Handout on Matrices

1 Introduction

The purpose of this note is to introduce the concept of matrices, and to provide the basic information about matrices that is useful for Investments.

1.1 Matrices Defined

An nxm matrix M is a collection of numbers arranged in n rows and m columns. For example, the matrix M below is a 3x2 matrix.

$$M = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$$

1.2 Special matrices

A matrix that has many rows but only one column is called a column vector. A matrix that has one row, but has many columns is a row vector. In the example below q is a column vector and v is a row vector.

$$q = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$
$$v = \begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix}$$

A matrix with the same number of rows and columns is said to be square.
The identity matrix, denoted I is a square matrix in which the elements in which the row i, column i elements are equal to 1, and all other elements are equal to 0. A 3x3 identity matrix is presented below:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2 Elementary Matrix Operations

2.1 Matrix Transpose

If A is an nxm matrix, then the transpose of A, denoted A' is an mxn matrix in which the row i column j element of A is the row j column i element of A' . Examples of matrices and their transposes are given below:

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix}$$
$$A' = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$
$$B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$$
$$B' = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$$

2.2 Matrix Addition and Subtraction

If A and B both have the same number of rows and columns, then they can be added or subtracted together. Denote the sum of $A + B$ as the matrix C. C will have the same number of rows and columns as A and B with elements given by just adding the elements of A and B. For example,

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{pmatrix} = \begin{pmatrix} 6 & 8 \\ 10 & 12 \end{pmatrix}$$

Subtraction is similar.

2.3 Multiplication

If A is an nxm matrix and B is an mxk matrix, then because the number of columns of A is the same as the number of rows of B, A and B can be multiplied. Call the result C. Then, C is an nxk matrix. The element in the ith row and jth column of C consists of multiplying all the elements in the ith row of A by the corresponding element of the jth column of B and adding them up. For example

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} * \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} (1*1)+(2*2) & (1*1)+(2*3) \\ (3*1)+(2*4) & (3*1)+(3*4) \\ (5*1)+(2*6) & (5*1)+(3*6) \end{pmatrix} = \begin{pmatrix} 5 & 7 \\ 11 & 15 \\ 17 & 23 \end{pmatrix}$$

Or if $k = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$ and $X = \begin{pmatrix} 0 \\ -2 \\ 3 \\ 6 \end{pmatrix}$ then $k'X = (1*0)+(2*-2)+(3*3)+(4*6) = 29$.

Facts about Matrix Multiplication:
1. Matrix A cannot be multiplied by matrix B (AxB) unless the number of columns of A is equal to the number of rows of B.
2. Matrix multiplication is associative:
$(A$x$B)$x$C + A$x$(B$x$C)$
3. Matrix multiplication is distributive:
Ax$(B + C) = A$x$B + A$xC
4. Matrix multiplication is *not* commutative. In other words, if you reverse the order of multiplication when multiplying matrices, it usually changes the answer.
Ax$B \neq B$xA
5. There's also "scalar multiplication" which is different from matrix multiplication. The matrix is multiplied by a single number, called a *scalar*. It just means that each element is multiplied by that number. For example, if

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

then

$$3A = \begin{pmatrix} 3 & 6 \\ 9 & 12 \end{pmatrix}$$

and this is scalar multiplication.

2.4 Inverses

If A is an nxn square matrix, then the inverse of A, written as A^{-1}, is the nxn square matrix that satisfies the equation

$$AA^{-1} = I$$

where I is the nxn identity matrix. It's hard to work out the matrix inverse for big matrices. But for small ones it's easy. First take the trivial case where $n = 1$. Then if $A = a$ (a simple number), clearly $A^{-1} = 1/a$.
For $n = 2$, there is a formula. If

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then

$$A^{-1} = \begin{pmatrix} \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}$$

And we can check the formula by multiplying the matrices together.

$$AA^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Think of a matrix inverse as a matrix analog of division.

3 Matrices and Investments

There are n assets 1, ...n with returns $R_1, R_2, ...R_n$, respecitvely. Let R be the nx1 vector of these returns, $R = (R_1, R_2, ...R_n)'$. The expected return on asset i is μ_i. The vector of expected returns on all assets, $E(R)$ is denoted $\mu = (\mu_1, \mu_2, ...\mu_n)'$. The variance and covariances of the assets are represented in a variance-covariance matrix, which is denoted Σ:

$$\Sigma = \begin{pmatrix} Var(R_1) & Cov(R_1, R_2) & ... & Cov(R_1, R_n) \\ Cov(R_1, R_2) & Var(R_2) & & Cov(R_2, R_n) \\ : & : & & : \\ Cov(R_1, R_n) & Cov(R_2, R_n) & & Var(R_n) \end{pmatrix}$$

Suppose that we have a portfolio with a weight w_i in asset i. Let w denote the vector of weights $(w_1, w_2, ...w_n)'$. Assume that these weights sum to one. Then the return on this portfolio is

$$r_p = w'R = w_1R_1 + w_2R_2... = w_nR_n$$

and we have

$$E(r_p) = w'\mu$$
$$Var(r_p) = w'\Sigma w$$