**Midterm Exam**

Fall 2009 Econ 180-636

Closed Book.
Formula Sheet Provided. Calculators OK.
Time Allowed: 2 Hours
All Questions Carry Equal Marks

1. Suppose that $Y$ has a probability density function

$$f_Y(y) = 1 - \frac{y}{2} \text{ for } 0 < y < C$$

where $C$ is a positive constant.
(a) Determine the value of $C$.
(b) Calculate $E(Y)$.
(c) Suppose $X = 1(Y < 1)$. Determine the distribution of $X$.
(d) Continuing with the definition of $X$ from part (c), determine $E(Y \mid X = 1)$.

2. Suppose that $X_1 \sim N(\mu_1, \sigma_1^2)$, $X_2 \sim N(\mu_2, \sigma_2^2)$ and $X = X_1$ with probability $p$ and $X = X_2$ otherwise.
(a) Find $E(X_1^3)$ (by Stein's Lemma or otherwise).
(b) Find $E(X^2)$.

3. In Baltimore, there are three kinds of days: rainy, nice and snowy. The transition matrix between these three kinds of weather is

$$\begin{pmatrix} 1/2 & 1/2 & 1/4 \\ 1/4 & 0 & 1/4 \\ 1/4 & 1/2 & 1/2 \end{pmatrix}$$

where the first column/row refers to rainy, the second refers to nice and the third refers to snowy. Find the steady state of the Markov chain.

4. Let $Y = \max(X_1, X_2)$ where $X_1 \sim \text{Exponential}(\beta_1)$ and $X_2 \sim \text{Exponential}(\beta_2)$ are independent random variables.
(a) Find the cdf and pdf of $Y$.
(b) Find $P(X_1 < X_2)$. Simplify as much as possible.

5. The height of individuals in a population is normally distributed with mean 170cm and variance 100 cm. The weight of individuals in this population is normally distributed with mean 80kg and variance 64 kg. The correlation between weight and height is 0.5. Find the distribution of height for an individual conditional on that person weighing 70kg.

6. Marie is getting married tomorrow, at an outdoor ceremony in the desert. In recent years, it has rained only 5 days each year. Alas, the weatherman has predicted rain for tomorrow. When it actually rains, the weatherman correctly forecasts rain 90% of the time. When it doesn't rain, he

incorrectly forecasts rain 10% of the time. What is the probability that it will rain on the day of Marie's wedding?

7. Suppose that the random variable $X$ has the pdf

$$f_X(x) = \frac{2}{\sqrt{2\pi}} e^{-x^2/2}$$

for $0 < x < \infty$.

(a) What is the variance of $X$ ?

(b) Let $Y = X^2$ (note that the support of $X$ is the positive real line only). Find the probability density of $Y$ .

8. (a) State and prove Chebyshev’s inequality.

(b) Newspaper articles are on average 1000 characters long with a standard deviation of 200 characters. Use Chebyshev's inequality to find a lower bound on the probability that a given article is between 600 and 1400 characters in length.

**Solutions**

1. (a) $\int_0^C (1-\frac{y}{2})dy = [y - \frac{y^2}{4}]_0^C = 1$

$\therefore C - \frac{C^2}{4} = 1$

$\therefore C = 2$

(b) $E(Y) = \int_0^2 y(1-\frac{y}{2})dy = \int_0^2 ydy - \frac{1}{2}\int_0^2 y^2 dy = [\frac{y^2}{2}]_0^2 - \frac{1}{2}[\frac{y^3}{3}]_0^2 = \frac{4}{2} - \frac{8}{6} = \frac{2}{3}$

(c) $P(X = 1) = P(Y < 1) = \int_0^1 (1-\frac{y}{2}) = \frac{3}{4}$

(d) $f(y \mid x = 1) = \frac{f(y)P(X = 1 \mid y)}{P(X = 1)} = \frac{(1 - y/2)1(y < 1)}{3/4}$

$\therefore E(Y \mid X = 1) = \int_0^1 y(1-\frac{y}{2})\frac{4}{3}dy = \frac{4}{9}$

2. (a) $0 = E((X_1 - \mu_1)^3) = E(X_1^3) - 3\mu_1 E(X_1^2) + 3\mu_1^2 E(X_1) - \mu_1^3$

$\therefore E(X_1^3) - 3\mu_1 E(X_1^2) + 3\mu_1^2 E(X_1) - \mu_1^3 = 0$

$\therefore E(X_1^3) - 3\mu_1(\mu_1^2 + \sigma_1^2) + 3\mu_1^3 - \mu_1^3 = 0$

$\therefore E(X_1^3) - 3\mu_1^3 - 3\mu_1\sigma_1^2 + 2\mu_1^3 = 0$

$\therefore E(X_1^3) - \mu_1^3 - 3\mu_1\sigma_1^2 = 0$

$\therefore E(X_1^3) = \mu_1^3 + 3\mu_1\sigma_1^2$

(b) $E(X^2) = p(\mu_1^2 + \sigma_1^2) + (1-p)(\mu_2^2 + \sigma_2^2)$

3. (0.4,0.2,0.4).

4. (a) The cdf is $(1 - e^{-\beta_1 y})(1 - e^{-\beta_2 y})$

The pdf is $(1 - e^{-\beta_1 y})\beta_2 e^{-\beta_2 y} + (1 - e^{-\beta_2 y})\beta_1 e^{-\beta_1 y}$

(b) The joint pdf is

$f(x_1, x_2) = \beta_1\beta_2 e^{-\beta_1 x_1} e^{-\beta_2 x_2}$

So $P(X_1 < X_2) = \int_0^\infty \int_0^{x_2} \beta_1\beta_2 e^{-\beta_1 x_1} e^{-\beta_2 x_2} dx_1 dx_2 = \beta_1\beta_2 \int_0^\infty \int_0^{x_2} e^{-\beta_1 x_1} dx_1 e^{-\beta_2 x_2} dx_2$

$\therefore P(X_1 < X_2) = \beta_1\beta_2 \int_0^\infty [\frac{1 - e^{-\beta_1 x_1}}{\beta_1}]_0^{x_2} e^{-\beta_2 x_2} dx_2 = \beta_1\beta_2 \int_0^\infty (\frac{1 - e^{-\beta_1 x_2}}{\beta_1}) e^{-\beta_2 x_2} dx_2$

$\therefore P(X_1 < X_2) = \beta_2\{\int_0^\infty e^{-\beta_2 x_2} dx_2 - \int_0^\infty e^{-(\beta_1 + \beta_2)x_2} dx_2\}$

$\therefore P(X_1 < X_2) = \beta_2\{[\frac{1 - e^{-\beta_2 x_2}}{\beta_2}]_0^\infty - [\frac{1 - e^{-(\beta_1 + \beta_2)x_2}}{\beta_1 + \beta_2}]_0^\infty\}$

$$\therefore P(X_1 < X_2) = \beta_2\{\frac{1}{\beta_2} - \frac{1}{\beta_1+\beta_2}\} = 1 - \frac{\beta_2}{\beta_1+\beta_2} = \frac{\beta_1}{\beta_1+\beta_2}$$

5. From the usual formula, the conditional distribution is N(163.75,75).

6. $$P(Rains \mid Forecast) = \frac{P(Forecast \mid Rains)P(Rains)}{P(Forecast)} = \frac{0.9\frac{5}{365}}{0.9\frac{5}{365} + 0.1\frac{360}{365}} = \frac{0.9*5}{0.9*5+0.1*360}$$

which is 1/9.

7. (a) $$E(X) = \int x \frac{2}{\sqrt{2\pi}} e^{-x^2/2} dx = \sqrt{\frac{2}{\pi}}$$

$$E(X^2) = \int_0^\infty x^2 \frac{2}{\sqrt{2\pi}} e^{-x^2/2} dx = 1$$

$$Var(X) = 1 - \frac{2}{\pi}$$

(b) $$f_Y(y) = \frac{2}{\sqrt{2\pi}} e^{-y/2} . \frac{1}{2\sqrt{y}} = \frac{1}{\sqrt{2\pi}} y^{-1/2} e^{-y/2}$$

8. (a) $$P(X \geq c) \leq \frac{E(X)}{c}$$

Proof: $E(X) = \int_0^\infty xf(x)dx = \int_0^c xf(x)dx + \int_c^\infty xf(x)dx$

$\therefore E(X) \geq \int_c^\infty xf(x)dx$

$\geq \int_c^\infty cf(x)dx = c\int_c^\infty f(x)dx = cP(X \geq c)$

$\therefore E(X) \geq cP(X \geq c) \Rightarrow P(X \geq c) \leq E(X)/c$

(b) $$P((Y-\mu)^2 \geq c\sigma^2) \leq \frac{1}{c}$$

$$\therefore P((Y-1,000)^2 \geq 40,000c) \leq \frac{1}{c}$$

$\therefore P((Y-1,000)^2 \geq 160,000) \leq 0.25$

$\therefore P(600 \leq Y \leq 1,400) \geq 0.75$