Midterm Exam

Fall 2010 Econ 180-636

Closed Book.
Formula Sheet Provided. Calculators OK.
Time Allowed: 2 Hours
All Questions Carry Equal Marks

1. There are three cards. The first is green on both sides, the second is red on both sides, and the third is green on one side and red on the other. We choose a card at random and we see one side of this card, which is also selected at random. The side that we see is green. What is the probability that the other side of this card is also green?

2. X is a uniform random variable on the unit interval. The distribution of Y conditional on X is uniform from X to 1, and so

$$f(y \mid x) = \frac{1}{1-x} 1(0 < x < y < 1)$$

Find the marginal probability density function of Y.

3. X and Y are two random variables, with a joint probability density

$$f(x, y) = 1$$

for $0 \le x \le 1$ and $0 \le y \le 1$. At all other points the density is equal to zero.

(a) What is $Cov(X, Y)$.

(b) What is $E(X^2 + Y^2)$?

4. Let $X_1, X_2, \ldots X_n$ be iid random variables with mean zero and variance 1. Let $Y_1, Y_2, \ldots Y_n$ be another set of iid random variables with mean zero and variance 1. Assume that $\{X_1, X_2, \ldots X_n\}$ and $\{Y_1, Y_2, \ldots Y_n\}$ are mutually independent. Let $\overline{X} = n^{-1}\Sigma_{i=1}^n X_i$ and $\overline{Y} = n^{-1}\Sigma_{i=1}^n Y_i$. What is the asymptotic distribution of $\overline{X}\overline{Y}$?

5. Suppose that the midday temperature in Baltimore is normally distributed with a mean of 65 degrees and a standard deviation of 10 degrees. Suppose that the midday temperature in Washington is normally distributed with a mean of 67 degrees and a standard deviation of 10 degrees. Also, let the correlation between the temperature in Baltimore and the temperature in Washington be 0.9. Given that it is 70 degrees in Baltimore, what is the probability that it is at least 71.5 degrees in Washington?

6. Let X and Y be two independent uniform random variables on the unit interval.
(a) Find the joint pdf of U=X-Y and V=X/Y. Be sure to include the support of U and V.
(b) Verify that the answer in (a) is indeed a valid pdf (i.e. nonnegative and integrates to 1).

7. Suppose that X is a Poisson random variable with parameter λ. Prove that $P(X \geq 2\lambda) \leq 1/\lambda$. Hint: the variance of X is λ.

8. Consider two stochastic processes. $U(t)$ is a standard Brownian motion, and $N(t)$ is a Poisson counting process with an arrival rate (parameter) of 1. As usual, both processes start at zero ($U(0) = N(0) = 0$). Find $P(U(1) > N(1))$.

In this calculation, you may assume that the probability that a standard normal random variable exceeds 4 is negligible. Note that the probability mass function for a Poisson random variable with parameter λ is $P(X = x) = e^{-\lambda} \dfrac{\lambda^x}{x!}$, $x = 0,1,2,....$

Solutions

1. By Bayes rule, it is $\dfrac{P(OneGreen \mid BothGreen)P(BothGreen)}{P(OneGreen)} = \dfrac{1*1/3}{1*1/3+1/2*1/3} = \dfrac{2}{3}/$

2. $f(x) = 1(0 < x < 1)$

$$f(y \mid x) = \frac{1}{1-x} 1(0 < x < y < 1)$$

$$f(x, y) = \frac{1}{1-x} 1(0 < x < y < 1)$$

$$f(y) = \int_0^y \frac{1}{1-x} dx = [-\log(1-x)]_0^y = -\log(1-y)$$

3. (a) X and Y are independent, so their covariance is zero.
(b) $E(X^2 + Y^2) = \int_0^1 \int_0^1 (x^2 + y^2) dxdy = \int_0^1 x^2 dx + \int_0^1 y^2 dy = 2/3$

4. There was a problem with this question. It was intended as a delta method question, but the delta method requires the derivative to be nonzero, and so unfortunately this doesn't work. From a literal application of the delta method, $\sqrt{n}\overline{X}\overline{Y} \to_d N(0,0)$ which is however not a distribution at all. In fact, $n\overline{X}\overline{Y}$ converges to the product of two independent normals. I gave all students full credit on this question.

5. Let B and W denote the temperature in Baltimore and Washington, respectively. We have
$W \mid B \sim N(67 + 0.9(70-65), 1-0.9^2) = N(71.5, 0.19)$
So the probability is 50%.

6. (a) $U = X - Y$ and $V = X / Y$

$X = \dfrac{UV}{V-1}$ and $Y = \dfrac{U}{V-1}$

$$\frac{\partial X}{\partial U} = \frac{V}{V-1}, \frac{\partial X}{\partial V} = \frac{-U}{(V-1)^2}, \frac{\partial Y}{\partial U} = \frac{1}{V-1}, \frac{\partial Y}{\partial V} = \frac{-U}{(V-1)^2}$$

$$\begin{pmatrix} \partial X / \partial U & \partial X / \partial V \\ \partial Y / \partial U & \partial Y / \partial V \end{pmatrix} = \frac{1}{(V-1)^2} \begin{pmatrix} V(V-1) & -U \\ V-1 & -U \end{pmatrix} \Rightarrow |J| = \frac{|U|}{(V-1)^2}$$

So the probability density of U and V is

$$f_{U,V}(u,v) = \frac{|u|}{(v-1)^2}$$

with support $-1 < U < 1$ and $0 < V < \infty$.
(b) The density is clearly nonnegative. The required double integral is

$$\int_0^\infty \int_{-1}^1 f_{U,V}(u,v) dudv = \int_0^\infty \int_{-1}^1 \frac{|u|}{(v-1)^2} dudv = \int_{-1}^1 |u| du \int_0^\infty \frac{1}{(v-1)^2} dv = 2\int_0^1 udu \int_1^\infty \frac{1}{(v-1)^2} dv = 2*\frac{1}{2}*1 = 1$$

7. $P(|X-\lambda|\geq\lambda) = P(X\geq 2\lambda) + P(X=0)$
Hence $P(X\geq 2\lambda) \leq P(|X-\lambda|\geq\lambda)$
By Chebychev's inequality,
$$P(|X-\lambda|\geq\lambda) = P((X-\lambda)^2 < \lambda^2) \leq \frac{Var(X)}{\lambda^2} = \frac{1}{\lambda}$$
Hence $P(|X-\lambda)\geq 2\lambda) \leq 1/\lambda$.

8. $P(U(1) > N(1)) = \Sigma_{j=0}^{\infty} P(N(1)=j)P(U(1)>j)$
$$\therefore P(U(1) > N(1)) = e^{-1}*0.5 + e^{-1}*0.1587 + e^{-1}*\frac{1}{2}*0.0228 + e^{-1}*\frac{1}{6}*0.0013 = 0.2466$$
omitting negligible terms.