Problem Set 4: Solutions.

1. (a) The joint pdf is

$$(2\pi\sigma^2)^{-n/2}\exp(-\Sigma_{i=1}^n \frac{x_i^2}{\sigma^2}) = (2\pi\sigma^2)^{-n/2}\exp(-\frac{1}{\sigma^2}\Sigma_{i=1}^n x_i^2)$$

This is of the form $g(t(x),\sigma^2)h(x)$ where $t(x)=\Sigma_{i=1}^n x_i^2$, $(2\pi\sigma^2)^{-n/2}\exp(-\frac{1}{\sigma^2}\Sigma_{i=1}^n x_i^2)$ and $h(x)=1$. So $\Sigma_{i=1}^n X_i^2$ is a sufficient statistic for σ^2.

(b) $\log(f(x,\sigma^2)) = -\frac{\log(2\pi\sigma^2)}{2} - \frac{x^2}{2\sigma^2}$

$$\therefore \frac{\partial \log(f(x,\sigma^2))}{\partial\sigma^2} = -\frac{1}{2\sigma^2} + \frac{x^2}{2\sigma^4}$$

$$\therefore E\{[\frac{\partial \log(f(x,\sigma^2))}{\partial\sigma^2}]^2\} = \frac{1}{4\sigma^4} + \frac{E(x^4)}{4\sigma^8} - \frac{2E(x^2)}{4\sigma^6} = \frac{1}{2\sigma^4}$$

So the CRLB for σ^2 is $2\sigma^4/n$.

2. (a) The likelihood function is $L(p) = (1-3p)^{n_0} p^{n_1}(2p)^{n_2}$

The log-likelihood function is $l(p) = n_0\log(1-3p) + n_1\log(p) + n_2\log(2p)$

FOC: $l'(p) = \frac{-3n_0}{1-3p} + \frac{n_1}{p} + \frac{2n_2}{2p} = 0$

$$\therefore \frac{3n_0}{1-3p} = \frac{n_1+n_2}{p} \Rightarrow 3pn_0 = n_1 + n_2 - 3n_1 p - 3n_2 p \Rightarrow [3n_0 + 3n_1 + 3n_2]p = n_1 + n_2$$

So the MLE of p is $\frac{n_1+n_2}{3(n_0+n_1+n_2)}$.

(b) $\log(f(x,p)) = 1(x=0)\log(1-3p) + 1(x=1)\log(p) + 1(x=2)\log(2p)$

$$\therefore \frac{\partial \log(f(x,p))}{\partial p} = 1(x=0)\frac{-3}{1-3p} + 1(x=1)\frac{1}{p} + 1(x=2)\frac{2}{2p}$$

$$\therefore E\{[\frac{\partial \log(f(x,p))}{\partial p}]^2\} = (1-3p)(\frac{-3}{1-3p})^2 + p(\frac{1}{p})^2 + 2p(\frac{1}{p})^2 = \frac{9}{1-3p} + \frac{1}{p} + \frac{2}{p} = \frac{9}{1-3p} + \frac{3}{p}$$

$$\therefore E\{[\frac{\partial \log(f(x,p))}{\partial p}]^2\} = \frac{3}{(1-3p)p}$$

So the CRLB is $\frac{p(1-3p)}{3n}$.

3 (a) $f(x,y) = f(y\,|\,x)f(x) = 2\frac{1}{\sqrt{2\pi\lambda x^2}}\exp(-\frac{y^2}{2\lambda x^2}) = 2\frac{1}{\sqrt{2\pi\lambda}}\frac{1}{x}\exp(-\frac{y^2}{2\lambda x^2})$

if $1/2 \le x \le 1$ and 0 otherwise.

(b) $f(z\,|\,x) = \frac{1}{\sqrt{2\pi\lambda}}\exp(-\frac{z^2}{2\lambda^2})$

$$f(z) = \int_{1/2}^{1} f(z \mid x) f(x) dx = \int_{1/2}^{1} \frac{1}{\sqrt{2\pi\lambda}} \exp(-\frac{z^2}{2\lambda^2}) 2dx = \frac{1}{\sqrt{2\pi\lambda}} \exp(-\frac{z^2}{2\lambda^2})$$

So Z is $N(0,\lambda)$.

(c) $l(\lambda) = \Sigma_{i=1}^{n} \log f(X_i, Y_i) = \Sigma_{i=1}^{n} \{\log(2) - \frac{1}{2}\log(2\pi\lambda) - \log(X_i) - \frac{Y_i^2}{2\lambda X_i^2}\}$

$$\therefore l(\lambda) = n\log(2) - \frac{n}{2}\log(2\pi\lambda) - \Sigma_{i=1}^{n}\{\log(X_i) + \frac{Y_i^2}{2\lambda X_i^2}\}$$

(d) From the FOC, $\hat{\lambda}_{MLE} = \frac{1}{n}\Sigma_{i=1}^{n}(\frac{Y_i}{X_i})^2$

4. The maximum likelihood estimator is $\bar{X}$ and the asymptotic variance is found as follows
$\log(f(x,\lambda)) = x\log(\lambda) - \lambda - \log(x!)$

$$\therefore \frac{\partial \log(f(x,\lambda))}{\partial\lambda} = \frac{x}{\lambda} - 1$$

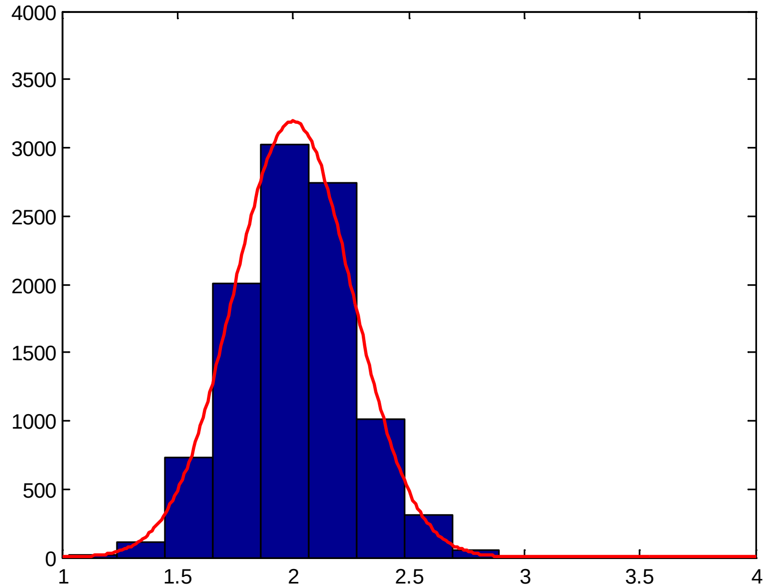
$$\therefore E\{[\frac{\partial \log(f(x,\lambda))}{\partial\lambda}]\}^2 = \frac{E(x^2)}{\lambda^2} + 1 - \frac{2E(x)}{\lambda} = \frac{\lambda+\lambda^2}{\lambda^2} + 1 - \frac{2\lambda}{\lambda} = \frac{1}{\lambda}$$

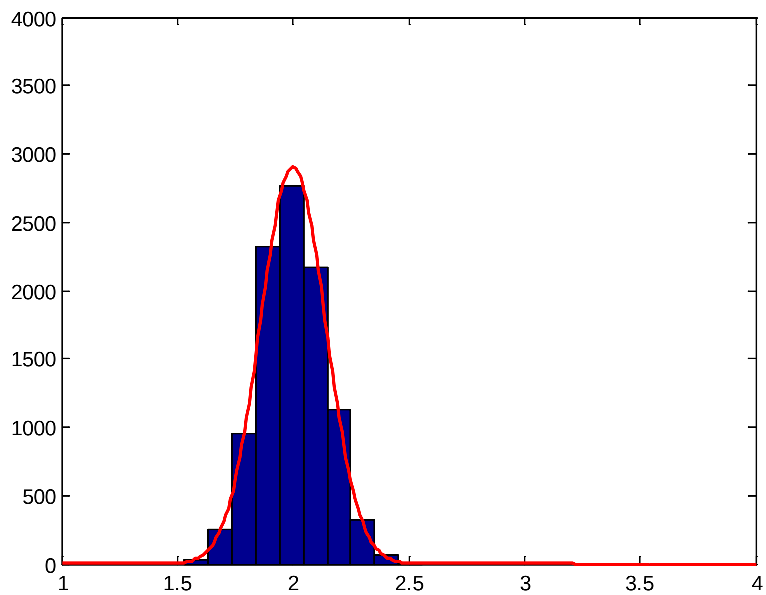
and so the approximate distribution of the MLE is $N(\lambda, \lambda / n)$.

Here is the program for $n = 30$ (just change the first line to make it $n = 100$).

```
n=30;
replics=10000;
for imc=1:replics;
    x=poissrnd(2,n,1);
    mle(imc)=mean(x);
end;
hist(mle);
axis([1 4 0 4000]);
hold on;
x=[1:0.01:4]';
y=normpdf(x,2,sqrt(2/n));
[q1,q2]=hist(mle);
y=y*10000*(q2(2)-q2(1));
plot(x,y,'Color','r','Linewidth',2);
```

And here are the results for the two cases.





We can see that the asymptotic distribution works well in both sample sizes, especially for the larger one.