Problem Set 3

Due Date: 10/31/2017 Econ 180.636

1. This question is designed to illustrate the law of large numbers and a rather pathological case where it doesn't work.

(a) An exponential random variable with parameter 1 has a mean of 1. Take the average of n=10 exponentially distributed random variables. Repeat this 10,000 times. What fraction of times is the average of these n random variables between 0.9 and 1.1? Repeat this for n=100 and for n=1,000. Hint: You can generate an exponential random variable either as minus the log of a uniform random variable, or using the exprnd function in the Statistics toolbox.

(b) A Cauchy random variable has median zero, but its mean does not exist. So it doesn't satisfy the conditions for the law of large numbers. Here we will demonstrate that the sample average of random variables drawn from a Cauchy distribution does not converge to zero (or to anything else for that matter). Take the average of n=10 Cauchy distributed random variables. Repeat this 10,000 times. What fraction of times is the average of these n random variables between -0.1 and +0.1? Repeat this for n=100 and for n=1,000. Hint: You can generate a Cauchy random variable either as the ratio of two independent standard normal random variables, or using the trnd function in the Statistics toolbox, as a Cauchy distribution is the same as a t distribution on 1 degree of freedom.

2. Suppose that $X_1, X_2, ...X_n$ are iid with mean zero and variance σ_X^2 and $Y_1, Y_2....Y_n$ are iid with mean zero and variance σ_Y^2. X_i and Y_i are mutually independent, so that $\sigma_{XY} \equiv E(X_i Y_i) = 0$. By the central limit theorem:

$$\begin{pmatrix} n^{1/2}[\frac{1}{n}\Sigma_{i=1}^n X_i^2 - 1] \\ n^{1/2}[\frac{1}{n}\Sigma_{i=1}^n Y_i^2 - 1] \\ n^{1/2}\frac{1}{n}\Sigma_{i=1}^n X_i Y_i \end{pmatrix} \to_d N(0, \begin{pmatrix} Var(X_i^2) & 0 & 0 \\ 0 & Var(Y_i^2) & 0 \\ 0 & 0 & \sigma_X^2 \sigma_Y^2 \end{pmatrix} \qquad (*)$$

Define the sample correlation as $\hat{\rho} = \frac{\Sigma_{i=1}^n X_i Y_i}{\sqrt{\Sigma_{i=1}^n X_i^2 \Sigma_{i=1}^n Y_i^2}}$.

Use the delta method to derive the limiting distribution of $n^{1/2}\hat{\rho}$ from equation (*).

3. Suppose that $X_1, X_2, ...X_n$ are iid with probability density function

$$f(x) = \theta x^{\theta-1}, \ 0 \le x \le 1.$$

(a) Derive the log likelihood and show that it depends on the data only through $\Sigma_{i=1}^n \log X_i$.

(b) Derive the maximum likelihood estimator for θ.

(c) Derive the method of moments estimator for θ.

(d) Now suppose that we observe data such that $n = 30$, $\Sigma_{i=1}^n X_i = 20$ and $\Sigma_{i=1}^n \log X_i = -13.67$. Calculate the MLE and method of moments estimates.

(e) In Matlab write a function that takes θ as input and calculates the log-likelihood, based in the data values in (d). The function should have only one input, θ. Plot the log-likelihood against θ for θ from 0.01 to 10 and discuss the shape of the log-likelihood function.

(f) Use the fminsearch function as demonstrated in class to numerically maximize the log-likelihood function.

4. Let $X_1, X_2, \ldots X_n$ be iid with one of two pdfs: if $\theta = 0$, then $f(x) = 1(0 < x < 1)$ while if $\theta = 1$ then $f(x) = \frac{1}{2\sqrt{x}} 1(0 < x < 1)$. Find the MLE of θ.

5. An *unfair* coin is tossed 80 times. It comes up heads 49 times and tails 31 times. Let p denote the probability of heads on each toss. What is the likelihood function at p=1/2? At p=3/4? What is the maximum likelihood estimate?

6. Based on past data and other information, your prior for the mean nominal annual stock return excluding dividends is normal with mean 8 percent and standard deviation 4 percent. Also you know that the standard deviation of stock returns is 16 percent. Then you are told the nominal annual stock returns in percentage points for the last ten years, which are as follows:

2007	5.5	2012	16.0
2008	-37.0	2013	32.4
2009	26.5	2014	13.7
2010	15.1	2015	1.4
2011	2.1	2016	11.9

Given your prior beliefs and these data what is your posterior estimate of the mean of annual stock returns?