Midterm Exam

Fall 2017 Econ 180-636

Closed Book.
Formula Sheet Provided. Calculators OK.
Time Allowed: 2 Hours
All Questions Carry Equal Marks

1. Suppose that Y has a probability density function

$$f_Y(y) = 1 - \frac{y}{2} \text{ for } 0 < y < C$$

where C is a positive constant.
(a) Determine the value of C.
(b) Calculate $E(Y)$.
(c) Suppose $X = 1(Y < 1)$. Determine the distribution of X.
(d) Continuing with the definition of X from part (c), determine $E(Y \mid X = 1)$.

2. Let $Y = \max(X_1, X_2)$ where $X_1 \sim \text{Exponential}(\beta_1)$ and $X_2 \sim \text{Exponential}(\beta_2)$ are independent random variables.
(a) Find the cdf and pdf of Y.
(b) Find $P(X_1 < X_2)$. Simplify as much as possible.

3. Let X be uniform on [0,1]. Suppose that a and b are constants such that 0<a<b<1. Define the random variables:
$Y = 1(0 < X < b)$
$Z = 1(a < X < 1)$
(a) What is the covariance between Y and Z (as a function of a and b)?
(b) What is $E(Y \mid Z)$?

4. Suppose that the random variable X has the pdf

$$f_X(x) = \frac{2}{\sqrt{2\pi}} e^{-x^2/2}$$

for $0 < x < \infty$.
(a) What is the variance of X?
(b) Let $Y = X^2$ (note that the support of X is the positive real line only). Find the probability density of Y.

5. (a) Suppose that X is Poisson with parameter λ: $P(X = x) = \dfrac{e^{-\lambda}\lambda^x}{x!}$. Use Chebyshev's inequality to prove that $P(X \geq 2\lambda) \leq \dfrac{1}{\lambda}$.

(b) Suppose that X is normal with mean μ and variance σ^2 . What is $E(\exp(X))$?

6. Let $B(s)$ be a standard Brownian motion as defined in class. Find $P(B(0.5) > 0 \mid B(1) = 1)$.

7. A random variable X has the density $f(x) = \sqrt{2}\exp(-\sqrt{2}x)$ for $x > 0$ and 0 otherwise. Find $E(X)$.

8. Suppose that there are two bags each containing black and white balls. The first bag contains three times as many white balls as blacks. The second bag contains three times as many black balls as white. Suppose we choose one of these bags at random. For this bag we select five balls at random, replacing each ball after it has been selected. The result is that we find 4 black balls and one white. What is the probability that we were using the bag with mainly white balls?

Solutions

1\. (a) $\int_0^C (1-\frac{y}{2})dy = [y-\frac{y^2}{4}]_0^C = 1$

$\therefore C - \frac{C^2}{4} = 1$

$\therefore C = 2$

(b) $E(Y) = \int_0^2 y(1-\frac{y}{2})dy = \int_0^2 ydy - \frac{1}{2}\int_0^2 y^2 dy = [\frac{y^2}{2}]_0^2 - \frac{1}{2}[\frac{y^3}{3}]_0^2 = \frac{4}{2} - \frac{8}{6} = \frac{2}{3}$

(c) $P(X=1) = P(Y<1) = \int_0^1 (1-\frac{y}{2}) = \frac{3}{4}$

(d) $f(y \mid x=1) = \frac{f(y)P(X=1 \mid y)}{P(X=1)} = \frac{(1-y/2)1(y<1)}{3/4}$

$\therefore E(Y \mid X=1) = \int_0^1 y(1-\frac{y}{2})\frac{4}{3}dy = \frac{4}{9}$

2\. (a) The cdf is $(1-e^{-\beta_1 y})(1-e^{-\beta_2 y})$

The pdf is $(1-e^{-\beta_1 y})\beta_2 e^{-\beta_2 y} + (1-e^{-\beta_2 y})\beta_1 e^{-\beta_1 y}$

(b) The joint pdf is

$f(x_1, x_2) = \beta_1\beta_2 e^{-\beta_1 x_1} e^{-\beta_2 x_2}$

So $P(X_1 < X_2) = \int_0^\infty \int_0^{x_2} \beta_1\beta_2 e^{-\beta_1 x_1} e^{-\beta_2 x_2} dx_1 dx_2 = \beta_1\beta_2 \int_0^\infty \int_0^{x_2} e^{-\beta_1 x_1} dx_1 e^{-\beta_2 x_2} dx_2$

$\therefore P(X_1 < X_2) = \beta_1\beta_2 \int_0^\infty [\frac{1-e^{-\beta_1 x_1}}{\beta_1}]_0^{x_2} e^{-\beta_2 x_2} dx_2 = \beta_1\beta_2 \int_0^\infty (\frac{1-e^{-\beta_1 x_2}}{\beta_1}) e^{-\beta_2 x_2} dx_2$

$\therefore P(X_1 < X_2) = \beta_2 \{\int_0^\infty e^{-\beta_2 x_2} dx_2 - \int_0^\infty e^{-(\beta_1+\beta_2)x_2} dx_2\}$

$\therefore P(X_1 < X_2) = \beta_2 \{[\frac{1-e^{-\beta_2 x_2}}{\beta_2}]_0^\infty - [\frac{1-e^{-(\beta_1+\beta_2)x_2}}{\beta_1+\beta_2}]_0^\infty\}$

$\therefore P(X_1 < X_2) = \beta_2 \{\frac{1}{\beta_2} - \frac{1}{\beta_1+\beta_2}\} = 1 - \frac{\beta_2}{\beta_1+\beta_2} = \frac{\beta_1}{\beta_1+\beta_2}$

3\. (a) $Cov(Y,Z) = E(YZ) - E(Y)E(Z) = P(Y=1, Z=1) - P(Y=1)P(Z=1)$

$P(Y=1, Z=1) = P(a < X < b) = b - a$

$P(Y=1) = b$

$P(Z=1) = 1 - a$

$P(Y=1)P(Z=1) = b(1-a) = b - ba$

$Cov(Y,Z) = b - a - b + ba = a(b-1)$

(b) If Z=0 then $E(Y) = P(X < b \mid X < a) = 1$

If Z=1 then $E(Y)=P(X<b\,|\,X>a)=\dfrac{b-a}{1-a}$

Hence $E(Y\,|\,Z)=Z(\dfrac{b-1}{1-a})+1$.

4. (a) $E(X)=\int x\dfrac{2}{\sqrt{2\pi}}e^{-x^2/2}dx=\sqrt{\dfrac{2}{\pi}}$

$$E(X^2)=\int_0^\infty x^2\frac{2}{\sqrt{2\pi}}e^{-x^2/2}dx=1$$

$$Var(X)=1-\frac{2}{\pi}$$

(b) $f_Y(y)=\dfrac{2}{\sqrt{2\pi}}e^{-y/2}.\dfrac{1}{2\sqrt{y}}=\dfrac{1}{\sqrt{2\pi}}y^{-1/2}e^{-y/2}$

5. (a) $P(X\geq 2\lambda)\leq\dfrac{E(X)}{2\lambda}=\dfrac{\lambda}{2\lambda}=\dfrac{1}{2}$

(b) $\exp(\mu+\dfrac{\sigma^2}{2})$.

6, $B(0.5)$ and $B(1)$ are jointly normally distributed with mean 0, variance 0.5 and 1, respectively and correlation $1/\sqrt{2}$.

Hence $B(0.5)\,|\,B(1)=1\sim N(\dfrac{1}{\sqrt{2}}\dfrac{1}{\sqrt{2}},\dfrac{1}{2}(1-\dfrac{1}{2}))=N(0.5,0.25)$.

So $P(B(0.5)>0\,|\,B(1)=1)=P(N(0.5,0.25)>0)=0.841$.

7. $E(X)=\int_0^\infty\sqrt{2}x\exp(-\sqrt{2}x)dx=[\dfrac{-\sqrt{2}x\exp(-\sqrt{2}x)}{\sqrt{2}}]_0^\infty-\int_0^\infty-\dfrac{\exp(-\sqrt{2}x)}{\sqrt{2}}\sqrt{2}dx$

$$=[x\exp(-\sqrt{2}x)]_0^\infty+\int_0^\infty\exp(-\sqrt{2}x)dx$$

$$=\int_0^\infty\exp(-\sqrt{2}x)dx=[-\frac{\exp(-\sqrt{2}x)}{\sqrt{2}}]_0^\infty=\frac{1}{\sqrt{2}}$$

8. $P(\text{4 black balls|bag 1})=5*(1/4)^4*(3/4)$

$P(\text{4 black balls|bag 2})=5*(3/4)^4*(1/4)$

$$P(\text{Bag 1|4 black balls})=\frac{P(\text{4 black balls|bag 1})*P(\text{Bag 1})}{P(\text{4 black balls|bag 1})*P(\text{Bag 1})+P(\text{4 black balls|bag 2})*P(\text{Bag 2})}$$

$$=\frac{5*(1/4)^4*(3/4)*(1/2)}{5*(1/4)^4*(3/4)*(1/2)+5*(3/4)^4*(1/4)*(1/2)}=\frac{3}{3+3^4}=\frac{1}{1+3^3}=\frac{1}{28}$$