# Pecuniary externalities and constrained inefficiency in an intermediated directed search model of the housing market

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**Abstract**

This paper studies the constrained inefficiency of house sale choices in a heterogeneous agent frictional model of the housing market. Trading frictions are modeled using a broker-intermediated directed search framework. Pecuniary externalities arise due to imperfect risk-sharing between agents and induce inefficiently high sales if house sellers are more constrained as a group than buyers. Under the same condition, a novel finding is that private sellers also list prices which are lower than the efficient list price.

*Keywords*: Directed search; Constrained inefficiency; Housing; Brokers; Pecuniary externalities; Sorting

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# 1 Introduction

The quantitative literature on housing markets in a heterogeneous agents environment (e.g. Hedlund [2016a,b], Garriga and Hedlund [2020]) has increasingly studied the 2006-2011 housing bust using models that incorporate frictions in house purchase and sale. This was intended to capture the price or consumption decline during the Great Recession, or to better understand the interaction of such frictions with other factors behind the housing crash (such as tighter credit or income shocks). This article considers whether selling a house rather than repaying mortgage debt is *constrained efficient* in an intermediated directed search, standard incomplete markets model of the housing market.

The analysis here is *normative* and *ex post*: A Social Planner chooses sale on behalf of owners in order to maximize a social welfare objective subject to market incompleteness (due to uninsurable idiosyncratic risk), financial frictions *and* trading frictions. Hence, this is a constrained Planner's problem in the tradition of Stiglitz [1982] and Geanakoplos and Polemarchakis [1986]. The Planner also takes the other decentralized choices made by agents as given when choosing whether to sell a house, i.e. the Planner only intervenes along the margin of sale choice. For tractability, I assume that the Planner's intervention to affect sale choice is not anticipated by agents in the economy.

Constrained inefficiency, expressed as the wedge between the efficient choice of sale and private sale choice, arises due to pecuniary externalities operating through a price index. The framework featuring directed search and heterogeneity implies that sellers choose market tightness, i.e. the broker-seller ratio, in submarkets differentiated by their characteristics such as income or asset holdings. Each submarket is characterized by a different list (posted) price. The key assumption that allows pecuniary externalities to operate is that all trades are intermediated by brokers. The quasi-Walrasian price index that clears the broker-intermediated housing market affects agents' choices in each individual submarket and thereby links the different submarkets together. This is the channel through which pecuniary externalities [1] associated with sale choices operate.

[1]The pecuniary externality studied here is distributive in nature and arises due to imperfect risk sharing. See Dávila and Korinek [2018] for a characterization of pecuniary externalities in models with financial frictions.

The overall sign of the pecuniary externality depends on whether sellers or buyers are more constrained as a group. Intuitively, if sellers tend to be more constrained, then house sales are inefficiently high as individual sellers do not internalize that the overall losses to other sellers from receiving lower prices outweigh the gains to buyers from having to pay a lower price.

When the model is enriched to consider defaultable debt, I find that default and foreclosure introduces an *additional* inefficiency associated with realized (*ex post*) lender foreclosure losses or deadweight costs.

Directed search models are generally constrained efficient with regard to market tightness choice (the buyer-seller ratio): the choice of market tightness by a Planner subject to the same search and matching frictions as private agents coincides with the decentralized tightness choice. Directed search models allow agents to trade off their returns from trade (posted prices) with the probability of being matched, which endogenously satisfies the Hosios condition for efficiency in search and matching models (see e.g. Rogerson et al. [2005], Wright et al. [2017]). Results on inefficiency in directed search models often hinge on informational frictions, e.g. private information about match-specific productivity for workers (Faig and Jerez [2005], Guerrieri [2008]), or product quality for sellers (Guerrieri et al. [2010], Guerrieri and Shimer [2014]).

I examine whether individual market tightness choices (the ratio of brokers to sellers in this framework) are constrained efficient. I show that pecuniary externalities introduce a wedge between private and constrained efficient market tightness choice. For example, if sellers are more constrained than buyers, the pecuniary externality would lead to a Planner choosing lower market tightness and thereby reducing the probability of a successful sale to keep the price index high and thereby benefit sellers in the economy. One could alternatively express this inefficiency in terms of list prices: private sellers set list prices too low relative to the efficient level.

To the best of my knowledge, the constrained inefficiency of tightness choice (or list prices) due to pecuniary externalities has not been shown before in the directed search literature. Trade intermediation by brokers is again at the heart of this result. In addition, modeling intermediation through brokers simplifies the analysis of matching with two-sided heterogeneity, as discussed below.

## 1.1 Related literature

Pecuniary externalities and inefficiency arising therefrom have been discussed extensively in various environments (see Dávila and Korinek [2018] and the references therein). The classification of pecuniary externalities as being distributive or collateral in nature was made by Dávila and Korinek [2018]. Though the emphasis in this literature has been on intervention *ex ante,* there is also a smaller literature that evaluates the use and implications of *ex post* intervention owing to the same source of inefficiency (e.g. Jeanne and Korinek [2020]). Pecuniary externalities in a standard incomplete markets model were first investigated by Davila et al. [2012].

A related paper (Contractor [2020]) derives the inefficiencies associated with mortgage default choices in a standard incomplete markets housing model where housing trades are Walrasian (frictionless). As in this paper, I employ a normative analysis to study the constrained efficient mortgage termination choice. The key difference is the treatment of the housing market, which explicitly incorporates search frictions using the directed search approach here. In addition, I study one-period mortgage debt contracts for simplicity here, as opposed to the treatment in Contractor [2020] that models mortgages as long-term debt contracts. Finally, the baseline results in this paper describe inefficiencies arising due to house sales. It also does not consider collateral externalities associated with collateral constraints for home buyers, which are incorporated in Contractor [2020].

Seminal papers introducing the directed search approach in the macro-labor literature are Moen [1997] and Acemoglu and Shimer [1999], who also discuss efficiency in relation to the canonical labor search and matching literature discussed in, e.g. Pissarides [2000]. Wright et al. [2017] is a recent survey that discusses efficiency and other applications of the directed search framework.

Macroeconomic models with housing are discussed in, e.g., Davis and Van Nieuwerburgh [2015] and Piazzesi and Schneider [2016]. A branch of literature using search models to study liquidity in the housing market includes Wheaton [1990], Ngai and Tenreyro [2014], Head et al. [2014]. These models do not either allow for or consider how credit or net worth affects housing. Guren and McQuade [2020] use a search model to study the feedback between foreclosures and house prices, but also does not allow for borrowing or saving. Recent quantitative models that use a heterogeneous agents directed search model include Hedlund [2016a,b], Garriga and

Hedlund [2020], Jerez et al. [2020]. These articles discuss sorting of market tightness choices by agents' financial state variables, which I consider as well. However, they do not discuss pecuniary externalities or constrained inefficiency, which is the primary focus of this paper.

### 1.2 Outline of paper

Section 2 describes the model and the stationary equilibrium. Section 3 examines how the market tightness choices of potential sellers vary with their risk-free asset holdings. Section 4 describes the pecuniary externality based inefficiency associated with the decision to sell. Section 5 discusses some extensions concerning frictional house purchases, the default option and the inefficiency associated with tightness choice. Section 6 concludes. Supplementary material is contained in two appendices. An outline of the material in the appendices can be found at the beginning of appendix A.

## 2 Model

The model is an infinite horizon standard incomplete markets model with housing. Heterogeneity arises *ex post* due to the saving and borrowing choices of agents facing uninsurable idiosyncratic risk, in addition to the tenancy/ownership choice. Trading on the supply side of the housing market is frictional, whereas for simplicity homebuyers are assumed to trade frictionlessly. All trades are intermediated by brokers.

There are **four** types of agents in the economy: (i) *renters*, who choose whether to own a house by taking out a mortgage or remain as tenants; (ii) *owners*, who decide whether to sell their house through a directed search mechanism or repay their mortgage; (iii) risk-neutral *financial intermediaries/lenders* who lend to homebuyers; and (iv) risk-neutral *brokers*, who purchase houses from owners in a competitive search process and in turn sell houses frictionlessly to buyers.

## 2.1 Environment

Time is discrete, continues forever and is indexed by $t = 0, 1, 2, ....$ There is a continuum of agents who receive an endowment $y$ drawn independently according to a Markov process with values in set $Y$. The probability that an endowment transitions from current level of $y$ to $y'$ is given by the transition matrix $\Pi(y'|y)$. In the following, I use primes to denote variables tomorrow.

Agents have period utility functions $u(c, \chi h)$, hence they receive benefits from consumption and housing services if they are owners (the housing preference parameter $\chi > 1$ for owners). For simplicity, it is assumed that rental housing yields housing services one-for-one with house size, i.e. $\chi = 1$. Utility is separable between consumption and housing services. All agents discount the future using discount factor $\beta$.

## 2.2 Housing market

Housing (owner-occupied and rental) is of a single size ($h = 1$), and agents can only own one house at a time. There is a fixed total housing stock and no construction sector. The owner-occupied and rental sectors of the housing market are segmented, so there is no convertibility between owner-occupied and rental housing space, which would otherwise tie down a relationship between house prices and rents (see e.g. Kaplan et al. [2019], Greenwald and Guren [2019]). This also implies that the homeownership rate is fixed in the economy. Rents are fixed and are assumed to be earned by absentee (un-modeled) landlords. Endogenous rents would introduce a channel of pecuniary externalities that would arise through rents, which I do not consider in this paper.

Heterogeneous home sellers trade their houses in a frictional decentralized market, following Garriga and Hedlund [2020]. The trading mechanism employs the directed search approach, wherein sellers post their prices and their trade counterparts direct their search accordingly. In common with most directed search models, sellers face a tradeoff between the price they post and their sale probability.

I follow the directed search housing literature (e.g. Hedlund [2016b], Garriga and Hedlund [2020]) and assume that all housing trades are intermediated by real estate brokers in order to account for two-sided heterogeneity on the side of buyers and sellers. As these papers discuss, in the absence of brokers, matching between heterogeneous sellers and buyers requires each party to forecast the dynamics of the entire distribution of income, assets and debt in order to calculate their trading probability in each submarket. Introducing brokers breaks this down to a one-sided heterogeneous matching problem, which considerably simplifies the analysis.

Hence, brokers buy houses from sellers in a frictional market, and can trade with other brokers and with buyers in a frictionless (quasi-Walrasian) market at price index $p$ that clears the broker-intermediated housing market in equilibrium. Introducing frictional trading on the demand side adds complexity without adding to the basic result of the model regarding pecuniary externalities associated with house sales, hence I make the Walrasian trading assumption on the demand side. Section 5.1 describes how the main result would be altered if home purchases were also frictional.

**Search in the housing market**

The probability $\eta(\theta)$ that a seller matches with a broker in any given submarket depends on the ratio of brokers to sellers in that submarket ($\theta$), i.e. the *market tightness*. I assume that $\eta^{'}(\theta) > 0$, $\eta^{''}(\theta) < 0$ and $C^2$, with $\eta(0) = 0$ and $\lim_{\theta\to\infty} \eta(\theta) = 1$. The corresponding probability that brokers match with a seller in a given submarket with tightness $\theta$ is then $\alpha(\theta) = \frac{\eta(\theta)}{\theta}$. Further, $\alpha(\theta)$ is strictly decreasing and $C^2$, with $\lim_{\theta\to\infty} \alpha(\theta) = 0$ and $\lim_{\theta\to 0} \alpha(\theta) = 1$. The elasticity $\epsilon(\theta) \equiv \frac{\eta^{'}(\theta)\theta}{\eta(\theta)}$ is assumed to be non-increasing.

As in Jerez et al. [2020], I extend the matching functions $\eta$ and $\alpha$ to domain $\Theta \equiv \mathbb{R}_+ \cup \{\theta_0\}$, with fictitious submarket $\theta_0 \in \mathbb{R}_-$ such that $\eta(\theta_0) = \alpha(\theta_0) = 0$. This fictitious submarket represents owners who choose not to sell but to repay their mortgage debt, hence it corresponds to non-participation in the frictional house sale market.

Brokers face transaction costs $\kappa_s$ and receive net revenue $\big(p - p(\theta)\big)$ when they match with a seller. Free entry of brokers into each submarket and the zero-profit condition require:

$$\kappa_s \geq \alpha(\theta)\Big(p - p(\theta)\Big) \tag{1}$$

The combination of directed search and free entry of brokers implies that market tightness and hence sale probabilities depend only on the price index $p$ and *not* on the distributions of owners and renters in the economy. This is referred to in the literature as *block recursivity* (see also Menzio and Shi [2010], Menzio et al. [2013], Wright et al. [2017] for a discussion in other contexts).

## 2.3 Financial markets

All households can save in a risk-free asset at rate $R$. Additionally, home buyers can finance their house purchases by borrowing.

Mortgages are one-period debt contracts, and I assume that owners can also refinance their debt each period. As I assume that mortgages are non-defaultable, the borrowing limit depends on the worst possible realization of income (denoted by $\underline{y}$). Thus, if $a^{'}$ represents saving (which is negative if an owner borrows), then the borrowing constraint faced by owners can be written as:

$$a^{'} \geq -\underline{y} \tag{2}$$

Loan price in the absence of default is then simply $R^{-1}$.

## 2.4 Choices and value functions

The choices of different types of agents can be summarized as follows:

- Current owners can *continue* as owners and make their mortgage payment (with or without refinancing) or *sell* their house. Unsuccessful sales lead to continuation and repayment.
- Renters can *buy* a house and become an owner, or continue to *rent*.

In the notation to follow, the short term constraint for tenants and sellers precludes borrowing, $a'(s) \geq 0$, i.e. the domain for saving choice is the set $a' \in \mathbb{R}_+$.

**Current owner**

As mentioned above, a current owner in each period chooses whether to continue with ownership by repaying, or sell his house ($\sigma = 1$). Upon successful sale, the agent loses possession of his house immediately and rents in that period. I denote the fixed rent by $\rho$ below. If sale is unsuccessful, the owner has to repay the mortgage and remain an owner. The state variables for an owner in each period include his endowment $y$ and asset position $a$. Let the vector of state variables be $s^o = (y, a)$. The expectations operator $E$ is defined over $y^{'}$ given $s^o$ using transition probabilities $\Gamma$.

### 2.4.1 Continuation and matched seller value functions

I represent the value of **continuing** with a mortgage contract by $V^c(s_t^o)$. Then, the Bellman equation for continuing is:

$$V^c(s_t^o) = \max_{\{c,a'\}} u(c, \chi h) + \beta \mathbb{E} \max \left\{ V^c(s_{t+1}^o), V^s(s_{t+1}^o) \right\} \tag{3}$$

subject to

$$c + \frac{a^{'}}{R} = y + a$$

and

$$a^{'} \geq -\underline{y}$$

If a potential seller is *successfully matched* and **sells** his house, he keeps the proceeds from selling his house after paying off his loan (if $a < 0$).The Bellman equation for a matched seller is:

$$v^s(y, a) = \max_{\{c,a' \geq 0\}} u(c, h) + \beta \mathbb{E} V^r(y', a') \tag{4}$$

subject to

$$c + \frac{a'}{R} + \rho = y + a + p\Big(\theta\big(s^o\big)\Big)$$

#### 2.4.2 Potential seller value function

An owner with state $s_t^o$ who chooses to sell need not be matched successfully, due to the frictional sale mechanism. His sale probability $\eta(s_t^o)$ depends on the market tightness, $\theta(s_t^o)$ and the list price chosen, $p\big(\theta(s_t^o)\big) \in \mathbb{R}_+$.

The potential seller then chooses market tightness $\theta(s_t^o)$ in order to maximize his expected payoff from choosing sale $\big(\sigma(s_t^o) = 1\big)$, which I denote by $V^s(s_t^o)$:

$$V^s(s_t^o) = \max_{\theta(s_t^o) \in \Theta} \eta\big(\theta(s_t^o)\big) v^s(s_t^o) + \Big(1 - \eta\big(\theta(s_t^o)\big)\Big) V^c(s_t^o) \tag{5}$$

subject to

$$p\big(\theta(s_t^o)\big) = p - \frac{\kappa_s}{\alpha\big(\theta(s_t^o)\big)} \geq -a - y \tag{6}$$

The constraint in equation $(6)$ is the zero profit condition for brokers, and is similar to other directed search models wherein the agent choosing market tightness has to ensure that his trading counterpart meets a certain reservation utility level [2]. In this model, free entry of brokers into active submarkets (which is the subset of $\Theta$ that contains the solutions to the above problem, i.e. those submarkets that attract both sellers and brokers) implies that they must be guaranteed zero profits, hence the list price must satisfy equation $(6)$ in active submarkets. Sellers therefore internalize the 'participation constraint' for brokers (the list price that yields brokers zero profits) when choosing submarket tightness.

Jerez [2014] shows that one could instead consider a problem wherein both sellers and brokers choose submarket tightnesses taking the price function $p(\theta)$ as given, which she refers to as the 'price taking approach'. Houses traded in different submarkets $\theta$ can be thought of as different commodities, and the price function $p : \Theta \to \mathbb{R}_+$ prices houses in different submarkets. In equilibrium, sellers and brokers take $p(\theta)$ as given and have rational expectations about the tightness levels in active submarkets.

Equation $(6)$ implies a tradeoff between list price and trade probability, as $\alpha'(\theta) < 0$; which is standard in directed search models. Note that one could rewrite the

[2] This is referred to in the directed search literature as the market utility approach

problem for a potential seller as the choice of a list price, with the market tightness being based on expressing equation $(6)$ in terms of $\theta$.

From the borrowing constraint in equation $(2)$, $-a - y \leq 0$, hence the price constraint is simply that $p(\theta) = p - \frac{\kappa_s}{\alpha\left(s_t^o\right)} \geq 0$.

If the constraint binds, $\theta$ solves the following equation:

$$\alpha(\theta) = \frac{\kappa_s}{p} \tag{7}$$

The owner value function for each state vector is defined as the upper envelope of the value functions associated with continuation and potential sale defined at the same state vector:

$$V^o(s_t^o) = \max\left\{V^c(s_t^o), V^s(s_t^o)\right\} \tag{8}$$

## Renter

The renter can either purchase a home through a mortgage $(\omega = 1)$, in which case he becomes an owner, or choose to remain a renter $(\omega = 0)$, which is referred to as tenancy. The relevant state variables for a renter are his income $y$ and asset level $a$, so a renter's state vector $s^r = (y, a)$.

A renter who chooses not to purchase a house remains a **tenant**, paying rent $\rho$, and has the value function:

$$V^t(s^r) = \max_{\{c, a' \geq 0\}} u(c, h) + \beta \mathbb{E} V^r(y', a') \tag{9}$$

subject to

$$c + \frac{a'}{R} + \rho = y + a$$

A renter who chooses to **buy** a house will do so by purchasing a mortgage. Given house value $p$ and transaction cost that is proportional to the house price index, $\kappa_b p$, his initial asset choice would be $a^{'}$.

It is assumed that a buyer enjoys homeownership utility premium in the period of purchase, i.e. he gets immediate possession of the house. Therefore, his value function would be:

$$V^b(s^r) = \max_{\{c,a'\}} u(c, \chi h) + \beta \mathbb{E} V^o(y', a') \tag{10}$$

subject to the borrowing constraint in equation $(2)$ and the budget constraint,

$$c + \frac{a'}{R} = -p(1+\kappa_b) + a + y$$

The renter value function for each state vector is defined as the upper envelope of the tenant's and homebuyer's value functions defined at the same state vector:

$$V^r(s^r) = \max \left\{ V^t(s^r), V^b(s^r) \right\} \tag{11}$$

## 2.5 Distributions of owners and renters

The distributions of owners $(\mu^o)$ and renters $(\mu^r)$ are defined over the relevant state space $Y \times \bar{A}$, where $\bar{A} = \{a : a \geq \underline{a}\}$.

Given the initial distributions of owners and renters ($\mu_0^o$, $\mu_0^r$ respectively), the policy functions $\Big\{a'(s), \theta(s), \omega(s), \sigma(s)\Big\}$ and the transition matrix for the Markov endowment process $\Pi$, this section describes the evolution of distributions from $\Big\{\mu^o, \mu^r\Big\}$ in a given period to new distributions denoted by $\Big\{T\mu^o, T\mu^r\Big\}$ in the next period, where $T$ is the updating operator.

The updating process for the distributions is:

- Renter distribution $(\mu^r)$:

$$T\mu^r(y', a') = \sum_{y \in Y} \int_{a \in \bar{A}} 1_{\{a'(y,a)=a'\}} * \Big(1 - \omega(y,a)\Big) * \Pi(y,y') * d\mu^r(y,a)$$

$$+ \sum_{y \in Y} \int_{a \in \bar{A}} 1_{\{a'(y,a)=a'\}} * \Big(\sigma(y,a) * \eta\big(\theta(y,a)\big)\Big) * \Pi(y,y') * d\mu^o(y,a) \tag{12}$$

- Owner distribution $(\mu^o)$ :

$$T\mu^o(y^{'}, a^{'}) = \sum_{y\in Y}\int_{a\in\bar{A}} 1_{\{a'(y,a)=a'\}} * \Bigg(1 - \Big(\sigma(y,a) * \eta\big(\theta(y,a)\big)\Big)\Bigg) * \Pi(y,y') * d\mu^o(y,a)$$
$$+ \sum_{y\in Y}\int_{a\in\bar{A}} 1_{\{a'(y,a)=a'\}} * \omega(y,a) * \Pi(y,y^{'}) * d\mu^r(y,a) \tag{13}$$

Here, $1_{\{a'(y,a)=a'\}}$ is an indicator for whether the savings policy function for an agent with state $(y, a)$ yields saving level $a'$.

## 2.6 Stationary equilibrium

Stationary equilibrium consists of price index $p$, distributions of owners and renters $\{\mu^o, \mu^r\}$, value functions $\Big\{V^c, V^s, v^s, V^t, V^b, V^r, V^o\Big\}$ and associated policy functions $\Big\{a'^r, a'^o, \theta, \omega, \sigma\Big\}$ that satisfy:

1. Owners and renters make their choices as described in section 2.4 given house price index $p$ and rent $\rho$
2. Given $p(\theta)$, $\theta \geq 0$ and $\kappa_s \geq \alpha(\theta)\Big(p - p(\theta)\Big)$ with complementary slackness
3. Price index equate demand and supply in the broker-intermediated housing market:
$$\int_{s^r} \omega(s^r; p) * d\mu^r(s^r) = \int_{s^o} \sigma(s^o; p) * \eta(s^o; p) * d\mu^o(s^o) \tag{14}$$
4. The distributions $\{\mu^j\}_{j=r,o}$ are consistent with individual sale $(\sigma)$ and purchase choices $(\omega)$ and evolve as in section 2.5.

Condition $2$ states that, given a list price function $p(\theta)$, all active submarkets entail brokers making zero profits. Implicit in the discussion in section 2.4 is the assumption that *all* submarkets have a list price function given by equation $(6)$. In other words, inactive submarkets also have the same price function, that agents internalize when choosing tightness. If agents were to deviate to an inactive submarket, they would need to internalize that the price function in that submarket would need to satisfy the zero-profit condition for brokers while choosing the tightness level. This corresponds to the restrictions imposed on out-of-equilibrium beliefs in directed search models (see e.g. Menzio and Shi [2010], Jerez et al. [2020]). Jerez et al. [2020] employ the price-taking approach and argue that this corresponds to assuming that inactive submarkets choose the lowest price that supports the equilibrium allocation in this model.

Condition 3 requires the broker intermediated housing market to clear. The supply of houses to brokers is from matched sellers, hence the RHS of equation (14) includes $\eta(\theta)$. As I have assumed that buyers can trade frictionlessly with brokers, the LHS of equation (14) simply aggregates over all buyers.

# 3 Equilibrium sorting patterns for tightness choice

In this section, I consider the variation of market tightness with asset level when the non-negativity constraint on list price does not bind. In order to facilitate this analysis, it would be convenient to establish differentiability results for the value functions $v^s$ and $V^c$, which is not straightforward. In appendix A, I follow the approach in Jerez et al. [2020], Clausen and Strub [2020] to establish differentiability results in the interior of the choice set.

Having established differentiability of the choice specific value functions, the tightness choice solves equation (5), which I rewrite below:

$$\max_{\theta(s_t^o)\in\Theta} \eta\Big(\theta(s_t^o)\Big)v^s(s_t^o) + \Big(1-\eta\big(\theta(s_t^o)\big)\Big)V^c(s_t^o)$$

Define the surplus from sale over repayment by $S(s_t^o) \equiv v^s(s_t^o) - V^c(s_t^o)$. The first-order condition for an interior choice of $\theta$ is:

$$\eta^{'}\big(\theta(s_t^o)\big)S(s_t^o) + \eta\big(\theta(s_t^o)\big)v_p^s(s_t^o)p^{'}(\theta) = 0 \tag{15}$$

Rewrite the above as:

$$p^{'}\big(\theta(s_t^o)\big) = \frac{-S(s_t^o)}{v_p^s(s_t^o)}\frac{\eta^{'}\big(\theta(s_t^o)\big)}{\eta\big(\theta(s_t^o)\big)} \tag{16}$$

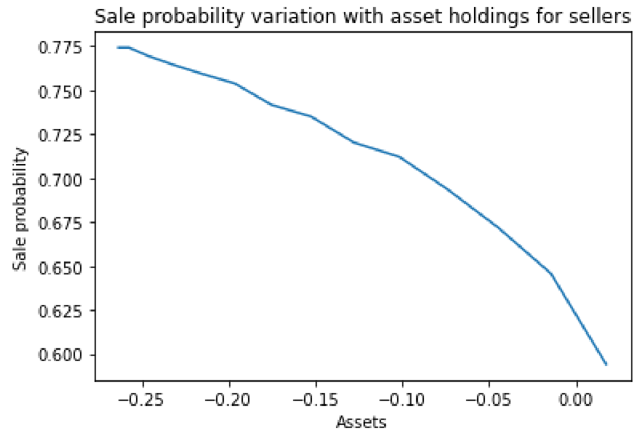


Figure 1: **Variation of sale probability with asset holdings**. *Sellers with more assets have a favourable outside option in terms of repayment, and therefore charge higher prices while trading off lower sale probabilities*

From equation $(6)$, $p^{'}(\theta) = \frac{\kappa_s}{\alpha^2(\theta)} * \alpha^{'}(\theta)$

Substituting this into the above expression, and using the definition of the elasticity of the matching function:

$$\epsilon(\theta) \equiv \frac{\eta^{'}(\theta)\theta}{\eta(\theta)}$$

the F.O.C for tightness becomes:

$$\frac{S(s_t^o)}{v_p^s(s_t^o)} = \frac{\Big(1-\epsilon\big(\theta(s_t^o)\big)\Big)}{\epsilon\big(\theta(s_t^o)\big)}\Big(p - p\big(\theta(s_t^o)\big)\Big) \tag{17}$$

Given $p(\theta)$, the RHS of equation $(17)$ is increasing in $\theta$. Hence, the variation of $\theta$ with asset levels depends on how the LHS varies with $a$. If $v^s(y, a)$ is concave in asset levels for given $y$, then the denominator is decreasing in $a$. Appendix A.3 provides conditions under which $v^s(y, .)$ is concave in $a$.

If the numerator is non-increasing in $a$ for given $y$, as is the case numerically, then the overall sign of the relationship between $\theta$ and $a$ will depend on which term on the LHS of equation $(17)$ dominates. Numerically, it turns out that the numerator of the LHS dominates, hence tightness decreases with asset level. In other words, lower debt (higher $a$) lowers market tightness and raises the list price. Hence, sellers with more assets have a favourable outside option in terms of repayment, and therefore charge higher prices while trading off lower sale probabilities. This can be seen in Figure **1** [3].

One can also show that, under some assumptions, owners choose to sell when their assets are below a certain threshold, and repay otherwise.

**Proposition 1:** If $V^c$, $V^s$ and $v^s$ are continuous in $a$ and $S(y,.)$ is non-increasing in $a$ for all $y$, then $\exists\hat{a}(y)$ for all $y$ such that $\sigma(y,a)=1$ when $a<\hat{a}(y)$, and $\sigma(y,a)=0$ otherwise.

*Proof:* See appendix B.2.

# 4 Decentralized and efficient sale choice

I now consider the difference between decentralized and socially efficient sale choice, i.e. the wedge between private and Planner sale choice.

## 4.1 Decentralized sale choice

Consider an owner with state $s^o=(y,a)$ who chooses whether to sell $\sigma(y,a)=1$ or not. From equation $(8)$, sale is chosen if:

$$V^s(y,a)>V^c(y,a)$$

Using the definition of $V^s(y,a)$ in equation $(5)$, if sale probability is positive, this condition becomes equivalent to:

$$v^s(y,a)>V^c(y,a)$$

In this environment with frictional matching for sellers, owners choose to sell if their expected surplus from sale over ownership (repayment) is positive. As discussed above, this surplus is generally decreasing in asset holdings, hence if sale is

[3] The figure is plotted using the parameter values from Contractor [2020], with the following exceptions: $\{\chi=1.5,\beta=0.95,\kappa=0.1\}$. The matching function, $\eta(\theta)=(1+\theta^{-\gamma})^{\frac{-1}{\gamma}}$, is from Jerez et al. [2020], with $\gamma=0.65$.

chosen for a given income level, it is usually when an owner is either indebted or has low positive asset holdings.

## 4.2 Social Planner's sale choice and the pecuniary externality

The Social Planner chooses sale on behalf of an owner with state $(y, a)$ in order to maximize a utilitarian social welfare function (SWF). The utilitarian SWF is akin to Davila et al. [2012], and can be motivated by the feature of standard incomplete markets models that agents are *ex ante* identical, prior to the realisation of idiosyncratic endowment shocks that generate heterogeneity through saving and asset holdings. Note that brokers and financial intermediaries are not included in this specification as they make zero profits (*ex ante* and realized).

The masses of owners and renters in the economy are denoted by $d\mu^o(y, a)$ and $d\mu^r(y, a)$, and are determined as part of the equilibrium. Extending the recursive definition of Davila et al. [2012] to the setting here, define the Planner's problem as:

$$\Omega\Big(\mu^o, \mu^r\Big) = \max_{\sigma} \sum_{y \in Y} \int_{a \in \bar{A}} u(c, h) * d\mu^r(y, a) + \sum_{y \in Y} \int_{a \in \bar{A}} u(c, h) * d\mu^o(y, a) \\ + \beta E \Omega\Big(\mu^{'o}, \mu^{'r}\Big) \tag{18}$$

The Planner makes sale choice in order to maximize the SWF, taking as given the other decentralized choices of ownership and saving/borrowing. The Planner internalizes the house price determination mechanism when choosing whether to sell or not. I assume that the Planner's intervention is not anticipated by agents. The Planner is also constrained by the market incompleteness, which arises here through the borrowing constraint faced by owners and the presence of uninsurable idiosyncratic risk by all agents in the economy. Hence, I focus on *constrained efficiency*.

It is easier to compare the Planner's sale choice to the decentralized sale choice using an alternative expression for the SWF objective (in appendix B.1) based on the definitions of choice-specific value functions. It is straightforward to show the equivalence of these two representations.

**Proposition 2:** The Planner chooses sale ($\sigma(y,a)=1$) for an owner with state $(y,a)$ if the following condition holds:

$$v^s(y,a) - V^c(y,a) + \text{Pecuniary externality} > 0 \tag{19}$$

*Proof:* See appendix B.3.

The pecuniary externality (PE) term here is:

$$\begin{aligned}\text{PE} = &\sum_{y\in Y}\int_{a\in\bar{A}}\Big(\sigma(y,a)*\eta\big(\theta(y,a)\big)*u_c(c,h)*\frac{\Delta p(\theta)}{\Delta p}*d\mu^o(y,a)*\frac{\Delta p}{\Delta S}\Big)\\ &-\sum_{y\in Y}\int_{a\in\bar{A}}\Big(\omega(y,a)*u_c(c,\chi h)*(1+\kappa_b)*d\mu^r(y,a)*\frac{\Delta p}{\Delta S}\Big)\end{aligned} \tag{20}$$

**Pecuniary externality associated with sale choice:**

In this model, the pecuniary externality is *distributive* in nature. It comprises differences in marginal utilities of consumption between agents (matched sellers and buyers). These enter into the expression as matched sales increase the supply of houses, driving the price index down and thereby affecting other matched sellers and buyers in the economy. The price impact of a successful sale in equation (20) is represented by $\frac{\Delta p}{\Delta S} \leq 0$.

The first term in equation (20) is the impact of a marginal sale on matched sellers, hence the product $\sigma(y,a)*\eta\big(\theta(y,a)\big)$. Further, matched sellers who are constrained by their debt position, from equation (2), are not affected by movements in the price index, i.e. $\frac{\Delta p(\theta)}{\Delta p}=0$. Otherwise, $\frac{\Delta p(\theta)}{\Delta p}=1$, from equation (6).

Distributive externalities in this environment arise when changes in net worth due to asset price changes affect buyers and matched sellers of the asset differently. With complete markets, changes in net worth would not matter as agents could perfectly hedge risk and equalize their marginal rates of substitution (cf. Dávila and Korinek [2018]), and the market clearing condition would imply that asset price changes would have zero aggregate impact. However, whenever agents face incomplete financial markets, they generally cannot equate their marginal rates of substitution across dates or states. This implies that, along with transaction costs incurred, distributive externalities do not wash out in the aggregate.

The market incompleteness here is due to the presence of uninsurable idiosyncratic risk and the inability to dissave. A Planner would then intervene to change asset prices, by suitably changing sale intensity, so as to benefit the agents in the economy who have higher marginal utility of consumption (the agents who are more constrained). For example, if sellers are more constrained than buyers, the Planner would choose to reduce sale intensity to raise prices and therefore benefit sellers as opposed to buyers. This can be understood from equation (20): the first term on the RHS is the pecuniary externality faced by sellers and is clearly higher, ceteris paribus, if the marginal utility of consumption is large (as would be the case for constrained sellers). If buyers tend to have lower marginal utilities of consumption, then the first term dominates and the overall PE term has a *negative* sign because $\frac{\Delta p}{\Delta S} \leq 0$. This implies, from equation (19) that the Planner would be *less* likely to choose to sell a house.

**Broker intermediation and pecuniary externalities**

Generally, pecuniary externalities have been analysed in environments where the asset market is Walrasian, i.e. where asset buyers and sellers can trade frictionlessly at a price that clears the market in equilibrium. The current framework differs in that sellers do not match directly with buyers; instead, sales are intermediated by brokers. Although the broker intermediation assumption is made in order to tractably incorporate two-sided heterogeneity (by yielding block recursivity) it also provides a means to link different submarkets together through the price index that clears the overall broker-intermediated market. Matched sellers increase the stock of houses that are traded to brokers, hence with downward sloping demand by buyers, the additional supply triggers a decline in the price index. Therefore, the introduction of brokers provides a *quasi-Walrasian* market clearing price that affects the net worth of agents in active submarkets.

### Distributive externalities and policy

Distributive externalities generally arise in environments with market incompleteness, asset trading and agent heterogeneity. As Dávila and Korinek [2018] discuss, there are various, mainly theoretical, articles that rely on distributive pecuniary externalities to motivate policy intervention (e.g. Lorenzoni [2008], He and Kondor [2016], Itskhoki and Moll [2019]). As distributive externalities rely on differences in agents' net worth, which are affected by asset prices, the rationale for policy interventions based on correcting these externalities is to transfer resources from less constrained to more constrained agents.

## 4.3 Corrective policy

Having established the presence of an inefficiency wedge associated with sale choice, I now consider policy that could implement the efficient sale choice. The policymaker could induce agents to make the efficient sale choice through the use of many different instruments. For example, sale can be penalized through the use of a tax on cash in hand, whereas repayment could be incentivized through the use of subsidies to net worth (or reductions in debt). Here, I describe the use of such subsidies to implement efficient sale choice. These debt reductions are only offered to owners who inefficiently choose to sell their house.

Owners who choose sale inefficiently can be made to choose repayment through a suitable subsidy policy. In particular, a potential seller with state $(y, a)$ and continuation value $V^c(y, a)$ receives a subsidy $\tilde{a}(y, a)$ so as to implement the efficient sale choice:

$$V^c(y, \tilde{a}) = V^s(y, a)$$

As the value function is continuous and nondecreasing in $a$ and the asset set is compact, one can use the intermediate value theorem to obtain a solution $\tilde{a}(y, a)$. As negative asset holdings correspond to debt, the subsidy policy can also be interpreted as a loan balance reduction for indebted owners who choose to sell inefficiently.

The subsidy (debt reduction) fraction is contingent on the agent state. It is greater for owners with low income and/or assets, as they require more inducement to choose efficient repayment over sale.

# 5 Extensions

The model presented above is the simplest environment in which to analyse directed search trading mechanisms that yield pecuniary externalities and inefficiencies associated with sale choice. I now discuss some possible extensions to the model and the differences in results that they generate.

## 5.1 Frictional home purchases

It is a simple matter to extend the model above to frictional home purchases, so that both sides of the market operate in a directed search environment intermediated by brokers (as in Garriga and Hedlund [2020]).

The problem for prospective buyers becomes analogous to that for sellers, described in section 2.4. Thus, a buyer with state $s_t^r$ chooses market tightness $\theta(s_t^r)$ in order to maximize his expected payoff from choosing ownership $(\omega(s_t^r) = 1)$:

$$\tilde{V}^b(s_t^r) = \max_{\theta(s_t^r)\in\Theta} \eta^b\big(\theta(s_t^r)\big)V^b(s_t^r) + \Big(1 - \eta^b\big(\theta(s_t^r)\big)\Big)V^t(s_t^r) \tag{21}$$

subject to

$$p\big(\theta(s_t^r)\big)(1+\kappa_b) = p + \frac{\kappa_s}{\alpha^b\big(\theta(s_t^r)\big)} \tag{22}$$

and

$$p(1+\kappa_b) \leq y + a$$

Thus, a prospective buyer chooses submarket tightness taking as constraints the zero profit requirement (participation constraint) for brokers, and the feasibility requirement for posted prices.

The broker-intermediated market clearing condition, which determines the equilibrium price index, becomes:

$$\int_{s^r} \omega(s^r;p) * \eta^b(s^r;p) d\mu^r(s^r) = \int_{s^o} \sigma(s^o;p) * \eta(s^o;p) * d\mu^o(s^o) \tag{23}$$

Finally, the expression for the pecuniary externality associated with sale choice is readily modified to:

$$\text{PE} = \sum_{y \in Y} \int_{a \in \bar{A}} \Big( \sigma(y,a) * \eta\big(\theta(y,a)\big) * u_c(c,h) * \frac{\Delta p(\theta)}{\Delta p} * d\mu^o(y,a) * \frac{\Delta p}{\Delta S} \Big)$$

$$- \sum_{y \in Y} \int_{a \in \bar{A}} \Big( \omega(y,a) * \eta^b\big(\theta(y,a)\big) * u_c(c,\chi h) * (1+\kappa_b) * d\mu^r(y,a) * \frac{\Delta p}{\Delta S} \Big) \tag{24}$$

## 5.2 Defaultable debt

I now consider how introducing defaultable debt alters the results.

To simplify the exposition, I revert to the assumption in the baseline model above that buyers can transact frictionlessly with brokers (taking the price index as given). Default leads to lenders possessing the house. Lenders are also assumed to transact frictionlessly with brokers at the given price index [4]. However, I assume that there is a deadweight loss associated with lender ownership, so lenders receive only a fraction $\zeta < 1$ of the sale price.

This requires a modification of the problem above, as lenders now account for default risk while pricing a loan. If an owner with state $(y,a)$ chooses borrowing level $a^{'}$, the loan pricing function $Q(y,a^{'})$ is:

$$Q(y,a^{'}) = \left\{ \begin{array}{ll} R^{-1}\mathbb{E}_{y^{'}|y}\Big[\sigma(y^{'},a^{'})\Big(1-\eta(\theta(y^{'},a^{'}))\min\{\frac{\zeta p^{'}}{a^{'}},1\}\Big) + \Big(1-\sigma(y^{'},a^{'})\Big(1-\eta\big(\theta(y^{'},a^{'})\big)\Big)\Big)\Big] & \text{if} \quad a^{'} < 0 \\ R^{-1} & \text{if} \quad a^{'} \geq 0 \end{array} \right\}$$

This amends the value functions of buyers and owners who repay as follows:

[4] This can be relaxed, as in Garriga and Hedlund [2020], but complicates the analysis considerably.

$$V^c(y,a) = \max_{a' \geq -\iota p} u(y + a - Q(y,a')a', \chi h) + \beta \mathbb{E} V^o(y', a')$$

$$V^b(y,a) = \max_{a' \geq -\iota p} u\Big(y + a - Q(y,a')a' - p\big(\theta(y,a)\big)(1+\kappa_b), \chi h\Big) + \beta \mathbb{E} V^o(y', a')$$

The maximum possible debt is then $\iota p$, where the credit constraint arises from informational and/or institutional frictions that are not modeled explicitly.

I assume that default occurs when potential sellers fail to be matched [5]. This modifies the value functions associated with potential sale to:

$$V^s(s_t^o) = \max_{\theta(s_t^o) \in \Theta} \eta\big(\theta(s_t^o)\big) v^s(s_t^o) + \Big(1 - \eta\big(\theta(s_t^o)\big)\Big) V^d(s_t^o)$$

subject to equation (6).

Further, default is assumed to lead to permanent tenancy, and the value function associated with default is given by:

$$V^d(y,a) = \max_{a'} u(y + a - \frac{a'}{R}, h) + \beta \mathbb{E} V^d(y', a')$$

In order to derive envelope conditions in this environment, one requires a slightly different approach to that detailed in appendix A.2. I describe the changes necessary in appendix A.4.

Interestingly, the surplus from successful sale now generally increases in $a$, given an income level. Following the argument in section 3, one can show that tightness choice is now increasing in asset holding. In other words, lower debt (higher $a$) now raises market tightness and lowers the list price. In this version of the model, sellers with more assets have a less favourable outside option in terms of default, and therefore charge lower prices while trading off higher sale probabilities.

[5] Alternatively, one could assume that unsuccessful sales lead to a choice between default or repayment.

### The Social Welfare Function and sale externalities

Now, I also assume that the Planner takes loan prices as given when choosing whether to sell or not. I assume that the Planner's intervention is unanticipated: agents do not account for a possible intervention that would reduce default risk in the future.

The Planner's problem is augmented to incorporate lender payoffs, given by:

$$\sum_{y \in Y} \int_{a \in \bar{A}} \sigma(y,a) * \Big(1 - \eta\big(\theta(y,a)\big)\Big) * 1_{(\zeta p + a < 0)} * \big(\zeta p + a\big) * d\mu^o(y,a) \tag{25}$$

Hence, the Planner's problem becomes:

$$\begin{aligned}\Omega\Big(\mu^o, \mu^r\Big) = \max_{\{\sigma\}} \sum_{y \in Y} \int_{a \in \bar{A}} u(c,h) * d\mu^r(y,a) + \sum_{y \in Y} \int_{a \in \bar{A}} u(c,h) * d\mu^o(y,a) \\ + \sum_{y \in Y} \int_{a \in \bar{A}} \sigma(y,a) * \Big(1 - \eta\big(\theta(y,a)\big)\Big) * 1_{(\zeta p + a < 0)} * \big(\zeta p + a\big) * d\mu^o(y,a) \\ + \beta E \Omega\Big(\mu^{'o}, \mu^{'r}\Big)\end{aligned} \tag{26}$$

In the above, $1_{(\zeta p + a < 0)}$ is an indicator function that takes the value $1$ if the lender recovery amount from sale is lower than the debt outstanding.

One can derive a version of equation $(20)$ that now also accounts for *ex post* losses incurred by lenders on their loans, in a manner similar to that described in appendix B.3.

**Proposition 3:** The Planner chooses sale $(\sigma(y,a) = 1)$ for an owner if the following condition holds:

$$V^s(y,a) - V^c(y,a) + \text{PE} + \text{Foreclosure deadweight cost} > 0 \tag{27}$$

*Proof:* See appendix B.3.

Then, the distributive PE term becomes:

$$\text{PE} = \sum_{y \in Y} \int_{a \in \bar{A}} \Big( \sigma(y,a) * \eta\big(\theta(y,a)\big) * u_c(c,h) * \frac{\Delta p(\theta)}{\Delta p} * d\mu^o(y,a) * \frac{\Delta p}{\Delta S} \Big)$$

$$+ \Big( \sum_{y \in Y} \int_{a \in \bar{A}} \sigma(y,a) * \Big(1 - \eta\big(\theta(y,a)\big)\Big) * 1_{(\zeta p + a < 0)} * \zeta * d\mu^o(y,a) * \frac{\Delta p}{\Delta S} \Big)$$

$$- \sum_{y \in Y} \int_{a \in \bar{A}} \Big( \omega(y,a) * u_c(c, \chi h) * (1 + \kappa_b) * d\mu^r(y,a) * \frac{\Delta p}{\Delta S} \Big) \tag{28}$$

The additional term represents the marginal effect of a change in prices following sale on lender liquidity. As lenders are risk-neutral and assumed to be unconstrained, the marginal value of their net worth is unity. This term, when prices decline following a sale, introduces another factor that favors fewer sales.

There is an additional inefficiency associated with lender loss in the event of an unsuccessful sale, which would lead to lenders recovering $\min\{\zeta p + a, 0\}$ from selling the house to brokers (this captures the deadweight costs of a foreclosure sale). This term is:

$$\text{Foreclosure deadweight cost} = \Big(1 - \eta\big(\theta(y,a)\big)\Big) * \min\{\zeta p + a, 0\} \tag{29}$$

As I discuss elsewhere (Contractor [2020]), renegotiation following an unsuccessful match would generally mitigate the foreclosure deadweight cost. In that sense then, this inefficiency associated with realized foreclosure losses can be interpreted as arising from frictions that impede renegotiation.

I do not consider collateral pecuniary externalities here, as the differentiability results apply to interior optima. However, one could potentially approximate the Lagrange multipliers on the collateral constraint by considering points corresponding to interior optima that are close to the bound. Collateral externalities would also favour fewer sales.

**Amplification mechanism with frictional trades**

The model in section 2 does not feature an amplification mechanism. Indeed, a decline in the price index induces potential sellers to lower their list price, which actually favors more repayment by making sale less attractive.

With defaultable debt, there is a feedback loop between debt overhang, foreclosure and low prices. Debt overhang prevents some potential sellers from lowering their list price as they are constrained to make their required mortgage payment. This would lead to foreclosure for highly indebted owners, and a lower price index through an increase in supply. A lower price index requires list prices to be lowered further in active submarkets, which leads to more indebted sellers being foreclosed upon and so on (see Garriga and Hedlund [2020] for a further discussion).

## 5.3 Inefficient tightness choice

The presence of pecuniary externalities also affects the efficiency of the submarket tightness choice. As section 2 discusses, the choice of market tightness by sellers can also be interpreted as a choice of list prices.

In order to demonstrate this, I work with the baseline model of section 2 and set up a constrained optimization problem similar to the one described in section 4. The Planner takes all other saving/borrowing, sale and ownership choices as given. He also internalizes the equilibrium house price determination mechanism. As before, I assume that the Planner's intervention is not anticipated by agents.

The Planner chooses a tightness function so as to maximize the utilitarian SWF, subject to the broker zero-profit and consumption non-negativity constraints for potential sellers, given in equation $(6)$.

**Proposition 4:** The Planner's interior F.O.C for constrained efficient tightness choice is given by:

$$\left\{ \Big( v^s(y,a) - V^c(y,a) \Big) + \frac{\eta\big(\theta(y,a)\big)}{\eta'\big(\theta(y,a)\big)} v^s_p(y,a) p'(\theta) \right\} + PE_\theta = 0 \tag{30}$$

*Proof:* See appendix B.5.

The pecuniary externality $(PE_\theta)$ associated with tightness choice is:

$$PE_\theta = \Bigg\{ -\sum_{y^{'}\in Y} \int_{a^{'}\in \bar{A}} \omega(y^{'},a^{'})\big(u_c(c,\chi h)\big) * d\mu^r(y^{'},a^{'})$$

$$+ \sum_{y^{'}\in Y} \int_{a^{'}\in \bar{A}} \Big(\sigma(y^{'},a^{'}) * \eta\big(\theta(y^{'},a^{'})\big) * u_c(c,h)\Big) * \frac{dp\big(\theta(y^{'},a^{'})\big)}{dp} * d\mu^o(y^{'},a^{'})\Bigg\}$$

$$* \frac{dp}{dS} \quad (31)$$

This is virtually identical to the expression in equation (20), which is unsurprising as the channel through which the inefficiency arises is the price index. However, a marginal change in $\theta(y,a)$ changes sale probability by $\eta^{'}\big(\theta(y,a)\big)$, and changes supply by $\sigma(y,a) * \eta^{'}\big(\theta(y,a)\big) * d\mu^o(y,a)$. This is captured in the expression above by the term $\frac{dp}{dS}$.

If $PE_\theta < 0$, then the Planner would reduce tightness and sale probability (as $\eta^{'} > 0$) relative to the decentralized tightness choice. This is because higher market tightness increases the probability of a successful sale, and more sales would lower the equilibrium price index $p$ and thereby adversely affect other agents in the economy. The condition that $PE_\theta < 0$ would tend to hold when sellers are more constrained than buyers. The reasoning is analogous to the case in section 4.2: if sellers are more constrained on average than buyers, they tend to have greater marginal utilities of consumption than buyers and hence the overall term in braces in equation (32) is positive. Since $\frac{dp}{dS} < 0$, so is $PE_\theta$.

One could alternatively interpret the result on market tightness in terms of list prices. As list prices are inversely related to market tightness, one could also state that, if sellers tend to be more constrained as a group than buyers, then private list prices are lower than the efficient list price benchmark.

Hence, in this model with directed search and choice of submarket tightness, pecuniary externalities due to incomplete markets and frictions lead to inefficient private submarket tightness choices. On this basis, one may conjecture that a similar inefficiency would arise in other directed search models that feature market incompleteness (such as a collateral constraint that may bind, or imperfect risk sharing) and, importantly, link various submarkets together through an intermediation mechanism such as the brokers considered here. Although brokers allow a tractable analysis of two-sided heterogeneous matching problems, a directed search model without brokers that features a collateral constraint would still require a specification of the (aggregate) price that enters into that constraint. If the aggregate price is affected by individual tightness (or alternatively, list price) choices, then one would expect a pecuniary externality that would similarly lead to inefficient tightness choices.

# 6 Conclusion

This paper uses a simple incomplete markets housing model with sale/repayment choice and intermediated frictional trades in order to study the constrained inefficiency of house sale decisions. The specific assumptions of broker intermediation and quasi-Walrasian market clearing in the broker-intermediated market link various submarkets together, thereby introducing a pecuniary externality channel. This particular model feature, which also facilitates analysis of a two-sided heterogeneous agent matching problem, allows one to extend inefficiency results obtained in frictionless housing models to an environment with matching frictions in the housing market.

In addition, I show that the choice of market tightness is also constrained inefficient in this setup, owing to the pecuniary externalities associated with tightness choice and sale probability. This pecuniary externality based inefficiency result has not, to my knowledge, been discussed previously in the directed search literature.

Extending this finding to specific environments with financial frictions and frictional trading, such as OTC markets, would be an interesting next step. Extending the model to allow default choice and collateral constraints would bring the model closer to the quantitative literature, e.g. Garriga and Hedlund [2020]. Quantifying the magnitude of the inefficiencies operational in this environment is the subject of future research.

# Appendices

Appendix A establishes differentiability results for the choice-specific value functions, and Appendix B contains proofs of the propositions and results in the paper.

## A Appendix: differentiability and concavity of value functions

In this appendix, I establish differentiability of the agents' value functions. In doing so, one cannot directly apply the results of Clausen and Strub [2020] to nonconcave problems, for the reasons described in Menzio et al. [2013], Jerez et al. [2020]. In particular, in order to show the differentiability of $V^s$, which from equation $(5)$ is a convex combination of $v^s$ and $V^c$, one needs to show the differentiability of these two value functions. Hence, it does not fall within the structure of the problems analysed by Clausen and Strub [2012, 2020]. I use a related approach based on subdifferentials, which I now describe.

### A.1 Fréchet sub- and superdifferentials

This appendix establishes differentiability of choice specific value functions along interior optima. In order to do so, I use the concepts of Fréchet sub- and superdifferentials, which I refer to alternatively as F sub-and superdifferentials. Below, I define and state some properties of Fréchet sub- and superdifferentials, that I shall employ in the proofs later. The exposition below is based heavily on Jerez et al. [2020] and Clausen and Strub [2012, 2020]. An alternative definition of Fréchet differentials based on limits superior and inferior can be found in Clausen and Strub [2012]. The two definitions are related in appendix F of Clausen and Strub [2016].

For a continuous function $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$, where $\Omega$ is an open set, the vector $p \in \mathbb{R}^n$ belongs to the F-superdifferential of $f$ at $x_0 \in \Omega$, $D^+ f(x_0)$, if and only if there exists a continuous function $\phi : \Omega \to \mathbb{R}$ which is differentiable at $x_0$ with $D\phi(x_0) = p$, $f(x_0) = \phi(x_0)$ and $f - \phi$ has a local maximum at $x_0$.

Similarly, $p \in \mathbb{R}^n$ belongs to the F-subdifferential of $f$ at $x_0 \in \Omega$, $D^- f(x_0)$, if and only if there exists a continuous function $\phi : \Omega \to \mathbb{R}$ which is differentiable at $x_0$ with $D\phi(x_0) = p$, $f(x_0) = \phi(x_0)$ and $f - \phi$ has a local minimum at $x_0$. $D^+ f(x_0)$ and $D^- f(x_0)$ are closed convex subsets of $\mathbb{R}$ . If $f$ is differentiable at $x_0$, then both $D^+ f(x_0)$ and $D^- f(x_0)$ are nonempty and $D^+ f(x_0) = D^- f(x_0) = Df(x_0)$.

Conversely, if for a function $f$, both $D^+ f(x_0)$ and $D^- f(x_0)$ are nonempty, then $f$ is differentiable at $x_0$ and $D^+ f(x_0) = D^- f(x_0) = Df(x_0)$, where $Df$ denotes the derivative of $f$. Finally, whenever $x_0$ is a local maximum of $f$ in $\Omega$, $0 \in D^+ f(x_0)$.

The following properties of F-sub and superdifferentials will be useful below. Let $f$ and $g$ be functions from $\mathbb{R} \to \mathbb{R}$.

1. If $f$ and $g$ are Fréchet sub (super) differentiable, then so is $f + g$ (*cf. Clausen and Strub* [*2012*], *Lemma* $2(i)$).

2. If $h(x) = \max\{f(x), g(x)\}$ is differentiable at $\bar{x}$ and $f(\bar{x}) = h(\bar{x})$, then $f$ is differentiable at $\bar{x}$ (*cf. Clausen and Strub* [*2020*], *Lemma* $2(iii)$).

3. If $f$ and $g$ are subdifferentiable, and $f + g$ is superdifferentiable, then $f, g$ and $f + g$ are differentiable (*cf. Clausen and Strub* [*2012*], *Lemma* $2(iii)$).

One can generalize $(3)$ above to the case of finite sums in order to obtain envelope results for stochastic dynamic programming problems, where one takes a convex combination of value functions using the Markov probabilities, as discussed after the statement of Theorem $3$ of Clausen and Strub [2012]. In addition, I shall sometimes use the results of Lemma $2$ of Clausen and Strub [2020] on 'Reverse Calculus'.

Finally, I state the following theorem that shall be used to establish the differentiability of value functions. Let $f(x) = \max_{y \in \Gamma(x)} F(x, y)$, where $F : X \times Y \to \mathbb{R}$ is continuous, $X, Y \subset \mathbb{R}^n$, and where $\Gamma$ is a nonempty, compact valued and continuous correspondence from $X$ to $Y$.

**Theorem 1:** Let $x_0$ be an interior point of $X$ and $y_0 \in \Gamma(x_0)$ satisfying: (i) $f(x_0) = F(x_0, y_0)$, (ii) there is a ball $B(x_0, \epsilon)$ in $X$ with center $x_0$ and radius $\epsilon > 0$ such that $\forall x \in B(x_0, \epsilon), y_0 \in \Gamma(x)$. Then $D_x^- F(x_0, y_0) \subseteq D^- f(x_0)$ and $D^+ f(x_0) \subseteq D_x^+ F(x_0, y_0)$, where $D_x^{\pm} F(x_0, y_0)$ denotes the F-super/subdifferential of the function $F(x, y_0)$.

*Proof*: See Jerez et al. [2020].

## A.2 Differentiability of the value functions

As in Jerez et al. [2020], I establish differentiability of value functions by showing that the F sub- and superdifferentials are nonempty along the optimal policies. I assume that there exist interior selections of saving/borrowing policies $\forall y$, denoted by $g_t^a, g_n^a, g_c^a, g_s^a$ denote the relevant saving/borrowing choices for tenants, new buyers, owners who continue and owners who sell respectively. I also assume that an interior selection of the tightness choice exists, denoted by $g_s^\theta$. The set of all feasible asset positions is $\bar{A}$, with minimum asset holding value denoted by $\underline{a}$, which is negative.

Lemmas 1 and 2 establish F-subdifferentiability of the value functions, while Lemma 3 and the subsequent discussion establishes F-superdifferentiability of the value functions.

**Lemma 1:** Let $a_0 > \underline{a}$. Then, $\forall y$ (i) $u_c(y + a_0 - \rho - \frac{g_t^a(y,a_0)}{R}, h) \in D_a^- V^t(y, a_0)$; (ii) $u_c(y + a_0 - \frac{g_c^a(y,a_0)}{R}, \chi h) \in D_a^- V^c(y, a_0)$; (iii) $u_c(y + a_0 - \frac{g_n^a(y,a_0)}{R}, \chi h) \in D_a^- V^b(y, a_0)$; (iv) $u_c(y + a_0 - \rho - \frac{g_s^a(y,a_0)}{R}, h) \in D_a^- v^s(y, a_0)$

*Proof.* I only prove (i), as the proofs of the other cases are almost identical, mutatis mutandis. As $a_0$ is interior, condition (i) of Theorem 1 is satisfied. Given that $g_t^a(y, a_0)$ is interior and the feasible correspondence is closed, there is an open interval centered at $a_0$ such that $g_t^a(y, a) \in \big(0, R(y + a)\big)$ for all $a$ in this open interval. Hence, condition (ii) of Theorem 1 is satisfied.

Construct the function:

$$F\big(y, a, g_t^a(y, a_0)\big) = u\big(y + a - \frac{g_t^a(y, a_0)}{R}, h\big) + \beta E V^r\big(y', g_t^a(y, a_0)\big)$$

Clearly, this function is differentiable w.r.t $a$ as the second term does not depend on $a$. The derivative w.r.t $a$ is $u_c(y + a_0 - \frac{g_t^a(y,a_0)}{R}, h)$. Hence, by Theorem 1, $u_c(y + a_0 - \frac{g_t^a(y,a_0)}{R}, h) \in D_a^- V^t(y, a_0)$.

$\square$

In the case of a seller choosing market tightness, I denote the optimal choice of $\theta$

given $(y, a)$ by $g_s^\theta(y, a)$. As the domain of choice for $\theta$,

$$D(a) = \{\theta \in \mathbb{R}_+ : p(\theta) \geq -a - y\}$$

is not compact, I follow Jerez et al. [2020] and transform the seller's choice of tightness into a choice of sale probability, $\eta$. One can then express the list price in terms of $\eta$:

$$\hat{p}(\eta) = p - \frac{\kappa_s}{\hat{\alpha}(\eta)}; \eta \in (0, 1)$$

and the domain is modified to:

$$\hat{D}(a) = \{\eta \in [0, 1] : \hat{p}(\eta) + a + y \geq 0\}$$

The sections of $\hat{D}$ are nonempty and compact for $\hat{p}(\eta) + a + y > 0$. One can then rewrite the seller's problem as:

$$\hat{V}^s(y, a) = \max_\eta \eta v^s(y, a) + (1 - \eta)V^c(y, a)$$

and the optimal choice of $\eta$ for owner with state $(y, a)$ is $g_s^\eta(y, a)$.

**Lemma 2:** Let $a_0 > \underline{a}$ and suppose $\exists \tilde{a}(y)$ given $y$ such that $\sigma(y, a) = 1$ if $a < \tilde{a}(y)$, and $\sigma(y, a) = 0$ otherwise. Then, (i) if $a_0 < \tilde{a}(y)$, then $\Big(1 - \eta\big(\theta(y, a_0)\big)\Big)u_c(y + a_0 - \frac{g_c^a(y,a_0)}{R}, \chi h) + \eta\big(\theta(y, a_0)\big)u_c(y + a_0 - \rho - \frac{g_s^a(y,a_0)}{R}, h) \in D_a^- V^o(y, a_0)$; (ii) if $a_0 \geq \tilde{a}(y)$, then $u_c(y + a_0 - \frac{g_c^a(y,a_0)}{R}, \chi h) \in D_a^- V^o(y, a_0)$.

*Proof.* For case (ii), $V^o(y, a_0) = V^c(y, a_0)$ and the result follows from Lemma 1 , part (ii). For case (i), as $g_s^\theta$ is interior, the optimal $g_s^\eta$ is also interior. Construct the function:

$$F\big(y, a, g_s^\eta(a_0)\big) = g_s^\eta(a_0)v^s(y, a) + \big(1 - g_s^\eta(a_0)\big)V^c(y, a)$$

which is well-defined in an open interval around $a_0$. Now, consider $p_s \in D_a^- v^s(y, a_0)$ and $p_c \in D_a^- V^c(y, a_0)$, which exist by Lemma 1 and are equal to $u_c(y + a_0 - \rho - \frac{g_s^a(y,a_0)}{R}, h)$ and $u_c(y + a_0 - \frac{g_c^a(y,a_0)}{R}, \chi h)$ respectively. Further, a convex combination of these subdifferentials belongs in $D_a^- F\big(y, a_0, g_s^\eta(a_0)\big)$, i.e.

$$\eta\big(g_s^\theta(y, a_0)\big)p_s + \Big(\eta\big(g_s^\theta(y, a_0)\big)\Big)p_c \in D_a^- F\big(y, a_0, g_s^\theta(a_0)\big)$$

From Theorem 1, $D_a^- F\big(y, a_0, g_s^\eta(a_0)\big) \subseteq D_a^- V^o(y, a_0) = D_a^- V^s(y, a_0)$. $\square$

One can similarly show that the subdifferential of $V^b$ is nonempty.

I now establish nonemptiness of the superdifferentials of value functions.

**Lemma 3:** Let $a_0 > \underline{a}$. Then, $\forall y$, $u_c(y + a_0 - \frac{g_c^a(y,a_0)}{R}, \chi h) \in \beta RE D_a^+ V^o\big(y^{'}, g_c^a(y, a_0)\big)$

*Proof.* Consider the following function of $a^{'}$:

$$F(y, a_0, a^{'}) = u(y + a_0 - \frac{a^{'}}{R}, \chi h) + \beta E_{y^{'}} V^o(y^{'}, a^{'})$$

As $g_c^a(y, a_0)$ is an interior optimum for the $V^c(y, a_0)$, $0 \in D_{a^{'}}^+ F\big(y, a_1, g_c^a(y, a_0)\big)$. As $u$ is $C^1$, $D_{a^{'}}^+ F = \frac{-u_c}{R} + \beta D_{a^{'}}^+ E V^o$. Thus, $u_c(y + a_0 - \frac{g_c^a(y,a_0)}{R}, \chi h) \in \beta R D_{a^{'}}^+ E V^o\big(y^{'}, g_c^a(y, a_0)\big)$. $\square$

**Proposition A1:** Let $a_0 > \underline{a}$. Then, $\forall y^{'}$, $V^o(y^{'}, a^{'})$ is differentiable w.r.t $a$ at $a^{'} = g_c^a(y, a_0)$. Further, $V^s(y^{'}, a^{'})$ and $V^c(y^{'}, a^{'})$ are also differentiable w.r.t $a$ at $a^{'} = g_c^a(y, a_0)$.

*Proof.* As $V^o(y, a_0) = \max\{V^c(y, a_0), V^s(y, a_0)\}$, it is subdifferentiable if $a_0 > \underline{a}$, from Lemma 1. Since $g_c^a(y, a_0)$ is interior, it exceeds $\underline{a}$.

Using Lemma 3 and applying property (3) of F sub-differentials in appendix A.1 extended to convex combinations using Markov probabilities of $V^o(., a^{'})$, $V^o(., a^{'})$ is differentiable in $a^{'}$ at $g_c^a(y, a_0)$.

Applying property (2) in appendix A.1, $V^c\big(y^{'}, g_c^a(y, a_0)\big)$ and $V^s\big(y^{'}, g_c^a(y, a_0)\big)$ are also differentiable w.r.t $a$ at $g_c^a(y, a_0)$, depending on whether $\sigma\big(y^{'}, g_c^a(y, a_0)\big) = 1$ or not.

$\square$

Similarly, one can establish the differentiability of $V^r(y^{'},.)$ w.r.t $a$ at $g_t^a(y,a_0)$, and hence, by applying property (2) in appendix A.1 again, $V^t\big(y^{'}g_t^a(y,a_0)\big)$ and $V^b\big(y^{'}g_t^a(y,a_0)\big)$ are also differentiable w.r.t $a$ at $g_t^a(y,a_0)\big)$, depending on whether $\omega\big(y^{'},g_t^a(y,a_0)\big)=1$ or not.

Further, if $a > \underline{a}$, Lemma $1$ derived the F-subdifferentials of $V^t$ and $V^b$, and as $V^r(y,a) = \max\{V^t(y,a), V^b(y,a)\}$, it is also subdifferentiable. By the same argument as in Lemma $3$, one can show that $u_c(y+a_0-\rho-\frac{g_s^a(y,a_0),h)}{R} \in \beta RED^{+}_{a^{'}}V^r\big(y^{'},g_s^a(y,a_0)\big)$. Hence, an application of property $(3)$ of F-subdifferentials to convex combinations of $V^r(.,a^{'})$ in appendix A.1 implies that $V^r(y^{'},.)$ is differentiable w.r.t $a$ at $g_s^a(y,a_0)$.

## A.3 Concavity of the value functions

Concavity of the value function $v^s$ is proved in intervals of the image of $g_s^a(y,a)$ corresponding to sale such that seller consumption is nondecreasing in the range of assets that lead to sale. Denote optimal consumption of a seller by the policy function $g_s^c(y,a)$.

**Proposition A2:** $\forall y$, $E_{y^{'}|y}V^r$ is concave in $a$ in the intervals $I$ of the image of $g_s^a$ iff $g_s^c$ is nondecreasing in $(g_s^a)^{-1}(I)$.

*Proof.* From the argument above, $V^r$ is differentiable w.r.t $a$ in $I$ for all $y$. Fix a value for $y$. If $a^{'} \in I$, then $\exists a > \underline{a}$ such that $g_s^a(y,a) = a^{'}$ and $E_{y^{'}|y}V_a^r\big(y^{'},g_s^a(y,a)\big) = \frac{u_c(g_s^c(y,a),h)}{\beta R}$. Let $(a_1^{'},a_2^{'}) \in I$ s.t. $a_1^{'} > a_2^{'}$, and suppose $\exists a_i$ such that $a_i^{'} = g_s^a(a_i)$ for $i=1,2$.

By the Mean value Theorem,

$$E_{y^{'}|y}V^r(y^{'},a_1^{'}) - E_{y^{'}|y}V^r(y^{'},a_2^{'}) = E_{y^{'}|y}V_a^r(y',b^{'})(a_1^{'}-a_2^{'}) = \frac{u_c(g_s^c(y,b),h)}{\beta R}(a_1^{'}-a_2^{'})$$

where $a_2^{'} < b^{'} < a_1^{'}$ and $b^{'} = g_s^a(y,b)$.

Given $y$, $W(y,a,a^{'}) = u(y,a,h;a^{'}) + \beta EV^r(y^{'},a^{'})$ satisfies increasing first differences in $(a,a^{'})$ as $u(y,a,h;a^{'})$ satisfies increasing first differences in $(a,a^{'})$, hence by Theorem $10.6$ of Sundaram [1996], $g_s^a(y,.)$ is nondecreasing in $a$. Thus, $a_2 < b < a_1$.

As $g_s^c(y,.)$ is nondecreasing in $a$ by assumption, we also have $g_s^c(y,a_2) < g_s^c(y,b) < g_s^c(y,a_1)$. As $u(.,h)$ is concave, $u_c(g_s^c(y,b),h) \leq u_c(g_s^c(y,a_2),h) = R\beta E_{y'|y} V_a^r(y^{'},a_2^{'})$.

Hence,

$$E_{y'|y} V^r(y^{'},a_1^{'}) - E_{y'|y} V^r(y^{'},a_2^{'}) \leq E_{y'|y} V_a^r(y^{'},a_2^{'})(a_1^{'} - a_2^{'})$$

and so $E_{y'|y} V^r(y^{'},.)$ is concave in $I$. The proof of the converse follows the reverse direction. □

**Proposition A3:** Let $I$ be an interval of $\bar{A}$ such that $g_s^a(y,I)$ is an interval, $\forall y$. Then, $v_s$ is concave in $I$ iff $g_s^c(y,.)$ is nondecreasing in $I$.

*Proof.* Let $a_1, a_2 \in I$, and let $\lambda_1, \lambda_2 \in [0,1]$ such that $\lambda_1 + \lambda_2 = 1$. Fix a level of $y$. As $g_s^a(y,I)$ is convex, $\lambda_1 a_1 + \lambda_2 a_2 \in I$, and $\lambda_1 g_s^a(y,a_1) + \lambda_2 g_s^a(y,a_2) \in g_s^a(y,I)$.

Further, $\big(\lambda_1 a_1 + \lambda_2 a_2, \lambda_1 g_s^a(y,a_1) + \lambda_2 g_s^a(y,a_2)\big)$ belongs to the graph of the feasible correspondence for a seller, as it is convex. Finally, $V^r(y,.)$ is concave in $g_s^a(y,I)$ from Proposition $A2$.

Then,

$$v^s(y, \lambda_1 a_1 + \lambda_2 a_2) \leq u\big(\lambda_1 a_1 + \lambda_2 a_2, \lambda_1 g_s^a(y,a_1) + \lambda_2 g_s^a(y,a_2), h\big) + \beta E V^r\big(y^{'}, \lambda_1 g_s^a(y,a_1) + \lambda_2 g_s^a(y,a_2)\big)$$

$$\leq \lambda_1 u(a_1, g_s^a(y,a_1), h) + \lambda_2 u(a_2, g_s^a(y,a_2), h) + \beta E V^r\big(y^{'}, \lambda_1 g_s^a(y,a_1)\big) + \beta E V^r\big(y^{'}, \lambda_2 g_s^a(y,a_2)\big)$$

$$= \lambda_1 v^s(y,a_1) + \lambda_2 v^s(y,a_2)$$

The second inequality follows from the concavity of $u$ and the concavity of $EV^r(y^{'},.)$ in the image of $g_s^a(y,a)$. Hence, $v^s(y,.)$ is concave in $I$. □

## A.4 Differentiability when debt is defaultable

As discussed in section 5.2, allowing for defaultable debt introduces a loan pricing function $Q(y, a^{'})$ that is not necessarily differentiable. This complicates our differentiability results above, as the proof of nonemptiness of F-superdifferentials for owner value function $V^o$ in Lemma 3 requires $Q$ to be differentiable at the interior optimum. Hence, one requires a slightly different approach.

In the following, I show that $Q(y, .)$ is differentiable using the techniques of Clausen and Strub [2020], particularly those utilized in Theorem 3 of that paper.

### A.4.1 Subdifferentials and differentiable lower support functions

One can relate the differentiable lower support function $L$ discussed in Clausen and Strub [2020] to the function $\phi$ described in the definition of F-subdifferentials in appendix A.1.

$L$ is a differentiable lower support function for a function $f$ at point $\bar{c}$ if (i) $f(c) \geq L(c) \forall c$, (ii) $f(\bar{c}) = L(\bar{c})$, and (iii) $L$ is differentiable at $\bar{c}$.

If $L$ is continuous, then clearly $f(c) - L(c) \geq f(\bar{c}) - L(\bar{c}) = 0$, so $f - L$ has a local minimum at $\bar{c}$. Then, $\phi$ is a lower support function for $f$ at $\bar{c}$, and $DL(\bar{c}) \in D^{-}f(\bar{c})$

### A.4.2 Differentiability of $Q$ w.r.t $a^{'}$ at an interior optimum

Lemma 1 of Clausen and Strub [2020] shows that if a function $f$ has lower and upper support functions that are differentiable at point $\bar{c}$, then $f$ is differentiable at $\bar{c}$.

In order to show that $Q(., a^{'})$ is differentiable w.r.t $a^{'}$, I sketch out how to construct suitable lower and upper support functions below. The exposition is based on section 3.2 of Clausen and Strub [2020].

Let $a^{'}$ be the asset holding choice for an owner with state $(y, a)$ who repays. Define function $\Phi(a^{'}; y, a)$ as follows:

$$\Phi(a^{'}; y, a) = u(y + a - Q(y, a^{'})a^{'}, \chi h) + \beta E V^o(y^{'}, a^{'})$$

**Lower support function**

First, I construct a lower support function for $\Phi(a^{'};y,a)$. For simplicity, I assume that $y \in [\underline{y},\bar{y}]$ with Markov transition function $G(y,.)$. Assume that there is a threshold function defined for each asset level, $y(a)$ such that sale is chosen when $y < y(a)$, and repayment is chosen when $y \geq y(a)$. One can then write:

$$V^{o}(y^{'},a^{'}) = \int_{\underline{y}}^{y(a^{'})} V^{s}(y^{'},a^{'})dG(y,y^{'}) + \int_{y(a^{'})}^{\bar{y}} V^{c}(y^{'},a^{'})dG(y,y^{'})$$

and

$$Q(y,a^{'}) = R^{-1} * \left\{ 1 + \int_{\underline{y}}^{y(a^{'})} \eta\big(\theta(y^{'},a^{'})\big)\Big(\min\{\frac{\chi p^{'}}{a^{'}},1\} - 1\Big)dG(y,y^{'}) \right\}$$

In order to obtain a lower support function for $\Phi(.;y,a)$ at $\bar{a^{'}}$, one requires a differentiable upper support function for $y(.)$, and lower support functions for $\theta(y^{'},.)$ and $V^{o}(y^{'},.)$ at $\bar{a^{'}}$.

For the upper support function, consider an owner choosing to save or borrow with state $(y^{'},\bar{a^{'}})$ who incorrectly perceives the state to be $\big(y(\bar{a^{'}}),\bar{a^{'}}\big)$, i.e. he expects his income to be at the threshold level, and chooses asset holding $\bar{a^{''}}$ accordingly. His value function is:

$$L(y^{'},a^{'};\bar{a^{'}}) = u(y^{'} + a^{'} - Q(y^{'},\bar{a^{''}},\chi h) + \beta E V^{o}(y^{''},\bar{a^{''}})$$

Similarly, one can construct a value function for an owner who decides to sell who also perceives the state to be $\big(y(\bar{a^{'}}),\bar{a^{'}}\big)$. Let us denote this by $\tilde{S}\big(y(\bar{a^{'}}),\bar{a^{'}}\big)$.

His cutoff between sale and repayment is then defined implicitly by:

$$L\Big(\bar{y}\big(a^{'};\bar{a^{'}}\big),a^{'};\bar{a^{'}}\Big) = \tilde{S}\Big(\bar{y}\big(a^{'};\bar{a^{'}}\big),a^{'};\bar{a^{'}}\Big)$$

This provides a cutoff $\bar{y}(.,.)$ for $y(.)$ that involves selling too often. Since $L$ and $\tilde{S}$ are differentiable w.r.t $a^{'}$, so is $\bar{y}\big(a^{'};\bar{a^{'}}\big)$.

In order to obtain lower support functions for $V^o$ and $Q$, consider an owner who perceives the state to be $(y^{'}, \bar{a^{'}})$ instead of $(y^{'}, a^{'})$. He chooses asset holding $\bar{a''}$ and tightness $\theta$ accordingly.

Define the following functions:

$$M^S(y^{'}, a^{'}; \bar{a^{'}}) = \eta\Big(\theta(y^{'}, \bar{a^{'}})\Big) M^s(y^{'}, \bar{a^{'}}) + \Big(1 - \eta\big(\theta(y^{'}, \bar{a^{'}})\big)\Big) M^c(y^{'}, \bar{a^{'}})$$

where

$$M^s(y^{'}, a^{'}; \bar{a^{'}}) = u\bigg(y + a^{'} - Q(y, \bar{a''})\bar{a''} + p\big(\theta(y^{'}, \bar{a^{'}})\big), h\bigg) + \beta E V^r(y^{''}, \bar{a''})$$

$$M^c(y^{'}, a^{'}; \bar{a^{'}}) = u\bigg(y + a^{'} - Q(y, \bar{a''})\bar{a''}, \chi h\bigg) + \beta E V^o(y^{''}, \bar{a''})$$

Also, define:

$$M^o(y^{'}, a^{'}; \bar{a^{'}}) = \int_{\underline{y}}^{\bar{y}\left(a^{'}; \bar{a^{'}}\right)} M^S(y^{'}, a^{'}; \bar{a^{'}}) dG(y, y^{'}) + \int_{\bar{y}\left(a^{'}; \bar{a^{'}}\right)}^{\bar{y}} M^c(y^{'}, a^{'}; \bar{a^{'}}) dG(y, y^{'})$$

and

$$\underline{Q}(y, a^{'}; \bar{a^{'}}) = R^{-1} * \left\{1 + \int_{\underline{y}}^{\bar{y}\left(a^{'}; \bar{a^{'}}\right)} \eta\big(\theta(y^{'}, \bar{a^{'}})\big)\Big(\min\{\frac{\chi p^{'}}{\bar{a^{'}}}, 1\} - 1\Big) dG(y, y^{'})\right\}$$

Then, $\underline{Q}(y, .; \bar{a^{'}})$ is a differentiable lower support function at $\bar{a^{'}}$, and so is $M^o(y^{'}, .; \bar{a^{'}})$ for all $y^{'}$ and $\bar{a^{'}}$.

Hence, the function of $a^{'}$:

$$\underline{\Phi}(a^{'}; y, a) = u(y + a - \underline{Q}(y, a^{'}; \bar{a^{'}})a^{'}, \chi h) + \beta E M^o(y^{'}, a^{'}; \bar{a^{'}})$$

is a lower support function for $\Phi(.; y, a)$ at $\bar{a^{'}}$.

Note the similarity of the constructions $M^s$, $M^c$ and $Q$ to the function $F$ in the proof of Lemma 1 in appendix A.2.

**Upper support function**

Define function $U$ for an arbitrary $y$ as follows:

$$U\big(a^{'}; y, a\big) = u(y + a - Q(y, a^{'})a^{'}, \chi h) + \beta EV^{o}\big(y^{'}, a^{'}\big)$$

The upper support function at $\bar{a^{'}}$ is then the constant function $U(\bar{a^{'}}; y, a)$.

If $\bar{a^{'}}(.,.)$ is the optimal policy and $\bar{a^{'}}(y, a) = \bar{a^{'}}$, then $\Phi(.; y, a)$ has differentiable upper and lower support functions at $\bar{a^{'}}$, so by Lemma 1 of Clausen and Strub [2020], it is differentiable at $\bar{a^{'}}$.

Now, repeatedly apply Lemma 2 of Clausen and Strub [2020] as follows. Using the summation property of their Lemma 2 (i), $u(y + a - Q(y, a^{'})a^{'}, \chi h)$ is differentiable at $\bar{a^{'}}$. Next, apply part (iv) of their Lemma 2 to $a^{'} \to u(y + a - Q(y, a^{'})a^{'}, \chi h)$, which establishes that $a^{'} \to Q(a^{'})a^{'}$ is differentiable at $\bar{a^{'}}$. Finally, apply part (ii) of their Lemma 2 to establish that $Q(y, .)$ is differentiable at $\bar{a^{'}}$.

Having thus shown the differentiability of $Q(y, .)$ at the interior optimum for a given $y$, we can then modify the proof of Lemma 3 in order to establish differentiability of value functions at interior optima.

# B Alternative SWF and proofs of propositions

## B.1 Alternative SWF

The transition operator for distribution $\mu^j$ is given by $\mu'^j = T\Big(\mu^j, \mu^{-j}, Q^j(s^j, s'^j)\Big)$ , where $Q(.,.,)$ is the transition matrix from state $s^j$ to $s'^j$ for an individual agent and $T(.,.,.)$ is the updating operator. Below, I do not use the superscript $j$ to differentiate between the state vector and state space $S^j$ for owners and renters, to avoid further cumbersome notation. I abuse notation in order to be concise by aggregating over the sum space using the summation symbol. The consumption, saving and loan choices are policy functions, as the use of the respective value functions indicates.

Aggregate welfare is:

$$\mathcal{W} = \sum_{y \in Y} \int_{\bar{A}} \Bigg( u(c,h) + \sum_{s' \in S'} \beta \Big[ \omega(y,a) * V^o(y',a') * Q^o(s,s')$$

$$+ \big(1 - \omega(y,a)\big) * V^r(y',a') * Q^r(s,s') \Big] \Bigg) * d\mu^r(y,a)$$

$$+ \Bigg( \sum_{y \in Y} \int_{\bar{A}} u(c, \chi h) + \sum_{s' \in S'} \beta \Big[ \sigma(y,a) * \eta\big(\theta(y,a)\big) * V^r(y',a') * Q^r(s,s') +$$

$$\Bigg(1 - \sigma(y,a) * \Big(1 - \eta\big(\theta(y,a)\big)\Big)\Bigg) * V^o(y',a') * Q^o(s,s') \Big] \Bigg) * d\mu^o(y,a) \qquad (32)$$

## B.2 Proposition 1: proof

**Proposition 1:** If $V^c$, $V^s$ and $v^s$ are continuous in $a$, $S(y,a)$ is non-increasing in $a$, then $\exists \hat{a}(y)$ such that $\sigma(y,a) = 1$ when $a < \hat{a}(y)$, and $\sigma(y,a) = 0$ otherwise.

*Proof.* $V^s(y,a) = \max\{v^s(y,a), V^c(y,a)\}$, and at $\bar{a}$, $V^s(y,\bar{a}) = v^s(y,\bar{a})$. As $V^s$, $v^s$ and $V^c$ are differentiable in $a$, one can apply the envelope theorem to obtain:

$$V_a^s(y,a) - V_a^c(y,a) = \eta\big(\theta(y,a)\big)\Big(v_a^s(y,a) - V_a^c(y,a)\Big)$$

The RHS of the above equation is negative, as $\eta$ is positive and $S(y,a)$ is decreasing in $a$ given $y$ under the assumption made. Thus, $V^s(y,a) - V^c(y,a)$ is decreasing in $a$. If $V^s(y,\underline{a}) - V^c(y,\underline{a}) < 0$, then $\hat{a}(y) = \underline{a}$, and if $V^s(y,a^{\max}) - V^c(y,a^{\max}) > 0$, then $\hat{a}(y) = a^{\max}$. Otherwise, as the value functions are continuous in $a$, by the intermediate value theorem, there $\exists \hat{a}(y)$ such that $V^o(y,a) = V^s(y,a)$ for all $a < \hat{a}(y)$, and $V^o(y,a) = V^c(y,a)$ otherwise.

□

## B.3 Sale choice

**Proposition 2:** The Planner chooses sale ($\sigma(y,a) = 1$) for an owner if the following condition holds:

$$v^s(y,a) - V^c(y,a) + \text{PE} > 0 \tag{33}$$

where:

$$\begin{aligned}\text{PE} = &\sum_{y\in Y}\int_{a\in\bar{A}}\Big(\sigma(y,a) * \eta\big(\theta(y,a)\big) * u_c(c,h) * \frac{\Delta p(\theta)}{\Delta p} * d\mu^o(y,a) * \frac{\Delta p}{\Delta S}\Big)\\ &-\sum_{y\in Y}\int_{a\in\bar{A}}\Big(\omega(y,a) * u_c(c,\chi h) * (1+\kappa_b) * d\mu^r(y,a) * \frac{\Delta p}{\Delta S}\Big)\end{aligned}$$

*Proof.* The approach here is to use a perturbation argument in the spirit of Davila et al. [2012]. Throughout the proof, I use the expression for the SWF in equation (32).

Let the Planner's sale choice be $\sigma(y,a)$. Consider a perturbation where the positive mass $d\mu^o(y,a)$ of owners with state $(y,a)$ for whom the unique optimal choice is to sell are now switched to continuation status, i.e. they switch from $\sigma(y,a) = 1$ to $\sigma(y,a) = 0$. This reduces housing supply by an amount $\Delta S = \eta\big(\theta(y,a)\big) * d\mu^o(y,a)$, and I denote the resulting price change by $\Delta p \geq 0$. Finally, let the change in felicity of an agent due to a price change be denoted by $\Delta_p u(c,h;p)$.

The resulting change in social welfare from the perturbation, denoted by $\Delta W$, is:

$$\Delta W = \Big(V^c(y,a) - V^s(y,a)\Big) * d\mu^o(y,a)$$

$$+ \sum_{y^{'}\in Y} \int_{a^{'}\in \bar{A}} \omega(y^{'},a^{'})\Big(\Delta_p u(c,\chi h;p)\Big) * d\mu^r(y^{'},a^{'})$$

$$+ \sum_{y^{'}\in Y} \int_{a^{'}\in \bar{A}} \Big(\sigma(y^{'},a^{'}) * \eta\big(\theta(y^{'},a^{'})\big) * \Delta_p u(c,h;p)\Big) * d\mu^o(y^{'},a^{'})$$

Using the first order approximation $u^{'}(c)\Delta c \approx \Delta u$, consider for example the change in utility of sellers due to the price change. Seller consumption is $c = y + a + p(\theta) - \frac{a^{'}}{R} - \rho$, so $\Delta c = \frac{\Delta p(\theta)}{\Delta p} * \Delta p$. Hence, $\Delta_p u(c,h;p) = u_c(c,h;p) * \frac{\Delta p(\theta)}{\Delta p} * \Delta p = -u_c(c,h;p) * \frac{\Delta p(\theta)}{\Delta p} * \frac{\Delta p}{\Delta S} * \eta\big(\theta(y,a)\big) * d\mu^o(y,a)$. Similarly, for buyers, the change in utility is $\Delta_p u(c,\chi h;p) = u_c(c,\chi h;p) * (1+\kappa_b) * \frac{\Delta p}{\Delta S} * \eta\big(\theta(y,a)\big) * d\mu^o(y,a)$.

Thus, the above expression becomes:

$$\Delta W \approx \Big(V^c(y,a) - V^s(y,a)\Big) * d\mu^o(y,a)$$

$$+\Bigg\{ \sum_{y^{'}\in Y} \int_{a^{'}\in \bar{A}} \omega(y^{'},a^{'}) * \Big(u_c(c,\chi h;p) * (1+\kappa_b)\Big) * d\mu^r(y^{'},a^{'}) \Bigg\} * \frac{\Delta p}{\Delta S} * \eta\big(\theta(y,a)\big) * d\mu^o(y,a)$$

$$-\Bigg\{ \sum_{y^{'}\in Y} \int_{a^{'}\in \bar{A}} \Big(\sigma(y^{'},a^{'}) * \eta\big(\theta(y^{'},a^{'})\big) * u_c(c,h;p)\Big) * d\mu^o(y^{'},a^{'}) \Bigg\}$$

$$* \frac{\Delta p}{\Delta S} * \eta\big(\theta(y,a)\big) * d\mu^o(y,a)$$

Rewriting the expression above, one obtains:

$$\Delta W \approx \Bigg[ \Big(V^c(y,a) - V^s(y,a)\Big)$$

$$+\Bigg\{ \sum_{y^{'}\in Y} \int_{a^{'}\in \bar{A}} \omega(y^{'},a^{'}) * \Big(u_c(c,\chi h;p) * (1+\kappa_b)\Big) * d\mu^r(y^{'},a^{'}) \Bigg\} * \frac{\Delta p}{\Delta S} * \eta\big(\theta(y,a)\big)$$

$$-\Bigg\{ \sum_{y^{'}\in Y} \int_{a^{'}\in \bar{A}} \Big(\sigma(y^{'},a^{'}) * \eta\big(\theta(y^{'},a^{'})\big) * u_c(c,h;p)\Big) * d\mu^o(y^{'},a^{'}) \Bigg\} * \frac{\Delta p}{\Delta S} * \eta\big(\theta(y,a)\big) \Bigg]$$

$$* d\mu^o(y,a)$$

As the Planner's optimal choice assigns all agents with state $(y, a)$ to sell, the perturbation should not increase welfare. Further, if there were no change in welfare, then sale and repayment would both be optimal to the Planner for agents with state $(y, a)$. As we have assumed that sale is the unique optimal choice for the Planner for agents with state $(y, a)$, the perturbation must yield strictly lower welfare. Hence, $\Delta W < 0$, and since $d\mu^o(y, a) > 0$, the expression in square brackets must be negative. Thus, the Planner chooses sale if the following condition holds:

$$\Delta W \approx \Bigg[ \Big( V^s(y,a) - V^c(y,a) \Big)$$

$$- \Bigg\{ \sum_{y' \in Y} \int_{a' \in \bar{A}} \omega(y', a') * \Big( u_c(c, \chi h; p) * (1 + \kappa_b) \Big) * d\mu^r(y', a') \Bigg\} * \frac{\Delta p}{\Delta S} * \eta\big(\theta(y,a)\big)$$

$$+ \Bigg\{ \sum_{y' \in Y} \int_{a' \in \bar{A}} \Big( \sigma(y', a') * \eta\big(\theta(y', a')\big) * u_c(c, h; p) \Big) * d\mu^o(y', a') \Bigg\} * \frac{\Delta p}{\Delta S} * \eta\big(\theta(y,a)\big) \Bigg]$$

$$> 0$$

Expanding the expression for $V^s(y, a)$ using equation (5) and factoring out $\eta\big(\theta(y, a)\big)$ that is assumed to be positive, the Planner chooses sale if the following condition holds:

$$\Delta W \approx \Bigg[ \Big( v^s(y,a) - V^c(y,a) \Big)$$

$$- \Bigg\{ \sum_{y' \in Y} \int_{a' \in \bar{A}} \omega(y', a') * \Big( u_c(c, \chi h; p) * (1 + \kappa_b) \Big) * d\mu^r(y', a') \Bigg\} * \frac{\Delta p}{\Delta S}$$

$$+ \Bigg\{ \sum_{y' \in Y} \int_{a' \in \bar{A}} \Big( \sigma(y', a') * \eta\big(\theta(y', a')\big) * u_c(c, h; p) \Big) * d\mu^o(y', a') \Bigg\} * \frac{\Delta p}{\Delta S} \Bigg]$$

$$> 0$$

□

## B.4 Sale choice with the default option

**Proposition 3:** The Planner chooses sale ($\sigma(y,a)=1$) for an owner if the following condition holds:

$$v^s(y,a) - V^c(y,a) + \text{PE} + \text{Non-pecuniary externality} > 0 \tag{34}$$

*Proof.* The approach here is to use a perturbation argument in the spirit of Davila et al. [2012]. Throughout the proof, I use the expression for the SWF in equation (32) augmented to include lender payoffs *ex post*:

$$\sum_{y\in Y}\int_{a\in\bar{A}} \sigma(y,a) * \Big(1-\eta\big(\theta(y,a)\big)\Big) * 1_{(\zeta p + a<0)} * \big(\zeta p + a\big) * d\mu^o(y,a)$$

Let the Planner's sale choice be $\sigma(y,a)$. Consider a perturbation where the positive mass $d\mu^o(y,a)$ of owners with state $(y,a)$ for whom the unique optimal choice is to sell are now switched to continuation status, i.e. they switch from $\sigma(y,a)=1$ to $\sigma(y,a)=0$. This reduces housing supply by an amount $\Delta S = d\mu^o(y,a)$, and I denote the resulting price change by $\Delta p \geq 0$. Finally, let the change in felicity of an agent due to a price change be denoted by $\Delta_p u(c,h;p)$.

The resulting change in social welfare from the perturbation, denoted by $\Delta W$, is:

$$\Delta W = \left(V^c(y,a) - V^s(y,a) + \Big(1-\eta\big(\theta(y,a)\big)\Big) * 1_{(\zeta p+a<0)} * \big(\zeta p + a\big)\right) * d\mu^o(y,a)$$

$$+ \sum_{y'\in Y}\int_{a'\in\bar{A}} \omega(y',a')\Big(\Delta_p u(c,\chi h;p)\Big) * d\mu^r(y',a')$$

$$+ \sum_{y'\in Y}\int_{a'\in\bar{A}} \Big(\sigma(y',a') * \eta\big(\theta(y',a')\big) * \Delta_p u(c,h;p)\Big) * d\mu^o(y',a')$$

$$+ \sum_{y'\in Y}\int_{a'\in\bar{A}} \sigma(y',a') * \Big(1-\eta\big(\theta(y',a')\big)\Big) * 1_{(\zeta p+a'<0)} * \big(\zeta\Delta p\big) * d\mu^o(y',a')$$

Using the first order approximation $u'(c)\Delta c \approx \Delta u$, consider for example the change in utility of sellers due to the price change. Seller consumption is $c = y + a +$

$p(\theta) - \frac{a^{'}}{R} - \rho$, so $\Delta c = \frac{\Delta p(\theta)}{\Delta p} * \Delta p$. Hence, $\Delta_p u(c,h;p) = u_c(c,h;p) * \frac{\Delta p(\theta)}{\Delta p} * \Delta p = -u_c(c,h;p) * \frac{\Delta p(\theta)}{\Delta p} * \frac{\Delta p}{\Delta S} * d\mu^o(y,a)$. Similarly, for buyers, the change in utility is $\Delta_p u(c,\chi h;p) = u_c(c,\chi h;p) * (1+\kappa_b) * \frac{\Delta p}{\Delta S} * d\mu^o(y,a)$.

Thus, the above expression becomes:

$$\begin{aligned}
\Delta W \approx & \left(V^c(y,a) - V^s(y,a) + \Big(1 - \eta\big(\theta(y,a)\big)\Big) * 1_{(\zeta p + a < 0)} * \big(\zeta p + a\big)\right) * d\mu^o(y,a) \\
& + \left\{ \sum_{y^{'} \in Y} \int_{a^{'} \in \bar{A}} \omega(y^{'},a^{'}) * \Big(u_c(c,\chi h;p) * (1+\kappa_b)\Big) * d\mu^r(y^{'},a^{'}) \right\} * \frac{\Delta p}{\Delta S} * d\mu^o(y,a) \\
& - \left\{ \sum_{y^{'} \in Y} \int_{a^{'} \in \bar{A}} \Big(\sigma(y^{'},a^{'}) * \eta\big(\theta(y^{'},a^{'})\big) * u_c(c,h;p)\Big) * d\mu^o(y^{'},a^{'}) \right\} \\
& * \frac{\Delta p}{\Delta S} * d\mu^o(y,a) \\
& - \left\{ \sum_{y^{'} \in Y} \int_{a^{'} \in \bar{A}} \sigma(y^{'},a^{'}) * \Big(1 - \eta\big(\theta(y^{'},a^{'})\big)\Big) * 1_{(\zeta p + a^{'} < 0)} * \zeta * d\mu^o(y^{'},a^{'}) \right\} \\
& * \frac{\Delta p}{\Delta S} * d\mu^o(y,a)
\end{aligned}$$

Rewriting the expression above, one obtains:

$$\begin{aligned}
\Delta W \approx & \Bigg[ \left(V^c(y,a) - V^s(y,a) + \Big(1 - \eta\big(\theta(y,a)\big)\Big) * 1_{(\zeta p + a < 0)} * \big(\zeta p + a\big)\right) \\
& + \left\{ \sum_{y^{'} \in Y} \int_{a^{'} \in \bar{A}} \omega(y^{'},a^{'}) * \Big(u_c(c,\chi h;p) * (1+\kappa_b)\Big) * d\mu^r(y^{'},a^{'}) \right\} * \frac{\Delta p}{\Delta S} \\
& - \left\{ \sum_{y^{'} \in Y} \int_{a^{'} \in \bar{A}} \Big(\sigma(y^{'},a^{'}) * \eta\big(\theta(y^{'},a^{'})\big) * u_c(c,h;p)\Big) * d\mu^o(y^{'},a^{'}) \right\} * \frac{\Delta p}{\Delta S} \\
& - \left\{ \sum_{y^{'} \in Y} \int_{a^{'} \in \bar{A}} \sigma(y^{'},a^{'}) * \Big(1 - \eta\big(\theta(y^{'},a^{'})\big)\Big) * 1_{(\zeta p + a^{'} < 0)} * \zeta * d\mu^o(y^{'},a^{'}) \right\} * \frac{\Delta p}{\Delta S} \Bigg] \\
& * d\mu^o(y,a)
\end{aligned}$$

As the Planner's optimal choice assigns all agents with state $(y,a)$ to sell, the perturbation should not increase welfare. Further, if there were no change in welfare, then sale and repayment would both be optimal to the Planner for agents with state $(y, a)$. As we have assumed that sale is the unique optimal choice for the Planner for agents with state $(y, a)$, the perturbation must yield strictly lower welfare. Hence, $\Delta W < 0$, and since $d\mu^o(y, a) > 0$, the expression in square brackets must be negative. Thus, the Planner chooses sale if the following condition holds:

$$
\begin{aligned}
\Delta W \approx \Bigg[ & \Big(V^s(y,a) - V^c(y,a) + \Big(1 - \eta\big(\theta(y,a)\big)\Big) * 1_{(\zeta p + a < 0)} * \big(\zeta p + a\big)\Big) \\
& - \Bigg\{ \sum_{y' \in Y} \int_{a' \in \bar{A}} \omega(y', a') * \Big(u_c(c, \chi h; p) * (1 + \kappa_b)\Big) * d\mu^r(y', a') \Bigg\} * \frac{\Delta p}{\Delta S} \\
& + \Bigg\{ \sum_{y' \in Y} \int_{a' \in \bar{A}} \Big(\sigma(y', a') * \eta\big(\theta(y', a')\big) * u_c(c, h; p)\Big) * d\mu^o(y', a') \Bigg\} * \frac{\Delta p}{\Delta S} \\
& + \Bigg\{ \sum_{y' \in Y} \int_{a' \in \bar{A}} \sigma(y', a') * \Big(1 - \eta\big(\theta(y', a')\big)\Big) * 1_{(\zeta p + a' < 0)} * \zeta * d\mu^o(y', a') \Bigg\} * \frac{\Delta p}{\Delta S} \Bigg] \\
& > 0
\end{aligned}
$$

$\square$

## B.5 Tightness choice

**Proposition 4:** The Planner's interior F.O.C for constrained efficient tightness choice is given by:

$$
\eta'\big(\theta(y,a)\big)\Big(v^s(y,a) - V^c(y,a)\Big) + \eta\big(\theta(y,a)\big) v_p^s(y,a) p'(\theta) + PE_\theta = 0
$$

where:

$$PE_\theta = \Bigg\{ -\sum_{y'\in Y}\int_{a'\in\bar{A}} \omega(y',a')\big(u_c(c,\chi h)\big) * d\mu^r(y',a') + \sum_{y'\in Y}\int_{a'\in\bar{A}} \Big(\sigma(y',a') * \eta\big(\theta(y',a')\big) * u_c(c,h)\Big) * \frac{dp\big(\theta(y',a')\big)}{dp} * d\mu^o(y',a')\Bigg\} * \frac{dp}{dS} \quad (35)$$

*Proof.* I use a perturbation argument similar to the proof of Propositions 2 and 3. Throughout the proof, I use the expression for the SWF in equation (32).

Let the Planner's tightness choice be $\theta(y,a)$. Consider a perturbation where the Planner increases tightness choice to $\theta(y,a)+\epsilon$, where $\epsilon > 0$. This increases sale probability by $\eta'\big(\theta(y,a)\big) * d\theta(y,a)$, which in turn increases housing supply by $dS = \eta'\big(\theta(y,a)\big) * d\theta(y,a) * d\mu^o(y,a)$. I denote the resulting price change by $dp \leq 0$.

The change in welfare from this perturbation, denoted by $dW$, is:

$$dW = \Big\{\eta'\big(\theta(y,a)\big)\Big(v^s(y,a) - V^c(y,a)\Big) + \eta\big(\theta(y,a)\big)v^s_p(y,a)p'(\theta)\Big\} * d\theta(y,a) * d\mu^o(y,a) + \sum_{y'\in Y}\int_{a'\in\bar{A}} \omega(y',a') * \big(d_\theta u(c,\chi h)\big) * d\mu^r(y',a') + \sum_{y'\in Y}\int_{a'\in\bar{A}} \Big(\sigma(y',a') * \eta\big(\theta(y',a')\big) * d_\theta u(c,h)\Big) * d\mu^o(y',a')$$

Using $u'(c)dc = du$, consider for example the change in utility of sellers due to the price change. Seller consumption is $c = y + a + p(\theta) - \frac{a'}{R} - \rho$, so $dc = \frac{dp(\theta)}{dp} * dp$. Hence, $du(c,h;p) = u_c(c,h;p) * \frac{dp(\theta)}{dp} * dp = u'(c,h;p) * \frac{dp(\theta)}{dp} * \frac{dp}{dS} * \eta'\big(\theta(y,a)\big) * d\mu^o(y,a)$. Similarly, for buyers, the change in utility from a price change is $du(c,\chi h;p) = u_c(c,\chi h;p) * (1+\kappa_b) * \frac{dp}{dS} * \eta'\big(\theta(y,a)\big) * d\mu^o(y,a)$.

Thus, the above expression becomes:

$$dW = \left\{ \eta^{'}\big(\theta(y,a)\big)\Big(v^s(y,a) - V^c(y,a)\Big) + \eta\big(\theta(y,a)\big) v_p^s(y,a) p^{'}(\theta) \right\} * d\theta(y,a) * d\mu^o(y,a)$$

$$\left\{ \sum_{y^{'} \in Y} \int_{a^{'} \in \bar{A}} -\omega(y^{'},a^{'}) * \big(u_c(c,\chi h)\big) * d\mu^r(y^{'},a^{'}) \right.$$

$$\left. + \sum_{y^{'} \in Y} \int_{a^{'} \in \bar{A}} \Big(\sigma(y^{'},a^{'}) * \eta\big(\theta(y^{'},a^{'})\big) * u_c(c,h)\Big) * \frac{dp\big(\theta(y^{'},a^{'})\big)}{dp} * d\mu^o(y^{'},a^{'}) \right\}$$

$$* \, d\theta(y,a) * d\mu^o(y,a) * \frac{dp}{dS} * \eta^{'}\big(\theta(y,a)\big)$$

As $\theta(y,a)$ is the unique optimal tightness choice for the Planner, the perturbation should not affect welfare. If $dW < 0$, then lowering tightness could raise welfare. On the other hand, if $dW > 0$, then raising tightness could increase welfare. Hence, it must be the case that $dW = 0$. If the constraint $\theta \geq 0$ is binding with Lagrange multiplier $\lambda(y,a)$, this condition becomes $dW = -\lambda(y,a)$.

Hence, the Planner's first order condition for tightness choice is:

$$\left\{ \eta^{'}\big(\theta(y,a)\big)\Big(v^s(y,a) - V^c(y,a)\Big) + \eta\big(\theta(y,a)\big) v_p^s(y,a) p^{'}(\theta) \right\} * d\theta(y,a) * d\mu^o(y,a)$$

$$\left\{ \sum_{y^{'} \in Y} \int_{a^{'} \in \bar{A}} -\omega(y^{'},a^{'}) * \big(u_c(c,\chi h)\big) * d\mu^r(y^{'},a^{'}) \right.$$

$$\left. + \sum_{y^{'} \in Y} \int_{a^{'} \in \bar{A}} \Big(\sigma(y^{'},a^{'}) * \eta\big(\theta(y^{'},a^{'})\big) * u_c(c,h)\Big) * \frac{dp\big(\theta(y^{'},a^{'})\big)}{dp} * d\mu^o(y^{'},a^{'}) \right\}$$

$$* \, d\theta(y,a) * d\mu^o(y,a) * \frac{dp}{dS} * \eta^{'}\big(\theta(y,a)\big) \leq 0$$

With positive density $d\mu^o(y,a)$, positive tightness choice $\theta(y,a)$, this can be written as:

$$\left\{\left(v^s(y,a)-V^c(y,a)\right)+\frac{\eta\big(\theta(y,a)\big)}{\eta^{'}\big(\theta(y,a)\big)}v^s_p(y,a)p^{'}(\theta)\right\}$$

$$\left\{\sum_{y^{'}\in Y}\int_{a^{'}\in\bar{A}}-\omega(y^{'},a^{'})*\big(u_c(c,\chi h)\big)*d\mu^r(y^{'},a^{'})\right.$$

$$\left.+\sum_{y^{'}\in Y}\int_{a^{'}\in\bar{A}}\Big(\sigma(y^{'},a^{'})*\eta\big(\theta(y^{'},a^{'})\big)*u_c(c,h)\Big)*\frac{dp\big(\theta(y^{'},a^{'})\big)}{dp}*d\mu^o(y^{'},a^{'})\right\}$$

$$*\frac{dp}{dS}\leq 0$$

which is the condition in equations $(30)-(32)$.

$\square$

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