Problem Set 3: Solutions

1. $E(W \mid v) = E(\alpha + u \mid v) = \alpha + E(u \mid v) = \alpha + \rho v$

$E(W \mid v > 0) = E(E(W \mid v) \mid v > 0)$

$\therefore E(W \mid v > 0) = \alpha + \rho E(v \mid v > 0)$

$\therefore E(W \mid v > 0) = \alpha + \rho\sqrt{\dfrac{2}{\pi}}$

2. For $t > 0$:

$P(X \geq a) = P(e^{Xt} \geq e^{at}) \leq \dfrac{E(e^{Xt})}{e^{at}}$ by Chebychev's inequality.

$\therefore P(X \geq a) \leq \dfrac{m(t)}{e^{at}}$

$\therefore P(X \geq a) \leq \exp(-at)m(t)$

For $t < 0$:

$P(X \leq a) = P(e^{Xt} \geq e^{at}) \leq \dfrac{E(e^{Xt})}{e^{at}}$ by Chebychev's inequality.

$\therefore P(X \leq a) \leq \dfrac{m(t)}{e^{at}}$

$\therefore P(X \leq a) \leq \exp(-at)m(t)$

3. Let n random numbers be ordered $X_{(1)} < X_{(2)} \ldots < X_{(n)}$.

$P(X_{(n-1)} \leq x) = nF(x)^{n-1} - (n-1)F(x)^n$

$\therefore P(X_{(n-1)} \leq 1.96) = nF(1.96)^{n-1} - (n-1)F(1.96)^n$

If the cdf $F(x)$ is standard normal, then $F(1.96) = 0.975$. Hence

$P(X_{(n-1)} \leq 1.96) = n0.975^{n-1} - (n-1)0.975^n$

$\therefore P(X_{(n-1)} > 1.96) = 1 - \{n0.975^{n-1} - (n-1)0.975^n\}$

The lowest value of n for which this exceeds 90 percent is 155.

4. $Cov(X,Y) = 0.5 * \sqrt{Var(X) * Var(Y)} = 0.5 * 2 = 1$

$E(X + 2Y + 1) = E(X) + 2E(Y) + 1 = 1 - 2 + 1 = 0$

$Var(X + 2Y + 1) = Var(X) + 4Var(Y) + 4Cov(X,Y) = 1 + 16 + 4 = 21$

5. (a) The conditional pdf is

$N(70 + \dfrac{0.6 * 2}{20}(190 - 180), 4(1 - 0.6^2)) = N(70.6, 2.56)$

(b) The conditional pdf is

$N(180 + \dfrac{0.6 * 20}{2}(68 - 70), 400(1 - 0.6^2)) = N(168, 256)$

6. (a) The joint pdf of X1 and X2 is

$$f(x_1,x_2)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-x_1^2/2\sigma^2}\frac{1}{\sqrt{2\pi}}e^{-x_2^2/2\sigma^2}=\frac{1}{2\pi\sigma^2}\exp(-\frac{x_1^2+x_2^2}{2\sigma^2})$$

The transformation is

$Y_1 = X_1^2 + X_2^2$ and $Y_2 = X_1/\sqrt{Y_1}$

If we partition the space of X1 and X2 into X2>0 and X2<0, then for each of these partitions, the transformation is one-to-one. Now consider X2>0.

To invert the transformation,

$$X_1 = Y_2\sqrt{Y_1}$$

$$X_2^2 = Y_1 - X_1^2 = Y_1 - Y_2^2 Y_1 = Y_1(1-Y_2^2) \Rightarrow X_2 = \sqrt{Y_1(1-Y_2^2)}$$

So

$$\begin{pmatrix}\partial X_1/\partial Y_1 & \partial X_1/\partial Y_2\\ \partial X_2/\partial Y_1 & \partial X_2/\partial Y_2\end{pmatrix}=\begin{pmatrix}Y_2\frac{1}{2}Y_1^{-1/2} & Y_1^{1/2}\\ \frac{1}{2}(Y_1(1-Y_2^2))^{-1/2}(1-Y_2^2) & \frac{1}{2}(Y_1(1-Y_2^2))^{-1/2}*(-2Y_1Y_2)\end{pmatrix}=\frac{1}{2Y_1^{1/2}}\begin{pmatrix}Y_2 & 2Y_1\\ (1-Y_2^2)^{1/2} & -2(1-Y_2^2)^{-1/2}Y_1Y_2\end{pmatrix}$$

$$\Rightarrow J=-\frac{1}{4Y_1}[2Y_1Y_2^2(1-Y_2^2)^{-1/2}+2Y_1(1-Y_2^2)^{1/2}]=-\frac{1}{2Y_1(1-Y_2^2)^{1/2}}[Y_1Y_2^2+Y_1(1-Y_2^2)]=$$

$$=-\frac{1}{2Y_1(1-Y_2^2)^{1/2}}[Y_1Y_2^2+Y_1-Y_1Y_2^2]=-\frac{1}{2(1-Y_2^2)^{1/2}}$$

$$\therefore |J|=\frac{1}{2(1-Y_2^2)^{1/2}}$$

$$\therefore f(y_1,y_2)=\frac{1}{2\pi\sigma^2}\exp(-\frac{y_2^2y_1+y_1(1-y_2^2)}{2\sigma^2})\frac{1}{2(1-y_2^2)^{1/2}}$$

$$\therefore f(y_1,y_2)=\frac{1}{2\pi\sigma^2}\exp(-\frac{y_1}{2\sigma^2})\frac{1}{2(1-y_2^2)^{1/2}}$$

If we repeat the exercise for X2<0, we get the same result, so combining these gives the joint density

$$f(y_1,y_2)=\frac{1}{2\pi\sigma^2}\exp(-\frac{y_1}{2\sigma^2})\frac{1}{(1-y_2^2)^{1/2}}$$

(b)The pdf can be factored as $\frac{1}{2\pi\sigma^2}\exp(-\frac{y_1}{2\sigma^2})*\frac{1}{(1-y_2^2)^{1/2}}$ so Y1 and Y2 must be independent.

The support of Y1 is from 0 to infinity. The support of Y2 is from -1 to +1.

7. For any real number t, $E((X-tY)^2)\geq 0$

$\therefore E(X^2)+t^2E(Y^2)-2tE(XY)\geq 0$

with strict inequality unless $X = tY$.

Suppose that $X \neq tY$

Consider the equation $E(X^2) + t^2 E(Y^2) - 2tE(XY) = 0$. This cannot have a real solution. But it has a real solution t if and only if $4E(XY)^2 - 4E(X^2)E(Y^2) \leq 0$.
Hence $|E(XY)| \leq \sqrt{E(X^2)E(Y^2)}$, as required.

If $X = tY$ then $|E(XY)| = \sqrt{E(X^2)E(Y^2)}$ and so $|E(XY)| \leq \sqrt{E(X^2)E(Y^2)}$.

8. (a) The transition matrix is $\begin{pmatrix} 0.9 & 0.5 \\ 0.1 & 0.5 \end{pmatrix}$

(b) The $P^2 = \begin{pmatrix} 0.86 & 0.7 \\ 0.14 & 0.3 \end{pmatrix}$. So it is sunny today, the probability that it will be sunny the day after tomorrow is 86 percent.

(c) The steady state solves $\begin{pmatrix} -0.1 & 0.5 \\ 0.1 & -0.5 \end{pmatrix}\begin{pmatrix} q \\ 1-q \end{pmatrix} = 0$.

So $-0.1q + 0.5*(1-q) = 0 \Rightarrow -q + 5 - 5q = 0 \Rightarrow 6q = 5 \Rightarrow q = 5/6$

So the steady state vector is (5/6,1/6)'.

9. (a) The transition matrix is $P = \begin{pmatrix} 0.28 & 0.18 & 0 \\ 0.18 & 0.28 & 0 \\ 0.54 & 0.54 & 1 \end{pmatrix}$.

(b) $P^4 = \begin{pmatrix} 0.0224 & 0.0223 & 0 \\ 0.0223 & 0.0224 & 0 \\ 0.9552 & 0.9552 & 1 \end{pmatrix}$ and so the probability that they are in the initial location after four moves is 2.224%.

(c) The cumulative distribution function of the length of the hunt is

$$P(T \leq t) = 0.54*(1 + 0.46^2 .... + 0.46^{t-1}) = 0.54\frac{1-0.46^t}{1-0.46} = 1 - 0.46^t$$

(d) The probability that the hunt lasts for t periods is $0.54*0.46^{t-1}$

(e) The expected length of the hunt is

$$\Sigma_{t=1}^{\infty} t*0.54*0.46^{t-1} = 0.54*\Sigma_{t=1}^{\infty} t*0.46^{t-1} = 0.54*\frac{1.}{(1-0.46)^2} = \frac{1}{0.54} = 1.8519 \text{ periods.}$$