Problem Set 3: Solutions

1. This is the Matlab program

```
%For part (a)
rand('seed',123); randn('seed',123);
x=mean(exprnd(1,10,10000))';
disp(mean(abs(x-1)<=0.1));
x=mean(exprnd(1,100,10000))';
disp(mean(abs(x-1)<=0.1));
x=mean(exprnd(1,1000,10000))';
disp(mean(abs(x-1)<=0.1));

%For part (b)
rand('seed',123); randn('seed',123);
x=mean(trnd(1,10,10000))';
disp(mean(abs(x)<=0.1));
x=mean(trnd(1,100,10000))';
disp(mean(abs(x)<=0.1));
x=mean(trnd(1,1000,10000))';
disp(mean(abs(x)<=0.1));
```

Here is a Table of results

	Exponential (part (a))	Cauchy (part (b))
n=10	0.2393	0.0660
n=100	0.6759	0.0576
n=1,000	0.9978	0.0684

Clearly on the left the law of large numbers is working. The probability of being between 0.9 and 1.1 is getting closer and closer to 1 as the sample size increases. Meanwhile, on the right, the law of large numbers is not working. The sample mean of Cauchy random variables does not converge in probability to anything, and indeed it does not even have a finite mean.

2. From the delta method,

$$\sqrt{n}\hat{\rho} \to N(0, \begin{pmatrix} -\frac{\sigma_{XY}}{2(\sigma_X^2\sigma_X^2)^{3/2}} & -\frac{\sigma_{XY}}{2(\sigma_X^2\sigma_X^2)^{3/2}} & \frac{1}{(\sigma_X^2\sigma_Y^2)^{1/2}} \end{pmatrix} \begin{pmatrix} Var(X_i^2) & 0 & 0 \\ 0 & Var(Y_i^2) & 0 \\ 0 & 0 & \sigma_X^2\sigma_Y^2 \end{pmatrix} \begin{pmatrix} -\frac{\sigma_{XY}}{2(\sigma_X^2\sigma_X^2)^{3/2}} \\ -\frac{\sigma_{XY}}{2(\sigma_X^2\sigma_X^2)^{3/2}} \\ \frac{1}{(\sigma_X^2\sigma_X^2)^{1/2}} \end{pmatrix})$$

But $\sigma_{XY} = 0$, so this simplifies to $\sqrt{n}\hat{\rho} \to N(0,1)$.

3. (a) The log-likelihood is

$$l(\theta) = \Sigma_{i=1}^n \log(\theta) + (\theta - 1)\log(X_i) = n\log(\theta) + (\theta - 1)\Sigma_{i=1}^n \log(X_i)$$

(b) The MLE of θ solves

$$l'(\tilde{\theta}) = 0$$

$$\therefore \frac{n}{\tilde{\theta}} + \Sigma_{i=1}^n \log(X_i) = 0$$

$$\therefore \tilde{\theta} = -\frac{n}{\Sigma_{i=1}^n \log(X_i)}$$

(c) $E(X) = \int_0^1 \theta x^\theta dx = \theta[\frac{x^{\theta+1}}{\theta+1}]_0^1 = \frac{\theta}{\theta+1}$

So the method of moments estimator solves $\bar{X} = \frac{\hat{\theta}}{\hat{\theta}+1}$. Accordingly $\hat{\theta} = \frac{\bar{X}}{1-\bar{X}}$.

(d) The MLE is 2.19. The method of moments estimator is 2.

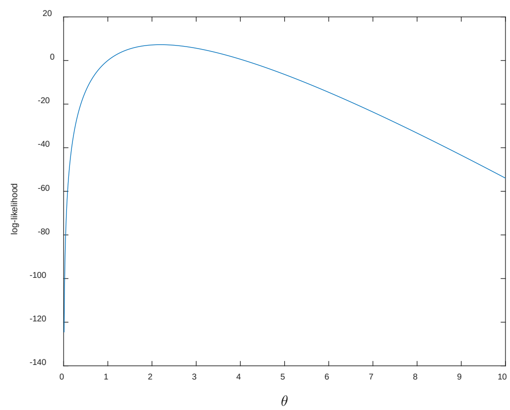
(e) and (f) Here is the log-likelihood function (up to a minus sign)

```
function l=likel(theta);
n=30; sumlogx=-13.67;
l=(n*log(theta))+((theta-1)*sumlogx);
l=-l;
```

Then, running the program

```
t=[0.01:0.01:10]';
plot(t,-likel(t));
ylabel('log-likelihood');
xlabel('\theta');
test=fminsearch('likel',6);
```

generates the log-likelihood which is concave and the MLE is 2.1946, very close to that obtained analytically.



4. The likelihood function is $\Pi_{i=1}^{n} 1(0 < X_i < 1)$ if $\theta = 0$. Since the support of the data is from 0 to 1, this means that the likelihood function is 1 if $\theta = 0$. Meanwhile, the likelihood function is $\Pi_{i=1}^{n} \frac{1}{2\sqrt{X_i}} 1(0 < X_i < 1) = \frac{1}{2^n} \Pi_{i=1}^{n} \frac{1}{\sqrt{X_i}}$ if $\theta = 1$. So the MLE is $1(\Pi_{i=1}^{n} \frac{1}{\sqrt{X_i}} > 2^n)$.

5. At $p = 1/2$, it is the probability of getting the observed outcome if this were the true probability. This is $\frac{80!}{49!31!} 0.5^{49} 0.5^{31}$. Similarly, at $p = 3/4$, the likelihood is $\frac{80!}{49!31!} 0.75^{49} 0.25^{31}$. The MLE is $49/80$.

6. The mean return is 8.76 percent. Hence the posterior mean is

$$\frac{10*4^2}{10*4^2+16^2} * 8.76 + \frac{16^2}{10*4^2+16^2} * 8 = 8.29$$