, it is easy to show that \succeq is separately convex and satisfies Archimedean property, separate continuity and restricted solvability. However, \succeq fails mixture continuity along the 45° line, hence also fails continuity.

4 Discussion and Remarks

In this section, we list three observations about our results. First, the results here (Theorems 1 and 2) can be generalized to a topological vector space setting by using an argument analogous to those in the proofs. Let X_0 be an arbitrary non-empty topological space and $\{X_i\}_{i=1}^n$ be a finite collection of non-empty, convex finite sets in a topological vector space.²¹ Let $A \subseteq \prod_{i=0}^n X_i$ be a set such that for all $i = 1, \ldots, n$ and all $x \in A$, $A_i(x)$ is convex. Then, A is open if and only if $A_i(x)$ is open for all $i = 0, 1, \ldots, n$ and all $x \in A$. Note that A can be considered as the graph of correspondence from X_0 into $X = \prod_{i=1}^n X_i$, or as a binary relation on X when $X_0 = X$. Moreover, generalizations of the results (Theorem 3) concerning the relationship

²⁰There is a rich and deep literature on the relationship among different continuity assumptions on functions; see for example, Ciesielski and Miller (2016) and Uyanik and Khan (2022) for a detailed discussion.

²¹See Footnote 14 for the definition of a locally finite set.

between different continuity postulates for functions can also be obtained.

Second, a major thrust of our paper has been to explore the notion of *separateness* within the choice theory register. Extending this analysis to producer theory appears to be a promising avenue for future work. In this context, Herberg (1973) develops a notion of point-wise convexity, similar to the idea of separate convexity in this paper.²² To the best of our knowledge, there is no work that has explored this issue further. Moreover, a weaker convexity assumption on production sets may give us a sufficient background to work with marginal rates.

Finally, non-convexity in preferences continues to be an important avenue for exploration.²³ Recent work by Halevy, Persitz, and Zrill (2017) suggests useful implications of non-convexity for techniques in revealed preferences. We hope that by bringing separate convexity to the picture, we can stimulate applications in this area.

5 Proofs of the Results

Proof of Theorem 1. Let X be a topological space, $Y \subseteq \mathbb{R}^n$ be a convex set with property \mathbf{A} , $F: X \to Y$ be a correspondence with separately convex upper sections and $I = \{1, \ldots, n\}$. For notational simplicity, define $A \subseteq X \times Y$ as $A = \operatorname{gr} F$. For all $(x, y) \in A$, let $A_0(x, y) = \{x' \in X \mid (x', y) \in A\}$, and for any $J \subseteq I$, let $A_J(x, y) = \{y' \in Y \mid y'_{-J} = y_{-J} \text{ and } (x, y') \in A\}$ and $A(y_J) = \{(x', y') \in A \mid y'_J = y_J\}$. The forward direction is clear. For the backward direction, let $Z = X \times Y$, and assume F has open lower sections and separately open upper sections, that is $A_0(z)$ is open and $A_I(z)$ is separately open for all $z \in A$.

Next, we show that for all $J \subsetneq I$ and all $(x, y) \in A$, if $A(y_{-J})$ is open in $Z(y_{-J})$, then $A(y_{-(J\cup\{i\})})$ is open in $Z(y_{-(J\cup\{i\})})$ for each $i \notin J$. Towards this end, pick $J \subsetneq I$, $(\bar{x}, \bar{y}) \in A$ and assume $A(\bar{y}_{-J})$ is open in $Z(\bar{y}_{-J})$. Pick $z = (x, y_{J\cup\{i\}}, \bar{y}_{-(J\cup\{i\})}) \in A$ and a straight line L_{y_i} in Y that contains $(y_{J\cup\{i\}}, \bar{y}_{-(J\cup\{i\})})$ and is parallel to the *i*-th coordinate axis in Y. Since $A_I(z)$ is separately open in Y, there exist $a_i, b_i \in \mathbb{R}$, $a_i \leq b_i$, such that $U = [a_i, b_i] \times \{y_J, \bar{y}_{-(J\cup\{i\})}\}$ is contained in $L_{y_i} \cap A_I(z)$ and is a neighborhood²⁴ of $(y_i, y_J, \bar{y}_{-(J\cup\{i\})})$ in the subspace L_{y_i} .

Since $(x, \tilde{y}_i, y_J, \bar{y}_{-(J\cup\{i\})}) \in A$ for all $\tilde{y}_i \in [a_i, b_i]$, the sets $A(a_i, \bar{y}_{-(J\cup\{i\})})$ and $A(b_i, \bar{y}_{-(J\cup\{i\})})$ are open in $Z(a_i, \bar{y}_{-(J\cup\{i\})})$ and $Z(b_i, \bar{y}_{-(J\cup\{i\})})$ respectively, and contain (x, y_J) , there exist neighborhoods $U^a(x, y_J) \subseteq A(a_i, \bar{y}_{-(J\cup\{i\})})$ and $U^b(x, y_J) \subseteq A(b_i, \bar{y}_{-(J\cup\{i\})})$ of (x, y_J) . Let $U(x, y_J) = U^a(x, y_J) \cap U^b(x, y_J)$. Then, $U(x, y_J)$ is a neighborhood of (x, y_J) and for all $(x', y'_J) \in$ $U(x, y_J), (x', y'_J) \in U^a(x, y_J) \subseteq A(a_i, \bar{y}_{-(J\cup\{i\})})$ and $(x', y'_J) \in U^b(x, y_J) \subseteq A(b_i, \bar{y}_{-(J\cup\{i\})})$.

²²See for example Figures 1–4 and Lemma 1 of Herberg (1973). As another simple example, the preference relation induced by the utility function $\max\{x_1, x_2\}$ is separately convex but not convex, in fact, its lower sections are convex. Note that the production set induced by the function above is convex.

²³Trockel (1984) writes on the role of non-convexity in the context of market demand and large economies.

 $^{^{24}}$ A set is a *neighborhood* of a point in a topological space if it contains the point in its topological interior.

Hence for all $(x', y'_J) \in U(x, y_J)$, $a_i, b_i \in A(x', y'_J, \bar{y}_{-(J \cup \{i\})})$. Then, it follows from the convexity assumption that for all $(x', y'_J) \in U(x, y_J)$, $[a_i, b_i] \subseteq A(x', y'_J, \bar{y}_{-(J \cup \{i\})})$. Note that $[a_i, b_i] \times U(x, y_J) \times \{\bar{y}_{-(J \cup \{i\})}\} \subseteq A$ and it is a neighborhood of z in the subspace $Z(\bar{y}_{-(J \cup \{i\})})$. Therefore, $A(\bar{y}_{-(J \cup \{i\})})$ is open in $Z(\bar{y}_{-(J \cup \{i\})})$

Note that $A_0(z)$ is open in X for all $z \in A$. By setting $J = \emptyset$, it follows from the argument above that for all $(x, y) \in X$, $A(y_{-1})$ is open in $Z(y_{-1})$. Iteratively adding one index i = 2, ..., ninto J and applying the argument above imply that A is open in Z.

Let X be a subset of \mathbb{R}^n . The closure of X is denoted by clX and its interior by intX. Since any lower dimensional subset of \mathbb{R}^n has an empty interior, it is more convenient to work with the concept of relative interior. Recall that a subset X of a (real) vector space is affine if for all $x, y \in X$ and $\lambda \in \mathbb{R}$, $\lambda x + (1 - \lambda)y \in X$. It is clear that A is affine if and only if $A - \{a\}$ is a subspace of X for all $a \in A$. The affine hull of X, affX, is the smallest affine set containing X. The relative interior of a subset X of \mathbb{R}^n is defined as

 $\operatorname{ri} X = \{ x \in \operatorname{aff} X \mid \exists N_{\varepsilon}, \text{ an } \varepsilon \text{ neighborhood of } x, \text{ such that } N_{\varepsilon} \cap \operatorname{aff} X \subseteq X \}.$

That is, the relative interior of X is the interior of X with respect to the smallest affine subspace containing X. The following result is due to Rockafellar (1970, p. 45).

Lemma 1. Let X be a non-empty and convex subset of \mathbb{R}^n . Then riX is non-empty, and for all $x \in \text{ri}X, y \in \text{cl}X$ and all $\lambda \in [0, 1), y\lambda x \in \text{ri}X$.

Proof of Theorem 2. The forward direction is obvious. For the backward direction, assume $F: X \to Y$ has convex and linearly open upper sections and open lower sections. First, we show that F has open upper sections, that is F(x) is open for all $x \in X$. Assume towards a contradiction that there exists $x_0 \in X$ such that $F(x_0)$ is not open (in Y), hence there exists $y \in F(x_0)$ such that y is not an interior point of $F(x_0)$. Note that $F(x_0)$ is convex and has infinitely many elements. Let \mathcal{H}_y denote the set of all supporting hyperplanes of $F(x_0)$ at y in the affine space generated by $F(x_0)$, and for all $h \in \mathcal{H}_y$, let H_h denote the closed half-space determined by h that contains $F(x_0)$.

Assume there exists $h \in \mathcal{H}_y$ such that $H_h^c \cap F(x_0) \neq \emptyset$. Since $H_h^c \cap F(x_0)$ is non-empty and convex, pick w in the relative interior of $H_h^c \cap F(x_0)$, which is nonempty by Lemma 1. Let L_{yw} be the straight line in Y that contains y and w. It follows from Lemma 1 that $y\delta w \notin F(x_0)$ for all $\delta \in [0, 1)$. Then, every open neighborhood of y in the subspace L_{yw} contains an element $y\delta w \notin F(x_0)$. Since $y \in F(x_0)$ this yields a contradiction with linear openness of $F(x_0)$.

Next, assume for all $h \in \mathcal{H}_y$, the set $H_h^c \cap F(x_0) = \emptyset$. This case happens only if Y is a polyhedron and y lies in one of the hyperplanes determining the relative boundary of Y defined

in property **B**. Let \mathcal{H}'_y be the set of all such hyperplanes. If there exists $h \in \mathcal{H}'_y$ and $y' \in h$ such that for all $\delta \in [0, 1)$, $y \delta y' \notin F(x_0)$, then this yields a contradiction with the linear openness of $F(x_0)$. Otherwise, if for all $h \in \mathcal{H}'_y$ and all $y' \in h$, there exists $\delta \in [0, 1)$ such that $y \delta y' \in F(x_0)$, then there exist finitely many points in these finitely many hyperplanes such that the convex hull of these points constitute a (closed) neighborhood of y. Hence y is an interior point of $F(x_0)$ which yields a contradiction with the assumption that y is not an interior point of $F(x_0)$. Therefore, F(x) is open for all $x \in X$.

It remains to show that $\operatorname{gr} F$ is open. Towards this end, pick $z_0 = (x_0, y_0) \in \operatorname{gr} F$. Since F has open upper sections and Y satisfies property **B**, there exist finitely many points p^1, \ldots, p^m in $F(x_0)$ such that $V = \operatorname{co}\{p^1, \ldots, p^m\}$ is a neighborhood of y_0 in Y.

For each k = 1, ..., m, we have $p^k \in F(x_0)$, hence $x_0 \in F^{-1}(p^k)$. For each k = 1, ..., m, since $F^{-1}(p^k)$ is open, there exists a neighborhood U^k of x_0 in X such that $U^k \subseteq F^{-1}(p^k)$. Then, the set $U = \bigcap_{k=1}^m U^k$ is a neighborhood of x_0 and $U \subseteq U^k \subseteq F^{-1}(p^k)$ for all k. Hence, for all $x \in U$ and all k = 1, ..., m, $p^k \in F(x)$. For all $x \in U$, since F(x) is convex, V = $co\{p^1, ..., p^m\} \subseteq F(x)$. Then, for all $x \in U$ and all $y \in V$, $x \in F^{-1}(y)$ and $y \in F(x)$. Therefore, $U \times V$ is a neighborhood of z_0 , hence z_0 is an interior point of $\operatorname{gr} F$.

Since a subset of the product space $X \times Y$ can be defined as the graph of a correspondence from X into Y, Theorems 1 and 2 provide characterizations of open sets when the sections of the set satisfy a suitable convexity assumption; see for example Halkin (1966), Uyanik and Khan (2023) and their references for characterizations of open sets in \mathbb{R}^n , and also see Fan (1966) for applications of sets with convex sections.

Proof of Proposition 1. Part (b) directly follows from the argument in the proof of Theorem 2 above as the first step of the proof shows that under convex upper sections, linearly open upper sections is equivalent to open upper sections. To prove part (a), assume F has separately convex and separately open upper sections. Pick $x \in X$. We need to show that F(x) is open. If Yis a product set, then define a correspondence $H : Y_i \twoheadrightarrow Y_{-i}$ for some $i = 1, \ldots, n$ such that $\operatorname{gr} H = F(x)$. Then, by Theorem 1, H has an open graph in the subspace Y, hence F(x) is open in Y. If Y is an open set in \mathbb{R}^n , then define a correspondence $G : \mathbb{R} \twoheadrightarrow \mathbb{R}^{n-1}$ such that $\operatorname{gr} G = F(x)$. Analogously, by Theorem 1, G has an open graph in \mathbb{R}^n . Since Y is open, therefore F(x) is open in Y.²⁵

Proof of Theorem 3. The one-directional strict inclusion relationship among the continuity postulates does not require a convexity assumption and is provided in the antecedent literature; see Ghosh, Khan, and Uyanik (2022, Proposition 9) for details and references.

²⁵Note that the convexity of the upper sections restricted to lines parallel to coordinate axis i is not needed in part (a). For clarity of the exposition, we do not relax this assumption in the statement of the proposition.

In part (a), the statement that under the suitable separate convexity assumption, separate upper (lower) continuity implies upper (lower) continuity follows from Proposition 1(a). Therefore, showing upper (lower) Archimedean implies lower (upper) separate continuity under the suitable separate convexity assumption completes the proof of part (a). Towards this end, assume \succeq has separately convex upper sections, satisfies lower Archimedean property but fails to have separately closed upper sections. Hence, there exists $x \in X$ and a line L parallel to a coordinate axis such that $A_{\succeq}(x) \cap L$ is not closed. Then, there exist $x \in X$, an index i, and a sequence $(y_i^k, \bar{y}_{-i}) \to (y_i, \bar{y}_{-i})$ on the line L_i parallel to coordinate *i* such that $(y_i^k, \bar{y}_{-i}) \succeq x$ for all k and $x \succ (y_i, \bar{y}_{-i})$. By separate convexity, $A_{\succeq}(x) \cap L_i$ is convex. Therefore, if $y_i^k \leq y_i \leq y_i^m$ for some k, m, then $(y_i, \bar{y}_{-i}) \succeq x$. Therefore, either $y_i^n > y_i$ for all n or $y_i^n < y_i$ for all n. Assume wlog that $y_i^n > y_i$ for all n. Then, separate convexity of \succeq implies that $(y'_i, \bar{y}_{-i}) \succeq x$ for all $y'_i \in (y_i, y^1_i]$. It follows from the Archimedean property and $x \succ (y_i, \bar{y}_{-i})$ that there exists $\delta \in (0,1)$ such that $x \succ (y_i, \bar{y}_{-i}) \delta(y_i^1, \bar{y}_{-i}) = (y_i \delta y_i^1, \bar{y}_{-i})$. Since $y_i \delta y_i^1 \in (y_i, y_i^1]$, this furnishes us a contradiction. Therefore, \succeq is separately upper continuous. The proof is analogous to the statement that when \succeq has separately convex lower sections, upper Archimedean implies lower separate continuity.

For part (b), given the relationships above, it remains to show the relationship between restricted solvability and separate continuity. In particular, we show that if \succeq has separately convex lower sections, then restricted solvability implies that \succeq has separately open lower sections. Towards this end assume \succeq has separately convex lower sections,²⁶ is restricted solvable but does not have separately open upper sections. That is, there exists $x \in X$ and a line Lparallel to a coordinate axis such that $A_{\prec}(x) \cap L$ is not open, hence $A_{\succeq}(x) \cap L$ is not closed. Then, there exist $x \in X$, an index i, and a sequence $y^k \to y$ on the line L_i parallel to coordinate i such that $y^k \succeq x$ for all k and $y \prec x$.

Since $y \prec x$, order denseness implies that there exists $x' \in X$ such that $y \prec x' \prec x$. By restricted solvability, there exists $z \in L_i$ such that $z \sim x'$. By transitivity and Sen (1969, Theorem I), $y \prec z \prec x \preceq y^k$ for all k. By convexity assumption and Debreu (1959, p.59), for all $z'_i \in (y_i, z_i], (z'_i, z_{-i}) \preceq z$, and by transitivity and Sen (1969, Theorem I), $(z'_i, z_{-i}) \prec x$. However, since $y^k \to y$, there exists $z'_i \in (y_i, z_i]$, such that $z'_i = y^k_i$ for some k and $(z'_i, z_{-i}) \succeq x$. This yields a contradiction. Therefore, \succeq has closed upper sections. The proof for closed lower sections of \succeq under separately convex upper sections is analogous.

Next, we prove the equivalence between separate convexity and separate monotonicity stated in Section 3.

²⁶This convexity assumption is equivalent to \succ having separately convex lower sections; see Debreu (1959, p.59) for details and further equivalences.

Claim 1. A complete and transitive binary relation \succeq is separately monotone if and only if it is monotone if and only if it has separately convex sections.

Proof of Claim 1. We first show that separate monotonicity implies monotonicity. Assume \succeq is separately monotone (and increasing in each coordinate). Pick $x \ge y$, i.e., $(x_i \ge y_i \text{ for all } i)$. Then, by separate monotonicity, $(x_1, y_{-1}) \succeq y$. Since $(x_1, x_2, y_{\{-1, -2\}}) \ge (x_1, y_{-1})$, by separate monotonicity and transitivity, $(x_1, x_2, y_{\{-1, -2\}}) \succeq y$. By iterating this argument, we obtain $x \succeq y$, hence \succeq is monotone (increasing). The proof of the case where \succeq is decreasing in each coordinate is analogous. The converse relationship is obvious.

To see that separate monotonicity implies separately convex sections, let \succeq be separately monotone (increasing), $(x_i, z_{-i}) \succeq a$, $(y_i, z_{-i}) \succeq a$ and $x_i > y_i$. By monotonicity, $(x_i, z_{-i}) \succeq$ (y_i, z_{-i}) , hence for all $w_i \in (y_i, x_i)$, $(w_i, z_{-i}) \succeq (y_i, z_{-i})$. By transitivity, for all $w_i \in (y_i, x_i)$, $(w_i, z_{-i}) \succeq a$. Hence, \succeq has separately convex upper sections. Now let $(x_i, z_{-i}) \preceq a$, $(y_i, z_{-i}) \preceq a$ and $x_i > y_i$. By monotonicity, $(x_i, z_{-i}) \succeq (y_i, z_{-i})$, hence for all $w_i \in (y_i, x_i)$, $(x_i, z_{-i}) \succeq (w_i, z_{-i})$. By transitivity, for all $w_i \in (y_i, x_i)$, $a \succeq (w_i, z_{-i})$. Hence, \succeq has separately convex lower sections.

To see that separately convex sections imply separate monotonicity, assume \succeq has separately convex sections, pick $i, z \in X$, and $(a_i, z_{-i}), (b_i, z_{-i}) \in X$. Assume wlog that $a_i > b_i$ and $(a_i, z_{-i}) \succeq (b_i, z_{-i})$. By convexity, for all $w_i \in (b_i, a_i), (a_i, z_{-i}) \succeq (w_i, z_{-i}) \succeq (b_i, z_{-i})$. Pick $x_i > a_i$. Let $(x_i, z_{-i}) \prec (a_i, z_{-i})$. Assume wlog that $(x_i, z_{-i}) \succeq (b_i, z_{-i})$. Then \succeq having convex lower sections imply that for all $w_i \in (b_i, x_i), (x_i, z_{-i}) \succeq (w_i, z_{-i})$. By transitivity and $a_i \in (b_i, x_i), (a_i, z_{-i}) \succ (a_i, z_{-i})$, which yields a contradiction. Hence, for all $x_i > a_i$, $(x_i, z_{-i}) \succeq (a_i, z_{-i})$. Analogously, for all $x_i < a_i, (a_i, z_{-i}) \succeq (x_i, z_{-i})$. Therefore, \succeq is separately monotone.

6 References

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