Problem Set 1: Solutions

1. (a) $f(x)$ is not nonnegative over the stated range. For example, $f(2) = -4$. So it cannot be a pdf.
(b) $f(x)$ is nonnegative over the stated range. If it integrates to 1, then $f(x)$ is a pdf.

$$f(x) = C(2x - x^2) \Rightarrow \int_0^{1,5} C(2x - x^2) = 1 \Rightarrow C\int_0^{1,5} (2x - x^2) = 1 \Rightarrow C[x^2 - \frac{x^3}{3}]_0^{1,5} = C(1.5^2 - \frac{1.5^3}{3}) = 1.125C$$

$\therefore C = 8/9$

2. Event A: The first statement is true
Event B: The second islander says that it is true.

By Bayes Rule

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

$$P(B) = P(B|A)P(A) + P(B|\bar{A})P(\bar{A})$$

$$\therefore P(A|B) = \frac{(1/3)*(1/3)}{(1/3)*(1/3) + (2/3)*(2/3)} = \frac{1}{5}$$

3.(i) $F_X(x) = \frac{x+1}{2}$

$\log(X+1)$ is monotone increasing.

$g^{-1}(.) = \exp(.) - 1$

$$\therefore F_Y(y) = \frac{\exp(y) - 1 + 1}{2} = e^y / 2$$

$f_Y(y) = e^y / 2$

Support is from $-\infty$ to $\log(2)$.

$$\int_{-\infty}^{\log(2)} \frac{e^y}{2} dy = [\frac{e^y}{2}]_{-\infty}^{\log(2)} = \frac{e^{\log(2)}}{2} = \frac{2}{2} = 1$$

(ii) $|X|$ is not monotone.

$$F_Y(y) = P(|X| \le y) = F_X(y) - F_X(-y) = \frac{y+1}{2} - \frac{-y+1}{2} = y$$

$\therefore f_Y(y) = 1$

Support is from 0 to 1.

$\int_0^1 1dy = [y]_0^1 = 1$

(iii) X^3 is monotone increasing.

$g^{-1}(.) = (.)^{1/3}$

$$\therefore F_Y(y) = \frac{y^{1/3} + 1}{2}$$

$\therefore\ f_Y(y) = \frac{1}{2}\frac{1}{3}y^{-2/3} = \frac{1}{6y^{2/3}}$

Support is from -1 to +1.

$\int_{-1}^{1} \frac{1}{6y^{2/3}}dy = [\frac{1}{2}y^{1/3}]_{-1}^{1} = \frac{1}{2} + \frac{1}{2} = 1$

4. $p(i) = 0.5(p(i+1) + p(i-1))$

$p(0) = 0$ and $p(15) = 1$

I want to prove $(i+1)p(i) = ip(i+1)$ by induction.

For $i = 0$, $p(0) = 0$ and so this works.

Suppose that $(i+1)p(i) = ip(i+1)$

$p(i+1) = 0.5(p(i+2) + p(i))$

$\therefore\ p(i+1) = 0.5(p(i+2) + \frac{i}{i+1}p(i+1))$

$\therefore 2p(i+1) = p(i+2) + \frac{i}{i+1}p(i+1)$

$\therefore 2p(i+1)(i+1) = (i+1)p(i+2) + ip(i+1)$

$\therefore [2i+2-i]p(i+1) = (i+1)p(i+2)$

$\therefore (i+2)p(i+1) = (i+1)p(i+2)$

and so the relation has been shown by proof by induction.

$p(15) = 1$. Working backwards, $p(n) = \Pi_{j=n}^{14} \frac{j}{j+1} = \frac{n}{15}$.

So $p(5) = 1/3$ is the probability that A wins starting with $\$5$.

5. $E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)dx = \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2}\sigma}\exp(-\frac{|x-\mu|}{\sigma/\sqrt{2}})dx$

$= \int_{-\infty}^{\mu} x^2 \frac{1}{\sqrt{2}\sigma}\exp(\frac{x-\mu}{\sigma/\sqrt{2}})dx + \int_{\mu}^{\infty} x^2 \frac{1}{\sqrt{2}\sigma}\exp(\frac{\mu-x}{\sigma/\sqrt{2}})dx$

Integration by parts

$\int_a^b f(x)g'(x)dx = [f(x)g(x)]_a^b - \int_a^b f'(x)g(x)dx$

Consider $\int_{-\infty}^{\mu} x^2 \frac{1}{\sqrt{2}\sigma}\exp(\frac{x-\mu}{\sigma/\sqrt{2}})dx$.

Let $f(x) = \frac{x^2}{\sqrt{2}\sigma}$ and $g(x) = \frac{\sigma}{\sqrt{2}}\exp(\frac{x-\mu}{\sigma/\sqrt{2}}) \Rightarrow g'(x) = \exp(\frac{x-\mu}{\sigma/\sqrt{2}})$.

$\therefore \int_{-\infty}^{\mu} \frac{x^2}{\sqrt{2}\sigma}\exp(\frac{x-\mu}{\sigma/\sqrt{2}})dx = [\frac{x^2}{\sqrt{2}\sigma}\frac{\sigma}{\sqrt{2}}\exp(\frac{x-\mu}{\sigma/\sqrt{2}})]_{-\infty}^{\mu} - \int_{-\infty}^{\mu} \frac{2x}{\sqrt{2}\sigma}\frac{\sigma}{\sqrt{2}}\exp(\frac{x-\mu}{\sigma/\sqrt{2}})dx$

$$= [\frac{x^2}{2}\exp(\frac{x-\mu}{\sigma/\sqrt{2}})]_{-\infty}^{\mu} - \int_{-\infty}^{\mu} x\exp(\frac{x-\mu}{\sigma/\sqrt{2}})dx = \frac{\mu^2}{2} - \int_{-\infty}^{\mu} x\exp(\frac{x-\mu}{\sigma/\sqrt{2}})dx$$

By another application of integration by parts,

$$\int_{-\infty}^{\mu} x\exp(\frac{x-\mu}{\sigma/\sqrt{2}})dx = [x\frac{\sigma}{\sqrt{2}}\exp(\frac{x-\mu}{\sigma/\sqrt{2}})]_{-\infty}^{\mu} - \int_{-\infty}^{\mu} \frac{\sigma}{\sqrt{2}}\exp(\frac{x-\mu}{\sigma/\sqrt{2}})dx$$

$$= \frac{\mu\sigma}{\sqrt{2}} - [\frac{\sigma^2}{2}\exp(\frac{x-\mu}{\sigma/\sqrt{2}})]_{-\infty}^{\mu} = \frac{\mu\sigma}{\sqrt{2}} - \frac{\sigma^2}{2}$$

$$\therefore \int_{-\infty}^{\mu} \frac{x^2}{\sqrt{2}\sigma}\exp(\frac{x-\mu}{\sigma/\sqrt{2}})dx = \frac{\mu^2}{2} - \frac{\mu\sigma}{\sqrt{2}} + \frac{\sigma^2}{2}$$

Now consider $\int_{\mu}^{\infty} x^2 \frac{1}{\sqrt{2}\sigma}\exp(\frac{\mu-x}{\sigma/\sqrt{2}})dx$. Proceeding similarly,

$$\therefore \int_{\mu}^{\infty} \frac{x^2}{\sqrt{2}\sigma}\exp(\frac{\mu-x}{\sigma/\sqrt{2}})dx = [-\frac{x^2}{\sqrt{2}\sigma}\frac{\sigma}{\sqrt{2}}\exp(\frac{\mu-x}{\sigma/\sqrt{2}})]_{\mu}^{\infty} - \int_{\mu}^{\infty}\{-\frac{2x}{\sqrt{2}\sigma}\frac{\sigma}{\sqrt{2}}\exp(\frac{\mu-x}{\sigma/\sqrt{2}})\}dx$$

$$= [-\frac{x^2}{2}\exp(\frac{\mu-x}{\sigma/\sqrt{2}})]_{\mu}^{\infty} - \int_{\mu}^{\infty}\{-x\exp(\frac{\mu-x}{\sigma/\sqrt{2}})\}dx = \frac{\mu^2}{2} - \int_{\mu}^{\infty}\{-x\exp(\frac{\mu-x}{\sigma/\sqrt{2}})\}dx$$

Again, using integration by parts,

$$\int_{\mu}^{\infty}\{-x\exp(\frac{\mu-x}{\sigma/\sqrt{2}})\}dx = \{[x\frac{\sigma}{\sqrt{2}}\exp(\frac{\mu-x}{\sigma/\sqrt{2}})]_{\mu}^{\infty} - \int_{\mu}^{\infty} \frac{\sigma}{\sqrt{2}}\exp(\frac{\mu-x}{\sigma/\sqrt{2}})dx\}$$

$$= -\frac{\mu\sigma}{\sqrt{2}} - [-\frac{\sigma^2}{2}\exp(\frac{\mu-x}{\sigma/\sqrt{2}})]_{\mu}^{\infty} = -\frac{\mu\sigma}{\sqrt{2}} - \frac{\sigma^2}{2}$$

$$\therefore \int_{\mu}^{\infty} \frac{x^2}{\sqrt{2}\sigma}\exp(\frac{x-\mu}{\sigma/\sqrt{2}})dx = \frac{\mu^2}{2} + \frac{\mu\sigma}{\sqrt{2}} + \frac{\sigma^2}{2}$$

Adding the pieces together,

$$\therefore E(X^2) = \mu^2 + \sigma^2$$

$$\therefore Var(X) = E(X^2) - E(X)^2 = \mu^2 + \sigma^2 - \mu^2 = \sigma^2$$

6. $P(X = x) = \Sigma_{x=0}^{n} \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$

$$m(t) = E(e^{tX}) = \Sigma_{x=0}^{n} e^{tx} \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x} = \Sigma_{x=0}^{n} \frac{n!}{x!(n-x)!} [pe^t]^x (1-p)^{n-x}$$

$$\therefore m(t) = (pe^t + 1 - p)^n$$

$$m'(t) = n(pe^t + 1 - p)^{n-1} pe^t$$

$$\therefore m'(0) = np$$

$$m''(t) = (pe^t + 1 - p)^{n-1} npe^t + (n-1)(pe^t + 1 - p)^{n-2} pe^t npe^t$$

$$\therefore m''(0) = np + (n-1)np^2$$

$$Var(X) = m''(0) - m'(0)^2 = np + (n-1)np^2 - n^2 p^2 = np + n^2 p^2 - np^2 - n^2 p^2 = np - np^2 = np(1-p)$$

7. Matlab code.

```
rand('seed',123);
x=randn(10000,1);
y=-log(x);
hist(y);
z=[0.1:0.1:12]';
hold on;
plot(z,10000*exp(-z),'Linewidth',2);
axis([0 12 0 10000]);
```

The theoretical pdf is $f(y) = e^{-y}$.

Resulting graph: