**Problem Set 2: Solutions**

1. (a) There are 36 possible outcomes, of which 3 sum to 10 (4-6, 5-5 and 6-4). So the marginal probability of the two numbers summing to 10 is 3/36=1/12.
(b) The joint probability of at least one 6 and the numbers summing to 10 is 2/36. So the conditional probability is (2/36)÷(3/36)=2/3.
(c) The expectation is (4*1/3)+(5*1/3)+(6*1/3)=5.
(d) The expectation is 4.5.
The variance is $((3\text{-}4.5)^2*1/4)+((4\text{-}4.5)^2*1/4)+((5\text{-}4.5)^2*1/4)+((6\text{-}4.5)^2*1/3)=5/4$.

2. $P(Y \geq \varepsilon) = P(\log(X) \geq \varepsilon) = P(X \geq \exp(\varepsilon)) = 1 - P(X \leq \exp(\varepsilon))$
$= 1-(\exp(\varepsilon)-1) = 2-\exp(\varepsilon)$

From Chebychev's inequality $P(Y \geq \varepsilon) = P(\log(X) \geq \varepsilon) = P(X \geq \exp(\varepsilon)) \leq \frac{E(X)}{\exp(\varepsilon)} = \frac{1.5}{\exp(\varepsilon)}$

so $P(Y \geq \varepsilon) \leq= \frac{1.5}{\exp(\varepsilon)}$. Note that this bound is no use at all if $\varepsilon < \log(1.5) = 0.4055$ because it only tells us that $P(Y \geq \varepsilon)$ is less than or equal to a number that is bigger than 1! True..but not much help.

3. The cdf is $F(x) = 1 - e^{-x/\lambda}$. So $P(X > 1) = 1-(1-e^{-1/\lambda}) = e^{-1/\lambda}$ and the density of X conditional on X>1 is $\frac{1}{\lambda}e^{-x/\lambda} \div e^{-1/\lambda} = \frac{1}{\lambda}e^{(1-x)/\lambda}$.

Hence $E(X \mid X > 1) = \int_1^\infty x\frac{1}{\lambda}e^{(1-x)/\lambda}dx = \frac{1}{\lambda}\int_1^\infty xe^{(1-x)/\lambda}dx = \frac{1}{\lambda}\{[-\lambda xe^{(1-x)/\lambda}]_1^\infty - \int_1^\infty[-\lambda e^{(1-x)/\lambda}]dx$

$\therefore E(X \mid X > 1) = \int_1^\infty e^{(1-x)/\lambda}dx - [xe^{(1-x)/\lambda}]_1^\infty = [-\lambda e^{(1-x)/\lambda}]_1^\infty - [xe^{(1-x)/\lambda}]_1^\infty = \lambda + 1$

4. The moment generating function of X is

$$m_X(t) = E(e^{tx}) = \int_0^\infty e^{\tau x}\frac{1}{\lambda}e^{-x/\lambda}dx = \frac{1}{\lambda}\int_0^\infty e^{(t-\frac{1}{\lambda})x}dx = \frac{1}{\lambda}[\frac{e^{(t-\frac{1}{\lambda})x}}{t-\frac{1}{\lambda}}]_0^\infty = -\frac{1}{\lambda}\frac{1}{t-\frac{1}{\lambda}} = -\frac{1}{t\lambda-1} = \frac{1}{1-t\lambda}$$

Similarly the moment generating function of Y is $\frac{1}{1-t\lambda}$. Hence the moment generating function of X+Y is $\frac{1}{(1-t\lambda)^2}$.

The moment generating function of the random variable with the pdf $\frac{ze^{-z/\lambda}}{\lambda^2}$ is

$$m_Z(t)=\int_0^\infty \frac{1}{\lambda^2} z e^{-z/\lambda} e^{zt}\,dz=\frac{1}{\lambda^2}\int_0^\infty z e^{z(t-\frac{1}{\lambda})}\,dz=\frac{1}{\lambda^2}\{[\frac{z}{t-\frac{1}{\lambda}} e^{z(t-\frac{1}{\lambda})}]_0^\infty-\int_0^\infty \frac{1}{t-\frac{1}{\lambda}} e^{z(t-\frac{1}{\lambda})}dz\}$$

$$=\frac{1}{\lambda^2}\frac{1}{(t-\frac{1}{\lambda})^2}=\frac{1}{(t\lambda-1)^2}=\frac{1}{(1-t\lambda)^2}$$

Thus X+Y and Z have the same mgfs and hence the same densities.

5. (a) $f_{X_1}(1.4\,|\,v)=\dfrac{1}{2v}\Rightarrow f_{X_1}(1.4)=\int_1^2 \dfrac{1}{2v}dv=\dfrac{\log(2)}{2}$.

By Bayes rule,

$$f(v\,|\,X_1=1.4)=\frac{\frac{1}{2v}*1}{\log(2)/2}=\frac{1}{v\log(2)}$$

$$\therefore E(V\,|\,X_1=1.4)=\int_1^2 v\frac{1}{v\log(2)}dv=\frac{1}{\log(2)}$$

(b) $P(X_2<1.4\,|\,v)=\dfrac{1.4}{2v}$

$$f(X_1=1.4, X_2<1.4\,|\,v)=f(X_1=1.4\,|\,v)P(X_2<1.4\,|\,v)=\frac{1}{2v}*\frac{1.4}{2v}=\frac{1.4}{4v^2}$$

The conditional density of V, by Bayes rule is

$$\frac{\frac{1.4}{4v^2}*1}{\int_1^2 \frac{1.4}{4s^2}*1ds}=\frac{\frac{1.4}{4v^2}}{\int_1^2 \frac{1.4}{4s^2}ds}$$

Hence $E(V\,|\,X_1=1.4, X_2<1.4)=\dfrac{\int_1^2 \frac{1.4}{4v}dv}{\int_1^2 \frac{1.4}{4v^2}dv}=\dfrac{\int_1^2 \frac{1}{v}dv}{\int_1^2 \frac{1}{v^2}dv}=\dfrac{[\log(v)]_1^2}{[-\frac{1}{v}]_1^2}=\dfrac{\log(2)}{-(\frac{1}{2}-1)}=\dfrac{\log(2)}{1/2}=2\log(2)$

(c) The question here was a little ill-posed. But I intended it to be assumed that contestant 1 will win if and only if $X_2<1.4$. The problem is that if $X_2<1.4$ then contestant 2 will bid the same as contestant 1 and so there will be a tie. Think of it as a tie-breaking rule that if they both submit the same bid, the one with the higher signal wins.

Contestant 1's bid is $1/\log(2)$. So if she wins, her expected profit is $2\log(2)-\dfrac{1}{\log(2)}=-0.0564$ which is a loss. The problem is that it isn't rational for contestant 1 to bid her expected value because it doesn't take account of the fact that she will only win if the other contestant has received a lower signal.

(d) The Matlab program is

```
rand('seed',123);
n=20000000;
v=rand(n,1)+1;
x1=(2*rand(n,1)).*v;
x2=(2*rand(n,1)).*v;
s1=(abs(x1-1.4)<0.02);
s2=(x2<1.4);
sum(v.*s1)/sum(s1)                    %First question
sum(v.*s1.*s2)/sum(s1.*s2)            %Second question
```

The results are

1.4422

1.3854

which are very close to $\frac{1}{\log(2)}$ and $2\log(2)$, respectively. Effectively, you have simulated the answer to the question, because conditioning on $X_1$ being between 1.38 and 1.42 is almost the same thing as conditioning on it being exactly 1.4 (which however could not be done directly).