Final Exam

Fall 2017 Econ 180-636

Closed Book.
Formula Sheet Provided. Non-Programmable Calculators OK.
Time Allowed: 3 Hours.
All Questions Carry Equal Marks.

1. Suppose that you observe a sample of 20 observations that are iid draws from a normal distribution with mean μ and variance σ^2. In your sample, $\bar{X} = 3$ and $\Sigma_{i=1}^{n}(X_i - \bar{X})^2 = 38$. Form a 90 percent exact confidence interval for μ.

2. Suppose that $X \sim N(6,2)$, $\varepsilon \sim N(0,3)$ and X and ε are independent. Let $Y = 4 + X + \varepsilon$.
   (a) What is the joint distribution of the 2x1 vector $(X,Y)'$?
   (b) What is the distribution of Y conditional on $X = 10$?

3. Consider an iid random sample $X_1, X_2, ... X_n$ from the density $f(x) = \frac{k}{\sqrt{\theta}} 5^{-x^2/\theta}$ with support $-\infty < x < \infty$ where $\theta > 0$ and $k = \sqrt{\frac{\ln(5)}{\pi}}$.
   (a) Find a scalar sufficient statistic for θ.
   (b) Derive the maximum likelihood estimate of θ.
   (c) Find the Cramer-Rao lower bound.

4. Consider an iid random sample $X_1, X_2, ... X_n$ from the density $f(x) = \frac{1}{\lambda}\exp(-x/\lambda)$ with support $x > 0$. Suppose that the sample size is $n = 20$, $\Sigma_{i=1}^{n} X_i = 95$ and $\Sigma_{i=1}^{n} X_i^2 = 500$.
   (a) Use a likelihood ratio test to test the hypothesis that $\lambda = 4$ at the 5 percent level.
   (b) Test the same hypothesis using a Lagrange multiplier test.

5. An urn contains 10 balls, of which q are black and the rest are red. Two balls are selected at random from (with replacement) from the urn. One is black, one is red. Let $L(q)$ denote the likelihood function.
   (a) What is $L(5)$?
   (b) What is $L(10)$?
   (c) Write down a formula for the likelihood as a function of q.
   (d) Hence determine the maximum-likelihood estimator of q.

6. Suppose that $X_1, X_2 ... X_{10}$ are iid normal data with mean zero (known) and variance σ^2. What is the rejection region for a Neyman-Pearson test of the hypothesis that $\sigma^2 = 1$ against the alternative that $\sigma^2 = 2$? Please give a complete answer….reject if this test statistic is above/below some specific number.

7. Suppose that $X_1, X_2 .... X_n$ are iid normal with mean μ and variance σ^2, so that the probability density function (pdf) of each observation is

$$f(x) = \frac{1}{\sigma}\phi(\frac{x-\mu}{\sigma})$$

where $\phi(z) = \frac{1}{\sqrt{2\pi}}\exp(-z^2/2)$ is the standard normal density. You estimate the pdf as

$$\hat{f}(x) = \frac{1}{\hat{\sigma}}\phi(\frac{x-\hat{\mu}}{\hat{\sigma}})$$

where $\hat{\mu}$ and $\hat{\sigma}^2$ are the maximum likelihood estimators of μ and variance σ^2, respectively. The true values of μ and σ^2 are 0 and 1, respectively. By the delta method, $\sqrt{n}(\hat{f}(0) - f(0)) \to_d N(0,W)$. Find an expression for W and simplify as much as possible.

8. Suppose that X is binomial with parameters N=3 and p. Suppose that we have a prior for p that is discrete:
P(p=0)=0.25
P(p=0.5)=0.5
P(p=1)=0.25
(a) What is the posterior mean of p if we observe X=1?
(b) What is the posterior mean of p if we observe X=0?

9. Let X and Y be random variables with a joint density:

$$f(x, y) = e^{-(x+y)}$$

for $x > 0$ and $y > 0$ and zero otherwise. Let $Z = X - Y$.
(a) Find the joint density of Z and X. Be careful to state the region over which the joint density is defined.
(b) Find the marginal density of X
(c) Hence or otherwise, find the expectation of Z given that $X = 1$.

10. Let $X(t)$ be a Poisson process with intensity λ. Find the probability that $X(t)$ is odd (1,3,5,….) as a function of t.

Solutions

1. $s^2 = \frac{38}{19} = 2$

An exact 90 percent confidence interval is $3 \pm \frac{1.729 * \sqrt{2}}{\sqrt{20}}$ where 1.729 is the critical value for a t on 19 degrees of freedom. This is from 2.45 to 3.55.

2. (a) $\begin{pmatrix} X \\ Y \end{pmatrix} \sim N(\begin{pmatrix} 6 \\ 10 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix})$

(b) The correlation is $2/\sqrt{10} = 1/\sqrt{2.5}$. From the formula for a conditional normal,

$$Y \mid X = 10 \sim N(10 + \frac{1}{\sqrt{2.5}}\sqrt{\frac{5}{2}} * (10-6), 5*(1-\frac{1}{2.5})$$

$$\therefore Y \mid X = 10 \sim N(14, 3)$$

3. (a) $f(x_1, x_2, \ldots x_n) = (\frac{k}{\sqrt{\theta}})^n 5^{-\frac{\Sigma X_i^2}{\theta}} = g(t(x), \theta)h(x)$. So $\Sigma_{i=1}^n X_i^2$ is a scalar sufficient statistic.

(b) $\log(f(x)) = \log(k) - \frac{1}{2}\log(\theta) - \frac{x^2}{\theta}\log(5)$

$$\log(f(x_1, \ldots x_n)) = n\log(k) - \frac{n}{2}\log(\theta) - \frac{\log(5)}{\theta}\Sigma_{i-1}^n X_i^2$$

$$\frac{d\log(f(x_1, \ldots x_n))}{d\theta} = -\frac{n}{2\theta} + \frac{\log(5)}{\theta^2}\Sigma_{i-1}^n X_i^2 = \frac{n}{2\theta^2}(\frac{2\log(5)}{n}\Sigma_{i-1}^n X_i^2 - \theta) = a(\theta)(W(x) - \theta)$$

So $\frac{2\log(5)}{n}\Sigma_{i-1}^n X_i^2$ is the MVUE and the MLE

(c) $\frac{d\log(f(x))}{d\theta} = -\frac{1}{2\theta} + \frac{x^2}{\theta^2}\log(5) = \frac{1}{2\theta^2}(2x^2\log(5) - \theta)$

$$\frac{d^2\log(f(x))}{d\theta^2} = \frac{1}{2\theta^2} - 2\frac{x^2}{\theta^3}\log(5)$$

$$E(X^2) = \int x^2 \frac{k}{\sqrt{\theta}}\exp(-\frac{x^2}{\theta/\ln(5)})$$

$$E(X^2) = \int x^2 \frac{k}{\sqrt{\theta}}\exp(-\frac{x^2}{2\sigma^2}) \text{ where } \sigma^2 = \frac{\theta}{2\ln(5)}$$

$$E(X^2) = \int x^2 \frac{\sqrt{\ln(5)/\pi}}{\sqrt{2\ln(5)\sigma^2}} \exp(-\frac{x^2}{2\sigma^2}) = \int x^2 \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{x^2}{2\sigma^2}) = \sigma^2$$

$$\therefore E(X^2) = \frac{\theta}{2\ln(5)}$$

$$\therefore E\{\frac{d^2 \log(f(x))}{d\theta^2}\} = \frac{1}{2\theta^2} - 2\frac{1}{\theta^3}\log(5)\frac{\theta}{2\log(5)} = \frac{1}{2\theta^2} - \frac{1}{\theta^2} = -\frac{1}{2\theta^2}$$

So the CRLB is $2\theta^2 / n$.

4. (a) The log likelihood is $l(\lambda) = -n\log(\lambda) - \frac{1}{\lambda}\Sigma_{i=1}^{n} x_i$.

$l(4) = -51.48$, $l(4.75) = -51.12$. So the LR statistic is 0.72 and the hypothesis is accepted.

(b) $l'(\lambda) = -\frac{n}{\lambda} + \frac{\Sigma_{i=1}^{n} X_i}{\lambda^2} = -\frac{20}{4} + \frac{95}{16} = \frac{15}{16}$

$$\log(f(x)) = -\log(\lambda) - \frac{x}{\lambda}$$

$$\frac{d\log(f(x))}{d\lambda} = -\frac{1}{\lambda} + \frac{x}{\lambda^2}$$

$$\frac{d^2\log(f(x))}{d\lambda^2} = \frac{1}{\lambda^2} - 2\frac{x}{\lambda^3}$$

$E(X) = \lambda$ (because it is exponential)

$$\therefore E\{\frac{d^2\log(f(x))}{d\lambda^2}\} = \frac{1}{\lambda^2} - 2\frac{\lambda}{\lambda^3} = -\frac{1}{\lambda^2}$$

$$\therefore I = 1/\lambda^2 = 1/16$$

$LM = \frac{1}{20} * (\frac{15}{16})^2 * 16 = 0.70$ and again the hypothesis is accepted.

5.(a) $L(5) = 1/2$

(b) $L(10) = 0$

(c) The likelihood function is $L(q) = 2(\frac{q}{10})(1 - \frac{q}{10}) = 0.02(10q - q^2)$.

(d) The parameter q can only take on integer values. Still, maximizing the likelihood with respect to q gives a maximum at exactly 5, so the MLE is 5.

6. The rejection region is

$$\{x:\frac{\Pi_{i=1}^{n}(2\pi)^{-1/2}\exp(-\frac{X_i^2}{4})}{\Pi_{i=1}^{n}(2\pi)^{-1/2}\exp(-\frac{X_i^2}{2})}\geq k_\alpha\}=\{x:\frac{\exp(-\Sigma_{i=1}^{n}\frac{X_i^2}{4})}{\exp(\Sigma_{i=1}^{n}-\frac{X_i^2}{2})}\geq k_\alpha\}=\{x:\exp(\Sigma_{i=1}^{n}\frac{X_i^2}{4})\geq k_\alpha\}=\{x:\Sigma_{i=1}^{n}X_i^2\geq k'_\alpha\}$$

If $\sigma^2=1$, $\Sigma_{i=1}^{n}X_i^2$ is $\chi^2(n)$ and in this case $n=10$. So I will reject if $\Sigma_{i=1}^{n}X_i^2>18.3$

7. The vector of derivatives of $f(x)$ with respect to (μ,σ^2) is

$$\begin{pmatrix} -\frac{1}{\sigma^2}\phi'(\frac{x-\mu}{\sigma}) \\ -\frac{1}{2}\frac{x-\mu}{\sigma^4}\phi'(\frac{x-\mu}{\sigma})-\frac{1}{2\sigma^3}\phi(\frac{x-\mu}{\sigma}) \end{pmatrix}=\begin{pmatrix} -\phi'(x) \\ -\frac{x}{2}\phi'(x)-\frac{1}{2}\phi(x) \end{pmatrix}$$

So $W=\begin{pmatrix} -\phi'(0) & -\frac{\phi(0)}{2} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} -\phi'(0) \\ -\frac{\phi(0)}{2} \end{pmatrix}=\phi'(0)^2+\frac{\phi(0)^2}{2}$

But $\phi'(0)=0$ and $\phi(0)=\frac{1}{\sqrt{2\pi}}$. Hence $W=\frac{1}{4\pi}$.

8. (a) $P(X=1|\,p=0)=0$

$P(X=1|\,p=0.5)=3*0.5*0.5^2=\frac{3}{8}$

$P(X=1|\,p=1)=0$

$\therefore P(X=1)=\frac{3}{8}\frac{1}{2}=\frac{3}{16}$

So $P(p=0\,|\,X=1)=0$, $P(p=0.5\,|\,X=1)=1$ and $P(p=0\,|\,X=1)=0$. The posterior mean is 0.5.

(b) $P(X=0\,|\,p=0)=1$

$P(X=0\,|\,p=0.5)=1/8$

$P(X=0\,|\,p=1)=0$

$\therefore P(X=0)=\frac{1}{4}+\frac{1}{8}*\frac{1}{2}=\frac{5}{16}$

So $P(p=0\,|\,X=0)=\frac{1*(1/4)}{5/16}=\frac{4}{5}$, $P(p=0.5\,|\,X=0)=\frac{(1/8)*(1/2)}{5/16}=\frac{1}{5}$ and

$P(p=1|\,X=0)=0$. The posterior mean is 0.1.

9. (a) The inverse function is $X=X$ and $Y=X-Z$, so from the multivariate transformation formula:

$$f(x,z)=e^{-(2x-z)}\left|\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right|=e^{-(2x-z)}$$

The support for X is $x > 0$ and the support for z is $-\infty$ to X .
(b) The marginal density of X is $f(x) = \int_{-\infty}^{\infty} e^{z-2x} dz = [e^{z-2x}]_{-\infty}^{\infty} = e^{-x}$.

(c) The conditional density of Z given X is $f(z \mid x) = \dfrac{f(x,z)}{f(z)} = \dfrac{e^{z-2x}}{e^{-x}} = e^{z-x}$.

So the conditional expectation is $\int_{-\infty}^{1} z e^{z-1} dz = [(z-1)e^{z-1}]_{-\infty}^{1} = 0$

10.From the Poisson pdf,

$$P(X(t) = 1,3,5,...) = \frac{e^{-\lambda t}(\lambda t)}{1!} + \frac{e^{-\lambda t}(\lambda t)^3}{3!} + \frac{e^{-\lambda t}(\lambda t)^5}{5!}...$$

$$= e^{-\lambda t}\frac{1}{2}[\Sigma_{k=0}^{\infty}\frac{(\lambda t)^k}{k!} - \Sigma_{k=0}^{\infty}\frac{(-\lambda t)^k}{k!}]$$

$$= \frac{e^{-\lambda t}}{2}\{e^{\lambda t} - e^{-\lambda t}\} = \frac{1-e^{-2\lambda t}}{2}$$