Final Exam

Fall 2010 Econ 180-636

Closed Book.
Formula Sheet Provided. Non-Programmable Calculators OK.
Time Allowed: 3 Hours.
All Questions Carry Equal Marks.

1. (a) What is the definition of a **consistent** test?
(b) What is the definition of a **uniformly most powerful** test?
(c) Construct an example to show that two random variables may be uncorrelated and yet not independent.

2. Let $X_1, X_2, ...X_n$ be iid uniform between 0 and θ, and let $Y_n = \max(X_1, X_2, ...X_n)$. We want to test the null hypothesis that $\theta = 1/2$ against the alternative that $\theta > 1/2$. Suppose that we decide to reject the null hypothesis whenever $Y_n > c$ for some cutoff c.
(a) What choice of c will ensure that the size of the test is 5 percent?
(b) Find the power of this test against the alternative that $\theta = 0.6$.

3. Let $X_1, X_2, ...X_n$ be iid random variables each of which is $N(0,1)$ with probability p and $N(1,1)$ otherwise. You observe a sample of size $n = 2$, and the observations are as follows: $X_1 = 0.5$ and $X_2 = 1$. What is the maximum likelihood estimate of p?

4. Consider the following data that are assumed to be iid (but not necessarily normal).

Observation	Value
1	-4
2	-17
3	1
4	3
5	-11
6	12
7	12
8	0
9	3
10	2
11	-2
12	7

(a) Estimate the mean of these data.
(b) Form a 95 percent (asymptotic) confidence interval for the mean of these data.

5. Suppose that $X_1, X_2, ...X_n$ are iid Poisson random variables with parameter λ.
(a) Find the Cramer-Rao bound for an estimator of λ.
(b) In a sample of size 30, you find that the average value $\bar{X} = 3$. Do a Wald test of the hypothesis that $\lambda = 1$.

6. Consider a three-state Markov chain with transition matrix $\begin{pmatrix} 0.1 & 0.9 & 0.1 \\ 0.2 & 0.1 & 0.8 \\ 0.7 & 0 & 0.1 \end{pmatrix}$. The states are indexed 0, 1 and 2. Assume that at time zero, the chain is in state 0 with probability 0.3, state 1 with probability 0.4, and state 2 with probability 0.3.
(a) Find the *joint* probability that the chain is in state 0 at time 0, state 1 at time 1 and state 2 at time 2.
(b) Find the steady state of the Markov chain.

7. Suppose that I observe a sample of size three of i.i.d. random variables with a mean of μ. The three observations are 2, 5 and 11, so that the sample average is 6. You want to form bootstrap confidence intervals for μ using the other percentile and percentile confidence intervals, in the terminology of Hall. You do this using trillions of bootstrap samples (an arbitrarily large number). You want the confidence interval to have coverage of $100 * \frac{25}{27} = 92.59\%$. What are the bootstrap confidence intervals? [Hint: you do not have time on the exam to actually simulate the trillions of bootstrap samples. Please find a shortcut.]

8. Suppose that there are two parameters θ_1 and θ_2. These can be estimated by estimators $\hat{\theta}_1$ and $\hat{\theta}_2$ that are mutually independent, and that $n^{1/2}(\hat{\theta}_1 - \theta_1) \to_d N(0,1)$ and $n^{1/2}(\hat{\theta}_2 - \theta_2) \to_d N(0,1)$
(a) What is the asymptotic distribution of $\frac{\hat{\theta}_1}{1-\hat{\theta}_2}$?
(b) Suppose that $n = 30$, $\hat{\theta}_1 = 1$ and $\hat{\theta}_2 = 0.9$. Using the result in part (a), find the asymptotic p-value of a test of the hypothesis that $\theta^* = 5$, where $\theta^* = \frac{\theta_1}{1-\theta_2}$.

9. Suppose that $X_1, X_2, ...X_n$ are iid uniform between 0 and θ. A Bayesian statistician has an (improper) prior for θ that is proportional to $1/\theta$ for $0 \le \theta \le 1$ (and zero otherwise). Find the posterior mean of θ.

10. X has a binomial distribution with parameters N = 1 and p = 1/2. Y, which is independent of X, has a normal distribution with mean μ and variance 1. Consider the estimator for μ of the form $W_1 = Y + 2X - 1$.
(a) Is W_1 unbiased?
(b) What is the variance of W_1?
(c) Consider the estimator $W_2 = E(W_1 | Y)$. Is W_2 unbiased? How does its variance compare to that of W_1?

Solutions

1. (a) One that rejects the null against a fixed alternative with probability one in the limit as the sample size gets large.
(b) A test that has greatest power against all alternatives, for tests with the required size.
(c)

	X=-1	X=0	X=1
Y=0	0	1/3	0
Y=1	1/3	0	1/3

2. (a) The cdf of Y_n is $(y/\theta)^n$. To get a 5 percent test,

$$1-(\frac{c}{0.5})^n = 0.05$$

$$(\frac{c}{0.5})^n = 0.95$$

$$c = 0.5*0.95^{1/n}$$

(b) $1-(\frac{0.5*0.95^{1/n}}{0.6})^n$

3. The likelihood is

$$[p\frac{1}{(2\pi)^{1/2}}\exp(-\frac{0.5^2}{2})+(1-p)\frac{1}{(2\pi)^{1/2}}\exp(-\frac{(0.5-1)^2}{2})][p\frac{1}{(2\pi)^{1/2}}\exp(-\frac{1^2}{2})+(1-p)\frac{1}{(2\pi)^{1/2}}\exp(-\frac{(1-1)^2}{2})]$$

$$=\frac{1}{2\pi}[p\exp(-\frac{1}{8})+(1-p)\exp(-\frac{1}{8})][p\exp(-\frac{1}{2})+(1-p)]$$

$$=\frac{1}{2\pi}\exp(-\frac{1}{8})[p\exp(-\frac{1}{2})+1-p]$$

which is maximized at $p=0$.

4. (a) 0.5

(b) s=8.46, so the asymptotic confidence interval is $0.5\pm\frac{1.96*8.46}{\sqrt{12}}=0.5\pm 4.79$ which is from -4.29 to 5.29.

5. (a) λ/n

(b) $W = 30*(3-1)*\frac{1}{3}*(3-1)=40$, which means that you reject.

6. (a) 0
(b) The equations are

$$0.1\pi_1+0.9\pi_2+0.1\pi_3=\pi_1$$

$$0.2\pi_1+0.1\pi_2+0.8\pi_3=\pi_2$$

$\pi_1 + \pi_2 + \pi_3 = 1$

Solving these equations gives $\pi_1 = 81/218, \ \pi_2 = 74/218, \pi_3 = 63/218$.

7. Other percentile is 2 to 11. Percentile is 1 to 10.

8. $n^{1/2}(\frac{\hat{\theta}_1}{1-\hat{\theta}_2} - \frac{\theta_1}{1-\theta_2}) \to_d N(0, \frac{1}{(1-\theta_2)^2} + \frac{\theta_1^2}{(1-\theta_2)^4})$

The test statistic is $\frac{n^{1/2}(10-5)}{\sqrt{10100}} = 0.2725$. The p-value is 0.7872.

9. The posterior density is $\frac{k}{\theta^{n+1}} 1(0 \le \max(X_i) \le \theta \le 1)$ for some constant k. Since this must integrate to one, $\int_{max(X_i)}^{1} \frac{k}{\theta^{n+1}} d\theta = 1$ and so $k = n\frac{\max(X_i)^n}{1-\max(X_i)^n}$.

The posterior mean is

$$\int_{max(X_i)}^{1} \frac{k\theta}{\theta^{n+1}} d\theta = \int_{max(X_i)}^{1} \frac{k}{\theta^{n}} d\theta = [-\frac{k}{(n-1)\theta^{n-1}}]_{\max(X_i)}^{1} = \frac{n}{(n-1)}[\frac{\max(X_i)^n}{1-\max(X_i)^n}][\frac{1}{\max(X_i)^{n-1}} - 1]$$

10. (a) $E(W_1) = \mu + 1 - 1 = \mu$. Yes it's unbiased.

(b) $1 + 4 * \frac{1}{2} * \frac{1}{2} = 2$

(c) $W_2 = Y$ and it has a variance of 1, which is smaller than the variance of W_1.