Final Exam

Fall 2008 Econ 180-636

Closed Book.
Formula Sheet Provided. Non-Programmable Calculators OK.
Time Allowed: 3 Hours.
All Questions Carry Equal Marks.

1. The CEO of a large construction company claims that the average wage of all its unskilled workers is \$20 per hour. The local Labor Union decides to test this claim against a two-sided alternative. The population is assumed to be normal with a population standard deviation of \$5.2 per hour. The wages for a random sample of 36 individuals are collected to test this claim.

(a) Determine the values of the sample mean wages for which you would reject the null hypothesis if a 5 percent level of significance were used.

(b) Determine the power of the test procedure set up in part (a) if the population mean wage of unskilled workers is in fact \$18 per hour.

2. (a) Define the p-value of a test.
(b) Define a sufficient statistic.
(c) Suppose that $X_1, X_2, ... X_n$ are iid $N(0, \sigma^2)$ where it is known that the mean is zero. Find a scalar sufficient statistic for σ^2.

3. Suppose that $X_1, X_2, ... X_{20}$ is an iid sample of 20 random variables from a Poisson distribution with probability density

$$P(X = x) = \frac{\lambda^x e^{-\lambda}}{x!}$$

The sample average of the 20 random variables is 5.

(a) What is the log-likelihood function?

(b) What is the maximum likelihood estimator of λ?

(c) What is the asymptotic variance of the estimator in (b)? You may find it helpful to note that $E(X) = \lambda$.

(d) Use a Wald test to test the hypothesis that $\lambda = 6$ at the 5 percent level.

4. A coin when tossed has a probability *p* of coming up heads. A Bayesian statistician has a prior for *p* that is
P(p=0)=0.2
P(p=0.5)=0.6
P(p=1)=0.2
The coin is tossed 5 times, and it comes up heads each time. What is the posterior mean estimate of *p*?

5. Suppose that $X \sim N(6,2)$, $\varepsilon \sim N(0,3)$ and X and ε are independent. Let $Y = 4 + X + \varepsilon$.

(a) What is the joint distribution of the 2x1 vector $(X, Y)'$?

(b) What is the distribution of Y conditional on $X = 10$?

6. X_1 and X_2 are independent normally distributed random variables with mean μ and variances σ_1^2 and σ_2^2, respectively. The variances are known and we are interested in estimating μ. Consider estimators of the form $W = \lambda X_1 + \gamma X_2$. For what values of λ and γ could W be a minimum variance unbiased estimator?

7. An urn contains 10 balls, of which q are black and the rest are red. Two balls are selected at random from (with replacement) from the urn. One is black, one is red. Let $L(q)$ denote the likelihood function.

(a) What is $L(5)$?

(b) What is $L(10)$?

(c) What is the maximum-likelihood estimator of q?

8. Let $X_1, X_2, ...X_n$ be iid with a density function $f(x) = \theta^{-1} x^{\frac{1-\theta}{\theta}}$ for $0 < x < 1$ and $\theta > 0$.

(a) Find the pdf of $Y_i = -\ln(X_i)$.

(b) Find $E(Y_i)$.

(c) What is the Cramer-Rao lower bound of any estimator of θ? Try to write it as a function of θ and n alone.

9. Let the random variable X have a probability density function $f(x) = \frac{1}{\lambda}\exp(-x/\lambda)$ for $x > 0$. Let X denote a single random variable from this distribution. Use the Neyman-Pearson Lemma to find the optimal rejection region for a test of the hypothesis that $\lambda = 2$ against the alternative that $\lambda = 4$ with a size of 10 percent.

10. Someone has shown that the vector of estimators $(\hat{\rho}, \hat{\sigma}^2)'$ of the parameter $(\rho, \sigma^2)'$ with $-1 < \rho < 1$ and $\sigma^2 > 0$ satisfies

$$\sqrt{n}\left(\begin{pmatrix} \hat{\rho} \\ \hat{\sigma}^2 \end{pmatrix} - \begin{pmatrix} \rho \\ \sigma^2 \end{pmatrix}\right) \to_d N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1/(1-\rho^2) & 0 \\ 0 & \kappa^2 \end{pmatrix}\right)$$

Derive the limiting distribution of $\hat{\sigma}^2 \hat{\rho}^4$ by the delta method, assuming that $\rho \neq 0$.

Final Solutions

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1. (a) The null would be rejected for wages less than 18.30 or more than 21.70.
(b) 63.7 percent.

2. (a) The p-value of a test is the significance level at which we are just on the margin of accepting or rejecting the hypothesis.
(b) A statistic $T(X)$ is a sufficient statistic for a parameter θ if the distribution of X given T does not depend on θ.
(c) The joint pdf is

$$(2\pi\sigma^2)^{-n/2}\exp(-\frac{\Sigma_{i=1}^n X_i^2}{2\sigma^2})$$

which means that $\Sigma_{i=1}^n X_i^2$ is a sufficient statistic.

3. (a) $l(\lambda)=\Sigma_{i=1}^n X_i\log(\lambda)-n\lambda-\Sigma_{i=1}^n\log(X_i!)$
(b) FOC: $\frac{1}{\lambda}\Sigma_{i-1}^n X_i-n=0\Rightarrow\lambda=\bar{X}=5$
(c) $\log f(\lambda)=-x\log(\lambda)-\lambda-\log(x!)$

$$\therefore\frac{\partial\log f(\lambda)}{\partial\lambda}=-\frac{x}{\lambda}-1$$

$$\therefore\frac{\partial^2\log f(\lambda)}{\partial\lambda^2}=\frac{x}{\lambda^2}$$

$$\therefore E(\frac{\partial^2\log f(\lambda)}{\partial\lambda^2})=\frac{\lambda}{\lambda^2}=\frac{1}{\lambda}$$

So $\sqrt{n}(\hat{\lambda}-\lambda)\to_d N(0,\lambda)$

(d) $W=\frac{20(6-5)^2}{5}=4$

The hypothesis is rejected.

4. P(p=0|5 heads)=0
P(p=0.5|5 heads)=P(5 heads|p=0.5)*P(p=0.5)/P(5 heads)=0.5^5*0.6/p(5 heads)
P(p=1|5 heads)=P(5 heads|p=1)*P(p=1)/P(5 heads)=1^5*0.2/p(5 heads)

So P(p=0.5|5 heads)= $\frac{0.5^5*0.6}{0.5^5*0.6+1*0.2}=\frac{3}{35}$

P(p=1|5 heads)= $\frac{1*0.2}{0.5^5*0.6+1*0.2}=\frac{32}{35}$

So the posterior mean is $\frac{3}{35}\frac{1}{2}+\frac{32}{35}1=\frac{67}{70}$.

5. (a) $\begin{pmatrix} X \\ Y \end{pmatrix} \sim N(\begin{pmatrix} 6 \\ 10 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix})$

(b) The correlation is $2/\sqrt{10} = 1/\sqrt{2.5}$. From the formula for a conditional normal,

$$Y \mid X = 10 \sim N(10 + \frac{1}{\sqrt{2.5}}\sqrt{\frac{5}{2}} * (10-6), 5*(1-\frac{1}{2.5})$$

$\therefore Y \mid X = 10 \sim N(14, 3)$

6. The estimator must be unbiased.

$E(W) = \lambda\mu + \gamma\mu = (\lambda + \gamma)\mu$

So $\gamma = 1 - \lambda$

The variance of the estimator is

$Var(W) = \lambda^2\sigma_1^2 + \gamma^2\sigma_2^2 = \lambda^2\sigma_1^2 + (1-\lambda)^2\sigma_2^2$

The FOC for minimizing this is

$$2\lambda\sigma_1^2 - 2(1-\lambda)\sigma_2^2 = 0 \Rightarrow \lambda\sigma_1^2 = (1-\lambda)\sigma_2^2 \Rightarrow \lambda(\sigma_1^2 + \sigma_2^2) = \sigma_2^2 \Rightarrow \lambda = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}$$

Correspondingly $\gamma = \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2}$.

7 (a) $L(5) = 1/2$

(b) $L(10) = 0$

(c) 5.

8. (a) $x = e^{-y}$

$f(y) = \theta^{-1}(e^{-y})^{\frac{1-\theta}{\theta}} e^{-y} = \theta^{-1}e^{-y/\theta}$

for $0 \le y \le \infty$.

(b) $E(Y) = \theta$, using integration by parts.

(c) $\log f(x) = -\log(\theta) + \frac{1-\theta}{\theta}\log(x)$

$$\frac{\partial \log f(x)}{\partial\theta} = -\frac{1}{\theta} - \frac{\log(x)}{\theta^2}$$

$$\frac{\partial^2 \log f(x)}{\partial\theta^2} = \frac{1}{\theta^2} + 2\frac{\log(x)}{\theta^3}$$

Hence $E(\frac{\partial^2 \log f(x)}{\partial\theta^2}) = \frac{1}{\theta^2} - 2\frac{\theta}{\theta^3} = -\frac{1}{\theta^2}$

and so the CRLB is θ^2 / n.

9. According to the Neyman-Pearson lemma, the optimal test rejects if

$$\frac{4^{-1}\exp(-X/4)}{2^{-1}\exp(-X/2)} > c$$

i.e. if

$$\frac{\exp(-X/4)}{2\exp(-X/2)} > c$$

i.e. if

$$\exp(X/4) > 2c$$

i.e. if

$$X > 4\log(c/2)$$

The cdf of X is $F(x) = 1 - e^{-x/\lambda}$. If $\lambda = 2$

$$P(X \le x) = 1 - e^{-x/2}$$

$$\therefore P(X > x) = 0.1 \Rightarrow 1 - e^{-x/2} = 0.9 \Rightarrow e^{-x/2} = 0.1 \Rightarrow x = -2 * \log(0.1) = 4.6052$$

So the hypothesis is rejected if X>4.6052.

10. By the delta method, $\sqrt{n}(\hat{\sigma}^2\hat{\rho}^4 - \sigma^2\rho^4) \to_d N(0, \frac{16\sigma^4\rho^6}{1-\rho^2} + \rho^8\kappa^{2})$