Midterm Exam

Fall 2011 Econ 180-636

Closed Book.
Formula Sheet Provided. Calculators OK.
Time Allowed: 2 Hours
All Questions Carry Equal Marks

1. The probability that a child has blue eyes is 0.25. Assume independence between children. Consider a family with three children.
   (a) If it is known that at least one child in this family has blue eyes, what is the probability that at least two children in the family have blue eyes?
   (b) If it is known that the youngest child in the family has blue eyes, what is the probability that at least two children in the family have blue eyes?

2. Two random variables X and Y are both standard normal with correlation 0.5. Given that X=1, what is the probability that Y is positive?

3. Suppose that $X_1, X_2, ..$ is a random sequence such that

$$P(X_n = \frac{1}{n}) = 1 - \frac{1}{n^2}$$
$$P(X_n = n) = \frac{1}{n^2}$$

   (a) Does X_n converge in probability to any constant? If so, what is it? If not, prove it.
   (b) Does X_n converge in quadratic mean to any constant? If so, what is it? If not, prove it.

4. Suppose that the pdf of X is $f(x) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2})$, $-\infty \le x \le \infty$. Find the pdf of X^2 (I am looking for the expression for the pdf, not the name of this random variable).

5. Suppose that $X_1, X_2, ...X_n$ is iid uniform on the unit interval. Let $\bar{X} = n^{-1}\Sigma_{i=1}^{n} X_i$. Find the distribution of $\log(\bar{X})$ in the limit as the sample size, n, goes to infinity.

6. Suppose that X is N(0,1) and Y is an independent N(0,10) random variable. Let W be an independent Bernoulli random variable such that $P(W = 1) = 0.5$. Define $Z = XW + Y(1-W)$. Find the kurtosis of Z. This means $\frac{E((Z - E(Z))^4)}{\{E((Z - E(Z))^2)\}^2}$.

7. X is uniform on the unit interval, and Y conditional on X is uniform between X and 1. What is the marginal distribution of Y?

8. Let X and Y be independent uniform random variables on the unit interval. Find the cumulative distribution function and probability density function of Z=X+Y.

Solutions

1. (a) P(No child has blue eyes)=27/64
P(One child has blue eyes)=27/64
P(Two children have blue eyes)=9/64
P(Three children have blue eyes)=1/64
P(At least two children have blue eyes|At least one has blue eyes)=10/37
(b) Of the other two children, the probability that neither has blue eyes is 9/16 and the probability that one or both of them has blue eyes is 7/16. Therefore the answer is 7/16.

2. $Y \mid X = 1 \sim N(0.5, 0.75)$. The probability that this exceeds zero is 0.72.

3. X_n converges in probability to zero. But for any μ,

$$E((X_n - \mu)^2) = (1 - \frac{1}{n^2})[\frac{1}{n} - \mu]^2 + \frac{1}{n^2}[n - \mu]^2 \to \mu^2 + 1$$

and so there is no μ for which this limit is zero. So X_n does not converge in quadratic mean to any constant.

4. X is N(0,1) and so is equally likely to be positive and negative. Let $Y = X^2$. Now $X = \sqrt{Y}$ if $X > 0$, or the negative square root otherwise.

$$f(y \mid x > 0) = \frac{2}{\sqrt{2\pi}} \exp(-\frac{y}{2}) \frac{1}{2\sqrt{y}} = \frac{1}{\sqrt{2\pi}} \exp(-\frac{y}{2}) \frac{1}{\sqrt{y}}$$

The density of Y conditional on X being negative is identical, and so

$$f(y) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{y}{2}) \frac{1}{\sqrt{y}}$$

5. $\sqrt{n}(\bar{X} - 0.5) \to_d N(0, 1/12)$

$\sqrt{n}(\log(\bar{X}) - \log(0.5)) \to_d N(0, (\frac{1}{0.5})^2 \frac{1}{12})$

$\therefore \sqrt{n}(\log(\bar{X}) - \log(0.5)) \to_d N(0, \frac{1}{3})$

6. $E(Z) = 0$

$E(Z^2) = E(E(Z^2 \mid W)) = 1 * 0.5 + 10 * 0.5 = 5.5$

$E(Z^4) = E(E(Z^4 \mid W)) = 3 * 0.5 + 300 * 0.5 = 151.5$

So the answer is $\frac{151.5}{5.5^2} = \frac{606}{121} = 5.01$

7. $f(y \mid x) = \frac{1(x \le y \le 1)}{1 - x}$

$f(x, y) = f(y \mid x) f(x) = \frac{1(x \le y \le 1)}{1 - x}$

$f(y) = \int_0^y \frac{1}{1 - x} dx = [-\ln(1 - x)]_0^y = -\ln(1 - y)$

8. $P(Z \le z) = P(X + Y \le z) = \int_0^z \int_0^{z-x} dydx = \int_0^z (z-x)dx = \frac{z^2}{2}$ if $z \le 1$

Meanwhile

$$P(Z > z) = P(X + Y > z) = \int_{z-1}^{1} \int_{z-x}^{1} dydx = \int_{z-1}^{1} (1 - z + x)dx = [(1-z)x + \frac{x^2}{2}]_{(z-1)}^{1} \quad \text{if } 1 \le z \le 2$$

$$= (1-z) + \frac{1}{2} - (1-z)(z-1) - \frac{(z-1)^2}{2} = \frac{1}{2} + 1 - z + \frac{(z-1)^2}{2}$$

So

$$P(Z \le z) = z - \frac{1}{2} - \frac{(z-1)^2}{2} = 2z - 1 - \frac{z^2}{2} \text{ if } 1 \le z \le 2$$

So the cdf is

$P(Z \le z) = 0$	$z < 0$
$z^2 / 2$	$0 \le z \le 1$
$2z - 1 - z^2 / 2$	$1 < z \le 2$
1	$z > 2$

And the pdf is

z	$0 \le z \le 1$
$2 - z$	$1 \le z \le 2$
0	otherwise