Final Exam

Fall 2011 Econ 180-636

Closed Book.
Formula Sheet Provided. Non-Programmable Calculators OK.
Time Allowed: 3 Hours.
All Questions Carry Equal Marks.

1. State (without proof) the weak law of large numbers and the central limit theorem for independently and identically distributed (iid) random variables.

2. Let $X_1, X_2, \ldots X_n$ be iid $N(\mu, 1)$ random variables. Let $\theta = e^{\mu}$.
   a. What is the maximum likelihood estimator (MLE) of θ?
   b. What is the asymptotic distribution of the MLE of θ?

3. Suppose that the joint probability density function of X and Y is

$$f(x, y) = \frac{3x + y}{7}$$

for $0 < x < 2$ and $0 < y < 1$
   a. Find the marginal density of X.
   b. Find the conditional density of Y given X.

4. Suppose that X has a binomial distribution with parameters $N = 1$ and $p = 1/2$. *Y*, which is independent of *X*, has a normal distribution with mean μ and variance 1. Consider the estimator for μ of the form $W_1 = Y + 2X - 1$
   a. Is W_1 unbiased?
   b. What is the variance of W_1?
   c. Consider the estimator $W_2 = E(W_1 \mid Y)$. Is W_2 unbiased? How does its variance compare to that of W_1?

5. Suppose that $X_1, X_2, \ldots X_n$ are iid normal with mean μ and variance 1 (known).
   a. Consider a sample of size $n = 36$. Find the acceptance region of a two-sided test of the hypothesis that $\mu = 10$ (the set of values of $\overline{X}$ for which the test accepts).
   b. Compute the power of the test in (a) against the alternative that $\mu = 10.5$.

6. Every quarter, the economy is either in an expansion or a recession and switches between these two states following a Markov Chain. If the economy is in an expansion this quarter, the probability that it will be

in an expansion next quarter is 90%. If the economy is in a recession this quarter, the probability that it will be in a recession next quarter is 60%.

a. Write the transition matrix for this Markov Chain.
b. In 2012Q2, the economy is in an expansion. What is the probability that it will in an expansion in 2012Q4?
c. What is the steady state distribution of this Markov chain?

7. A random variable X is uniform between 0 and θ. Your prior for θ is in turn uniform between 0 and 1. You observe two draws of X, which are 0.5 and 0.7, respectively. Find the posterior density of θ.

8. Suppose that X_1 and X_2 are two iid random variables with mean μ and (known) variance 1. The sample values of X_1 and X_2 are -1 and +2, respectively. You estimate μ by the sample mean which is 0.5.
   a. Assume that X_1 and X_2 are normally distributed. Form a 95% confidence interval for μ.
   b. Use the "other percentile" bootstrap to form a 95% confidence interval for μ.
   c. Use the "percentile" bootstrap to form a 95% confidence interval for μ.

9. Suppose that X is a random variable with a probability density

$$f(x) = \frac{x}{b} e^{(-\frac{x^2}{2b})}, 0 \leq x < \infty$$

You observe three independent draws from this random variable, which are $X_1 = 1$, $X_2 = 4$ and $X_3 = 2$. Note that $E(X^2) = 2b$.
   a. What is the maximum likelihood estimator of b?
   b. What is it's asymptotic distribution?

10. Suppose that X is a chi-squared random variable on v degrees of freedom. This means that it has a probability density.

$$f(x) = \frac{x^{(v-2)/2} e^{-x/2}}{2^{v/2} \Gamma(v/2)}$$

You observe a sample size of one, drawn from this density, where v is unknown. What is the rejection region of a uniformly most powerful test of the null hypothesis that $v = 1$ against the alternative that $v = 2$ with a 5 percent size? [Note: $\Gamma(.)$ is Euler's gamma function; $\Gamma(0.5) > 0$ and $\Gamma(1) > 0$]

Solutions

1. If $X_1, X_2, \ldots X_n$ are iid random variables with mean μ and variance $\sigma^2 < \infty$, then $\bar{X} = n^{-1}\Sigma_{i=1}^{n} X_i \to_p \mu$. The CLT states that if, further, $X_1, X_2, \ldots X_n$ have $2+\delta$ finite moments for some $\delta > 0$, then $n^{1/2}(\bar{X} - \mu) \to_d N(0, \sigma^2)$.
2. The MLE of θ is $e^{\bar{X}}$. The asymptotic distribution of $\bar{X}$ is $n^{1/2}(\bar{X} - \mu) \to_d N(0,1)$. So by the delta method, $n^{1/2}(e^{\bar{X}} - \theta) \to_d N(0, e^{2\mu})$.
3. The marginal density is

$$f(x) = \int_0^1 \frac{3x+y}{7} dy = [\frac{3xy}{7} + \frac{y^2}{14}]_0^1 = \frac{3x}{7} + \frac{1}{14} = \frac{6x+1}{14}$$

The conditional density is

$$f(y \mid x) = \frac{f(x,y)}{f(x)} = \frac{3x+y}{7} * \frac{14}{6x+1} = \frac{6x+2y}{6x+1}.$$

4. Yes, $E(W_1) = \mu + 1 - 1 = \mu$. $Var(W_1) = Var(Y) + 4Var(X) = 1 + 4 * 0.5 * 0.5 = 2$. $E(W_1 \mid Y) = Y$ and its variance is 1. This is lower than the variance of W_1.

5. The acceptance region is $10 \pm \frac{1.96}{\sqrt{36}} = 10 \pm 0.3267$. Under the alternative, $\bar{X} \sim N(10.5, 1/36)$. The probability of being in the rejection region is 85.08% (power of the test).

6. (a) The transition matrix is $\begin{pmatrix} 0.9 & 0.4 \\ 0.1 & 0.6 \end{pmatrix}$; (b) 85 percent; (c) 80 percent of the time in expansions and 20 percent in recessions.

7. The posterior is $f(\theta \mid X) \propto f(X \mid \theta) f(\theta) = \frac{1(\theta > 0.5)}{\theta} \frac{1(\theta > 0.7)}{\theta} = \frac{1}{\theta^2} 1(\theta > 0.7)$. So the posterior is $k\frac{1}{\theta^2}$ with support on $[0.7, 1]$. As $\int_{0.7}^1 \theta^{-2} d\theta = [-\theta^{-1}]_{0.7}^1 = \frac{1}{0.7} - 1 = \frac{3}{7}$, the posterior is $\frac{7}{3\theta^2}$.

8. The confidence interval is $0.5 \pm \frac{1.96}{\sqrt{2}} = 0.5 \pm 1.386$, or from -0.886 to 1.886. Both bootstrap intervals are from -1 to +2.

9. The log-likelihood is $l(b) = \Sigma_{i=1}^{n} \log(X_i) - n\log(b) - \frac{1}{2b}\Sigma_{i=1}^{n} X_i^2$.

   The FOC is $-\frac{n}{b} + \frac{1}{2b^2}\Sigma_{i=1}^{n} X_i^2 = 0 \Rightarrow \frac{n}{b} = \frac{1}{2b^2}\Sigma_{i=1}^{n} X_i^2 \Rightarrow \hat{b}_{MLE} = \frac{1}{2n}\Sigma_{i=1}^{n} X_i^2 = \frac{21}{6} = 3.5$

   As $l''(b) = \frac{n}{b^2} - \frac{2}{2b^3}\Sigma_{i=1}^{n} X_i^2$

   $\therefore E(l''(b)) = \frac{n}{b^2} - \frac{2}{2b^3} 2bn = -\frac{n}{b^2}$

   and so, the asymptotic distribution is $\sqrt{n}(\hat{b} - b) \rightarrow_d N(0, b^2)$

10. By the Neyman-Pearson lemma, the rejection region is of the form

    $\{x : \frac{x^{(2-2)/2} e^{-x/2}}{2^{2/2}\Gamma(2/2)} \div \frac{x^{(1-2)/2} e^{-x/2}}{2^{1/2}\Gamma(1/2)} \geq k\}$

    $= \{x : \frac{x^{1/2} 2^{1/2}\Gamma(1/2)}{2\Gamma(1)} \geq k\}$

    $= \{x : x \geq k'\}$.

    The upper 5th percentile of a $\chi^2(1)$ is 3.84. So the rejection rule is to reject if and only if X>3.84.