Solutions to Problem Set 2

1. Here I was looking for something more intuitive than formal definitions. Identification is the question of whether the setup of the problem allows you to reverse-engineer what the value of a parameter is, with an infinitely large sample size. If you have an endogenous right-hand-side variable and an irrelevant instrument, you could have millions of observations, and it would be no use. If you observe price and quantity in a market, but have no other information, then you could again observe millions of observations, but you could never tell whether you had a demand curve or a supply curve. These are examples of where identification fails. There is no information that allows you to tell apart the true parameters from incorrect false parameter values.

2. The efficient GMM estimator is

$$\hat{\beta} = \arg\min_{\beta} \begin{pmatrix} \Sigma(y_i - \beta x_i)x_i \\ \Sigma(y_i - \beta x_i)z_i \end{pmatrix}' W \begin{pmatrix} \Sigma(y_i - \beta x_i)x_i \\ \Sigma(y_i - \beta x_i)z_i \end{pmatrix}$$

The FOC is

$$\begin{pmatrix} \Sigma x_i^2 \\ \Sigma x_i z_i \end{pmatrix}' W \begin{pmatrix} \Sigma(y_i - \hat{\beta} x_i)x_i \\ \Sigma(y_i - \hat{\beta} x_i)z_i \end{pmatrix} = 0$$

$$\therefore \Sigma x_i^2 w_{11}\Sigma(y_i - \hat{\beta}x_i)x_i + \Sigma x_i^2 w_{12}\Sigma(y_i - \hat{\beta}x_i)z_i + \Sigma x_i z_i w_{12}\Sigma(y_i - \hat{\beta}x_i)x_i + \Sigma x_i z_i w_{22}\Sigma(y_i - \hat{\beta}x_i)z_i = 0$$

$$\therefore \Sigma x_i^2 w_{11}\Sigma y_i x_i + \Sigma x_i^2 w_{12}\Sigma y_i z_i + \Sigma x_i z_i w_{12}\Sigma y_i x_i + \Sigma x_i z_i w_{22}\Sigma y_i z_i = \hat{\beta}\{(\Sigma x_i^2)^2 w_{11} + 2\Sigma x_i^2 w_{12}\Sigma x_i z_i + (\Sigma x_i z_i)^2 w_{11}\}$$

$$\therefore \hat{\beta} = \frac{\Sigma x_i^2 w_{11}\Sigma y_i x_i + \Sigma x_i^2 w_{12}\Sigma y_i z_i + \Sigma x_i z_i w_{12}\Sigma y_i x_i + \Sigma x_i z_i w_{22}\Sigma y_i z_i}{(\Sigma x_i^2)^2 w_{11} + 2\Sigma x_i^2 w_{12}\Sigma x_i z_i + (\Sigma x_i z_i)^2 w_{11}}$$

where $W = V^{-1}$ and $V = \begin{pmatrix} \Sigma x_i^2 & \Sigma x_i z_i \\ \Sigma x_i z_i & \Sigma z_i^2 \end{pmatrix}$.

3. $\Sigma_{i=1}^n h(Y_i, \hat{\theta}) = \Sigma_{i=1}^n h(Y_i, \theta_0) + \Sigma_{i=1}^n \frac{dh(Y_i, \theta_0)}{d\theta}(\hat{\theta} - \theta_0)$.

$$\therefore n^{-1/2}\Sigma_{i=1}^n h(Y_i, \hat{\theta}) = n^{-1/2}\Sigma_{i=1}^n h(Y_i, \theta_0) + n^{-1}\Sigma_{i=1}^n \frac{dh(Y_i, \theta_0)}{d\theta} n^{1/2}(\hat{\theta} - \theta_0)$$

$$\therefore n^{-1/2}\Sigma_{i=1}^n h(Y_i, \hat{\theta}) = n^{-1/2}\Sigma_{i=1}^n h(Y_i, \theta_0) - n^{-1}\Sigma_{i=1}^n \frac{dh(Y_i, \theta_0)}{d\theta}\{n^{-1}\Sigma_{i=1}^n \frac{dh(Y_i, \theta_0)}{d\theta}' W n^{-1}\Sigma_{i=1}^n \frac{dh(Y_i, \theta_0)}{d\theta}\}^{-1}$$

$$n^{-1}\Sigma_{i=1}^n \frac{dh(Y_i, \theta_0)}{d\theta}' W n^{-1/2}\Sigma_{i=1}^n h(Y_i, \theta_0)$$

using the expression for $\hat{\theta} - \theta_0$ derived in the lecture notes. Suppose that $n^{-1}\Sigma_{i=1}^n \frac{dh(Y_i, \theta_0)}{d\theta} \to_p D$ and $n^{-1/2}\Sigma_{i=1}^n h(Y_i, \theta_0) \to_d E^{1/2}\phi$ where ϕ is $N(0, I)$ and $W = E^{-1}$. Then

$$n^{-1/2}\Sigma_{i=1}^n h(Y_i, \hat{\theta}) \to_d E^{1/2}\phi - D(D'E^{-1}D)^{-1}D'E^{-1/2}\phi = E^{1/2}(I - E^{-1/2}D(D'E^{-1}D)^{-1}D'E^{-1/2})\phi$$

$$\therefore S(\hat{\theta}) \to_d \phi'\tilde{E}'E^{1/2}E^{-1}E^{1/2}\tilde{E}\phi = \phi'\tilde{E}'\tilde{E}\phi$$

where $\tilde{E} = I - E^{-1/2}D(D'E^{-1}D)^{-1}D'E^{-1/2}$. But $\tilde{E}$ is an idempotent matrix and $trace(\tilde{E}) = k - p$, so the rank of $\tilde{E}$ is also $k-p$ and this means that $S(\hat{\theta}) \rightarrow_d \chi^2(k)$.

4. In this case, $V(\hat{\theta}) = (D'E^{-1}D)^{-1}$ and $V(\tilde{\theta}) = (D'D)^{-1}D'ED(D'D)^{-1}$

$I - E^{-1/2}D(D'E^{-1}D)^{-1}D'E^{-1/2}$ is idempotent and therefore positive semidefinite

$\therefore (D'D)^{-1}D'E^{1/2}\{I - E^{-1/2}D(D'E^{-1}D)^{-1}D'E^{-1/2}\}E^{1/2}D(D'D)^{-1}$ is positive semidefinite

$\therefore (D'D)^{-1}D'ED(D'D)^{-1} - (D'E^{-1}D)^{-1}$ is positive semidefinite

$\therefore V(\tilde{\theta}) - V(\hat{\theta})$ must be positive semidefinite.

5. Here is the program

```
randn('seed',123);
rand('seed',123);
replics=1000;
nboot=1000;
for imc=1:replics;
    z=filter(1,[1;-0.95],randn(100,1));
    y=z(2:100);
    x=z(1:99);
    bhat=inv(x'*x)*x'*y;
    res=y-(x*bhat);
    sigs=mean(res.^2);
    se=sqrt(sigs*inv(x'*x));
    for j=1:nboot;
        resboot=res(ceil(rand(99,1)*99));
        zboot(1,1)=z(1);
        for t=1:99; zboot(t+1,1)=(bhat*zboot(t))+resboot(t); end;
        yboot=zboot(2:100);
        xboot=zboot(1:99);
        bhatboot(j)=inv(xboot'*xboot)*xboot'*yboot;
        sigsboot=mean((yboot-(xboot*bhatboot(j))).^2);
        seboot(j)=sqrt(sigsboot*inv(xboot'*xboot));
    end;
    tboot=(bhatboot-bhat)./seboot;
    bhatboot=sort(bhatboot);
    tboot=sort(tboot);
    op=[bhatboot(25) bhatboot(975)];
    p=[(2*bhat)-bhatboot(975) (2*bhat)-bhatboot(25)];
    pt=[bhat-(se*tboot(975)) bhat-(se*tboot(25))];
    asy=[bhat-(1.96*se) bhat+(1.96*se)];
    if op(1)<0.95 & op(2)>0.95; cop(imc)=1; else; cop(imc)=0; end;
    if p(1)<0.95 & p(2)>0.95; cp(imc)=1; else; cp(imc)=0; end;
    if pt(1)<0.95 & pt(2)>0.95; cpt(imc)=1; else; cpt(imc)=0; end;
    if asy(1)<0.95 & asy(2)>0.95; casy(imc)=1; else; casy(imc)=0; end;
 end;
disp('Coverage Rates');
disp('Other Percentile');
disp(mean(cop));
disp('Percentile');
disp(mean(cp));
disp('Percentile-t');
```

```
disp(mean(cpt));
disp('Asymptotic');
disp(mean(casy));
```

And here are the results
Coverage Rates

```
Other Percentile
   0.8960
Percentile
   0.9260
Percentile-t
   0.9540
Asymptotic
   0.9500
```

In this case, the bootstrap is not doing well. The other percentile is too wide; the percentile and percentile-t confidence intervals are too short. In this case, the asymptotic interval is right on the nose, but that wouldn't be true if we put in an intercept or trend.

6. Here is a program that does the continuous updating estimator with standard errors and the Stock-Wright confidence set. It can easily be adapted to give the other estimators.

Main Program

```
r=xlsread('euler.xls','Sheet1','D2:D117');
c=xlsread('euler.xls','Sheet1','B2:B117');
rf=xlsread('euler.xls','Sheet1','C2:C117');
z=[ones(113,1) c(3:115)./c(2:114) c(2:114)./c(1:113) r(3:115) r(2:114)
rf(3:115) rf(2:114)];


est=fminsearch(@(est) cu(est,r,rf,c,z),[0.95;5]);

delta=est(1); gamma=est(2);
bigt=116; nints=size(z,2);
w=(c(4:bigt)./c(3:bigt-1)).^(-gamma);
h=(delta.*w.*r(4:bigt))-1;
f=kron(h,ones(1,nints)).*z;
h=(delta.*w.*rf(4:bigt))-1;
f=[f kron(h,ones(1,nints)).*z];
h1=w.*r(4:bigt);
h2=-w.*r(4:bigt).*log(c(4:bigt)./c(3:bigt-1));
f1=kron(h1,ones(1,nints)).*z; f2=kron(h2,ones(1,nints)).*z;
h1=w.*rf(4:bigt);
h2=-w.*rf(4:bigt).*log(c(4:bigt)./c(3:bigt-1));
f1=[f1 kron(h1,ones(1,nints)).*z]; f2=[f2 kron(h2,ones(1,nints)).*z];
fdm=f-kron(ones(bigt-3,1),mean(f));
E=(fdm'*fdm)/(bigt-3);
D=[mean(f1);mean(f2)]';
vcov=inv(D'*inv(E)*D)/bigt;
disp('CU Estimates and Standard Errors');
```

```
[est sqrt(diag(vcov))]

deltas=[0.9:0.01:1.1]; gammas=[1:1:135];
for i=1:length(deltas); for j=1:length(gammas);
        est=[deltas(i);gammas(j)];
        if cu(est,r,rf,c,z)<chi2inv(0.95,14); s(i,j)=1; else s(i,j)=0; end;
end; end;
g=[[1 1 1];[0.8 0.8 0.8]];
pcolor(deltas,gammas,s');
colormap(g);
shading flat;
ylabel('\gamma');
xlabel('\delta');
```

Continuous Updating Objective Function

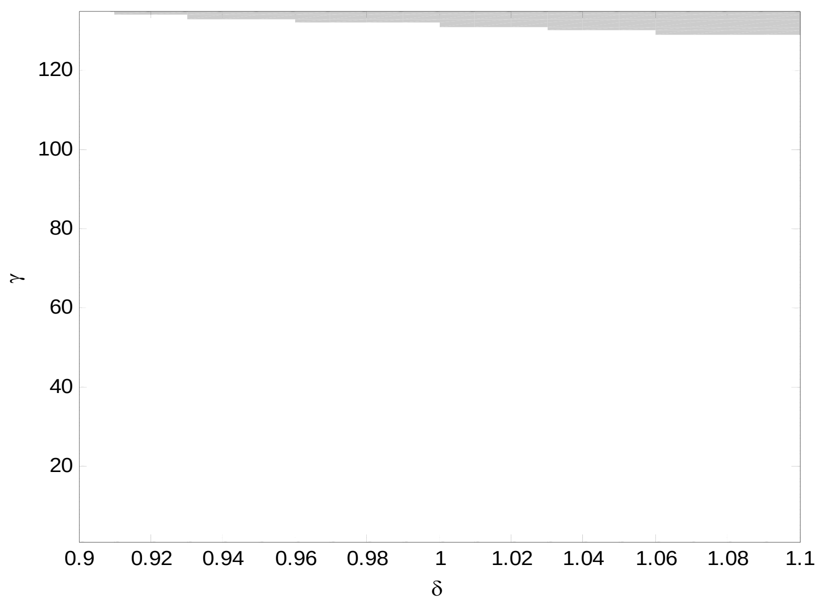
```
function s=cu(est,r,rf,c,z);

delta=est(1); gamma=est(2);
bigt=length(r); nints=size(z,2);
w=(c(4:bigt)./c(3:bigt-1)).^(-gamma);
h=(delta.*w.*r(4:bigt))-1;
f=kron(h,ones(1,nints)).*z;
h=(delta.*w.*rf(4:bigt))-1;
f=[f kron(h,ones(1,nints)).*z];
fdm=f-kron(ones(bigt-3,1),mean(f));
w=inv((fdm'*fdm)/(bigt-3));
s=bigt*mean(f)*w*mean(f)';
```

Running this the results are
CU Estimates and Standard Errors

0.9514	0.0071
-1.0724	0.2155

and the confidence set is



7. The mean and variance-covariance matrix of gross bond and stock returns are

$$\mu = (1.0201, 1.0819)'$$

and

$$\Sigma = \begin{pmatrix} 0.0035 & 0.0013 \\ 0.0013 & 0.0346 \end{pmatrix}$$

respectively. The HJ bound is

$$\sigma^2(M_{t+1}) \geq (i - E(M_{t+1})\mu)'\Sigma^{-1}(i - E(M_{t+1})\mu)$$

I take values of $E(M_{t+1})$ from 0.8 to 1.2 and compute the bound for $\sigma(M_{t+1})$ (the square root for each of these values). Varying $E(M_{t+1})$ over this range sweeps out a parabola. Here it is: