Solutions to Problem Set 1

1. (a) From the delta method, $T^{1/2}(\hat{\phi}^2 - \phi^2) \Rightarrow_d N(0,(1 - \phi^2)4\phi^2)$

(b) $y_t = \phi y_{t-1} + u_t = \phi(\phi y_{t-2} + u_{t-1}) + u_t = \phi^2 y_{t-2} + u_t + \phi u_{t-1} = \alpha y_{t-2} + v_t$

(c) $T^{1/2}(\hat{\alpha} - \alpha) \Rightarrow_d N(0, \frac{E(y_{t-2}^2)}{E(y_{t-2}^2)^2})$

noting that there are no other non-zero terms in the numerator. Hence

$$T^{1/2}(\hat{\alpha} - \alpha) \Rightarrow_d N(0, \frac{\sigma^2}{1 - \phi^2} \frac{\sigma^2(1 + \phi^2) + 2\phi \sigma^2}{1 - \phi^2 \phi \sigma^2})$$

$$\therefore T^{1/2}(\hat{\alpha} - \alpha) \Rightarrow_d N(0,(1 - \phi^2)(1 + 3\phi^2))$$

Since $1 + 3\phi^2 > 4\phi^2$ (given that $|\phi| < 1$), the variance in (c) is bigger than that in (a).

2. Here’s the program:

```matlab
replics=1000;
bigt=400;
a=0.9;
randn('seed',132);
for imc=1:replics;
    u=randn(bigt,1);
    eps=filter(1,[1;-a],u);
    y=1+eps;
    bhat=mean(y);
    res=y-bhat;
    a0=(res'*res)/bigt;
    a1=a0+(0.5*res(2:end,:)'*res(1:end-1,:)/bigt)+(0.5*res(1:end-1,:)'*res(2:end,:)/bigt);
    a2=a0;
    for i=1:26;
        w=(27-i)/27;
        a2=a2+(w*res(i+1:end,:)'*res(1:end-i,:)/bigt)+(w*res(1:end-i,:)'*res(1+1:end,:)/bigt);
    end;
    vcov1=a0/bigt;
    vcov2=a1/bigt;
    vcov3=a2/bigt;
    ci1=[bhat-(1.96*sqrt(vcov1)) bhat+(1.96*sqrt(vcov1))];
    ci2=[bhat-(1.96*sqrt(vcov2)) bhat+(1.96*sqrt(vcov2))];
    ci3=[bhat-(1.96*sqrt(vcov3)) bhat+(1.96*sqrt(vcov3))];
    ci4=[bhat-(2.15*sqrt(vcov3)) bhat+(2.15*sqrt(vcov3))];
    if ci1(1)<=1 & ci1(2)>=1; cover(imc,1)=1; else cover(imc,1)=0; end;
    if ci2(1)<=1 & ci2(2)>=1; cover(imc,2)=1; else cover(imc,2)=0; end;
    if ci3(1)<=1 & ci3(2)>=1; cover(imc,3)=1; else cover(imc,3)=0; end;
    if ci4(1)<=1 & ci4(2)>=1; cover(imc,4)=1; else cover(imc,4)=0; end;
end;
disp('Average Effective Coverage');
disp(mean(cover));
```
And here are the results:
0.3490  0.4710  0.8590  0.8900

The first two methods do horribly on coverage. The longer lag length essentially solves the problem, though it is better to use Kiefer-Vogelsang critical values. Even with this, there is still some shortfall in coverage.

3. As $\Sigma = RR'$,
$$R' \Sigma^{-1} = R'(RR')^{-1} = R'(R')^{-1} R^{-1} = R^{-1} = S$$

4. Here’s the program:
clear
close all
m=xlsread('pset1.xlsx','Macrodata','B2:D577');
y=m(13:end,:);
bigt=size(y,1);
x=ones(bigt,1);
for j=1:12; x=[x m(13-j:end-j,:)];
end;
bhat=inv(x'*x)*x'*y;

for imc=1:1000;
    sigs=iw(u'*u,bigt-36); %iw is an inverse Wishart
    alphahat=reshape(bhat,111,1);
    vcov=kron(sigs,inv(x'*x));
    alpha=alphahat+(chol(vcov)'*randn(111,1));
    r=chol(sigs)';
    A=reshape(alpha,37,3)';
    A=[A(:,2:37); eye(33) zeros(33,3)];
    for j=1:61; c=A^(j-1); d(:,:,j)=c(1:3,1:3)*r;
    end;
    IR1(imc,:)=squeeze(d(1,3,:));
    IR2(imc,:)=squeeze(d(2,3,:));
    IR3(imc,:)=squeeze(d(3,3,:));
end;
IR1=sort(IR1);
IR2=sort(IR2);
IR3=sort(IR3);
subplot(2,2,1);
plot(mean(IR1),'Linewidth',2);
hold on;
plot(IR1(25,:),'r-','Linewidth',2);
plot(IR1(975,:),'r-','Linewidth',2);
title('Effect on Unemployment');
axis([0 60 -0.2 0.2]);
subplot(2,2,2);
plot(mean(IR2),'Linewidth',2);
And here are the results:

Note that the data ends in 2007. This VAR would give crazy results covering the Great Recession. The reason is clear. The Fed cut interest rates in 2007 and 2008. The VAR thinks that this was because of a monetary policy easing shock. There followed a massive recession and a fall in inflation, but this was not because monetary policy had been eased. Perhaps larger VARs might avoid this unreasonable result.

5. Yes. Here is an example
\[ Y_t = \begin{pmatrix} 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 0 \end{pmatrix} \epsilon_t + \begin{pmatrix} 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} \end{pmatrix} \epsilon_{t-1} \]

Direct calculations reveal that \( Y_t \) is iid \((0, I)\) as required. However, it is not an invertible moving average representation.